*Exercise Sheet 11 - Model Answers*

1. (a)

$$\begin{aligned} S\ n\ (\lambda u.\ u) &= (\lambda mh.\ h(mfx))\ n\ (\lambda u.\ u) \\ &\longrightarrow_\beta (\lambda u.\ u)\ (n\ f\ x) \\ &\longrightarrow_\beta n\ f\ x \end{aligned}$$

   (b)

$$\begin{aligned} T(\lambda u.\ x) &= (\lambda gh.\ h(gf))\ (\lambda u.\ x) \\ &\longrightarrow_\beta (\lambda h.\ h(((\lambda u.\ x))\ f)) \\ &\longrightarrow_\beta \lambda h.\ hx \end{aligned}$$

$$\begin{aligned} S0 &= (\lambda mh.\ h(mfx))0 \\ &\longrightarrow_\beta \lambda h.\ h(0fx) \\ &\longrightarrow_\beta \lambda h.\ hx \end{aligned}$$

2. (a) $f$ can have any type $A$ whatsoever, and $0 \stackrel{\text{def}}{=\!=} \lambda f{:}A\ x{:}\sigma.\ x{:}\sigma$ is well typed, with type $A \to \sigma \to \sigma$.

   (b) To make the application $fx$ well typed, $f$ must have type $\sigma \to B$ for some $B$. Then $1 \stackrel{\text{def}}{=\!=} \lambda f{:}\sigma \to B\ x{:}\sigma.\ f{:}\sigma \to B\ x{:}\sigma$ is well typed, with type $(\sigma \to B) \to \sigma \to B$.

   (c) We must match

$$A \to \sigma \to \sigma$$
$$(\sigma \to B) \to \sigma \to B$$

   To match the final results without imposing any restrictions on $\sigma$, we take $B = \sigma$, after which we find we must take $A = \sigma \to B = \sigma \to \sigma$. Hence we type $f{:}\sigma \to \sigma$. 0 and 1 then have type $(\sigma \to \sigma) \to \sigma \to \sigma$.
   We then have that

$$2 \stackrel{\text{def}}{=\!=} \lambda f{:}\sigma \to \sigma\ x{:}\sigma.\ f{:}\sigma \to \sigma\ (f{:}\sigma \to \sigma\ x{:}\sigma)$$

   is well typed since $fx$ and $f(fx)$ both have type $\sigma$, and again has type $(\sigma \to \sigma) \to \sigma \to \sigma$.

3. (a) $+\ m\ n$ is got by adding 1 to $n$ $m$ times.

$$+ \stackrel{\text{def}}{=\!=} \lambda mn.\ m\ \mathsf{succ}\ n$$

   (b) $m + n$ is $n$ if $m$ is 0, and otherwise $1 + ((m - 1) + n)$.

$$\mathsf{Y}_{\mathbf{nat}\to\mathbf{nat}\to\mathbf{nat}}\ \lambda P.\ \lambda mn.\ \mathsf{zero?}_{\mathbf{nat}}\ m\ n\ (\mathsf{succ}\ (P\ (\mathsf{pred}\ m)\ n))$$

   (No marks deducted if you missed out the types.)