Exercises 1
Cbits and vectors

1. Write as many notations as you can for the ket $|9⟩^4$.

2. Calculate the matrices for the following operators.
   (a) The 2-Cbit operator $X_1X_0$ that flips both Cbits. (Note – don’t confuse this with the swap operator.)
   (b) The 3-Cbit operator $S_{21}$ that swaps the states of Cbits 2 and 1.

3. Design a 3-Cbit gate $A_{210}$ that flips Cbit 0 if at least one of Cbits 1,2 has state $|1⟩$. (It leaves Cbits 1 and 2 unchanged.)
   (a) Write down an expression for $A_{210} |xyz⟩$.
   (b) Show that it is reversible.
   (c) Write down its matrix.
   (d) Draw a circuit diagram to construct it using NOT gates and Toffoli gates. (Use as many of each as you need).
   (e) Show that that configuration of gates really does compute $A_{210}$.

Hint: Think of the bits 0 and 1 as logical false and true. Then multiplication $xy$ is logical “and”, $x ∨ y$; addition $x ⊕ y$ modulo 2 is “exclusive or”; $x ⊕ 1$ is the same as complementation $\bar{x}$ and gives logical “not”. If “inclusive or” is written $x ∨ y$ then de Morgan’s law says that $\bar{x} ∨ y = \bar{x} ∧ \bar{y}$.

Solutions

1. $|9⟩$, $|1001⟩$, $|1⟩|0⟩|0⟩|1⟩$, $|1⟩ ⊗ |0⟩ ⊗ |0⟩ ⊗ |1⟩$ and

$$
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
$$

2. (a) $X_1X_0$ exchanges $|00⟩ = |0⟩$ with $|11⟩ = |3⟩$, and $|01⟩ = |1⟩$ with $|10⟩ = |2⟩$. Its matrix is

$$
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
$$

(b) $S_{21}$ exchanges $|2⟩$ with $|4⟩$ and $|3⟩$ with $|5⟩$ and fixes all the rest. Its matrix is

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$
3. (a) 
\[ A_{210}|xyz\rangle = |x\rangle|y\rangle|z \oplus (x \lor y)\rangle \]

(b) \( A_{210} \) is its own inverse.
\[ A_{210}A_{210}|xyz\rangle = A_{210}|x\rangle|y\rangle|z \oplus (x \lor y)\rangle = |x\rangle|y\rangle|z \oplus (x \lor y) \oplus (x \lor y)\rangle = |xyz\rangle \]

(c) \( A_{210} \) fixes (leaves unchanged) those states \(|xyz\rangle\) in which \( x = y = 0 \). These two fixed states \(|000\rangle = |0\rangle_3 \) and \(|001\rangle = |1\rangle_3 \) correspond to 1s on the diagonal of the matrix. If either of \( x \), \( y \) is 1 then it exchanges the two states \(|xy0\rangle \) and \(|xy1\rangle \): thus \(|2\rangle \leftrightarrow |3\rangle \), \(|4\rangle \leftrightarrow |5\rangle \) and \(|6\rangle \leftrightarrow |7\rangle \). These exchanges correspond to six 1s off the diagonal. The matrix is
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

(d) Using de Morgan’s law we can calculate
\[ z \oplus (x \lor y) = z \oplus \tilde{x} \land \tilde{y} = z \oplus (\tilde{x}\tilde{y} \oplus 1) \]
and so
\[ A_{210}|xyz\rangle = |x\rangle|y\rangle|z \oplus (\tilde{x}\tilde{y} \oplus 1)\rangle \]
\[ = X_0|x\rangle|y\rangle|z \oplus (\tilde{x}\tilde{y})\rangle \\
= X_2X_1X_0|x\rangle|\tilde{y}\rangle|z \oplus (\tilde{x}\tilde{y})\rangle \\
= X_2X_1X_0T_{210}(|\tilde{x}\rangle|\tilde{y}\rangle|z\rangle) \\
= X_2X_1X_0T_{210}X_2X_1|x\rangle|y\rangle|z\rangle . \]
(Note that because \( X \) and \( T \) are reversible, so is \( A \).)

Hence \( A_{210} \) can be implemented using the following circuit diagram. Remember that the operations \( X_2X_1X_0T_{210}X_2X_1 \) are applied from right to left, while the diagram is read from left to right.

(e) We have already shown this.