06-22457 *Quantum Computing and Cryptography*
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Exercises 11
Introduction to Quantum Error Correction

1. Verify some of the properties of projectors in the slides. If A is an involution, we define

$$\mathsf{P}_0^{\mathsf{A}} = \frac{1}{2}(1 + \mathsf{A}), \qquad \mathsf{P}_1^{\mathsf{A}} = \frac{1}{2}(1 - \mathsf{A}).$$

   Show that

$$\mathsf{P}_x^{\mathsf{A}}\mathsf{P}_y^{\mathsf{A}} = \begin{cases} \mathsf{P}_x^{\mathsf{A}} & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

$$\mathsf{P}_0^{\mathsf{A}} + \mathsf{P}_1^{\mathsf{A}} = 1$$

$$\mathsf{A}\mathsf{P}_x^{\mathsf{A}} = (-1)^x \, \mathsf{P}_x^{\mathsf{A}}$$

$$\mathsf{A}|\psi\rangle = |\psi\rangle \Longleftrightarrow \mathsf{P}_0^{\mathsf{A}}|\psi\rangle = |\psi\rangle \Longleftrightarrow \mathsf{P}_1^{\mathsf{A}}|\psi\rangle = 0$$

$$\mathsf{A}|\psi\rangle = -|\psi\rangle \Longleftrightarrow \mathsf{P}_1^{\mathsf{A}}|\psi\rangle = |\psi\rangle \Longleftrightarrow \mathsf{P}_0^{\mathsf{A}}|\psi\rangle = 0$$

2. When measuring an involution A, the state before measurement was $|0\rangle\mathsf{P}_0^{\mathsf{A}}|\psi\rangle + |1\rangle\mathsf{P}_1^{\mathsf{A}}|\psi\rangle$. When the ancilla is measured, what are the probabilities of getting results 0 and 1? What are the resulting states for the data Qbits, as normalized vectors? (Remember that since the projectors $\mathsf{P}_x^{\mathsf{A}}$ are not unitary, $\mathsf{P}_x^{\mathsf{A}}|\psi\rangle$ need not have unit length.)

Solutions

1.

$$\begin{aligned} \mathsf{P}^{\mathsf{A}}_x \mathsf{P}^{\mathsf{A}}_y &= \frac{1}{4}\left(1+(-1)^x \mathsf{A}\right)\left(1+(-1)^y \mathsf{A}\right) \\ &= \frac{1}{4}\left(\left(1+(-1)^{x+y}\right)1 + \left((-1)^x+(-1)^y\right)\mathsf{A}\right) \end{aligned}$$

If $x \neq y$ this is 0, while if $x = y$ it is

$$\frac{1}{4}\left((2)\,1 + 2\,(-1)^x \mathsf{A}\right) = \frac{1}{2}\left(1+(-1)^x \mathsf{A}\right) = \mathsf{P}^{\mathsf{A}}_x.$$

(Or you can check the cases separately.)

$$\begin{aligned} \mathsf{P}^{\mathsf{A}}_0 + \mathsf{P}^{\mathsf{A}}_1 &= \frac{1}{2}(1+\mathsf{A}) + \frac{1}{2}(1-\mathsf{A}) = 1 \\ \mathsf{A}\mathsf{P}^{\mathsf{A}}_x &= \frac{1}{2}\mathsf{A}\left(1+(-1)^x \mathsf{A}\right) = \frac{1}{2}\left(\mathsf{A}+(-1)^x 1\right) \\ &= (-1)^x \frac{1}{2}\left(1+(-1)^x \mathsf{A}\right) = (-1)^x \mathsf{P}^{\mathsf{A}}_x \end{aligned}$$

$$\begin{aligned} \mathsf{P}^{\mathsf{A}}_x|\psi\rangle = |\psi\rangle &\iff \frac{1}{2}\left(1+(-1)^x \mathsf{A}\right)|\psi\rangle = |\psi\rangle \\ &\iff |\psi\rangle + (-1)^x \mathsf{A}|\psi\rangle = 2|\psi\rangle \\ &\iff (-1)^x \mathsf{A}|\psi\rangle = |\psi\rangle \\ &\iff \mathsf{A}|\psi\rangle = (-1)^x |\psi\rangle \end{aligned}$$

$$\begin{aligned} \mathsf{P}^{\mathsf{A}}_x|\psi\rangle = 0 &\iff \frac{1}{2}\left(1+(-1)^x \mathsf{A}\right)|\psi\rangle = 0 \\ &\iff |\psi\rangle + (-1)^x \mathsf{A}|\psi\rangle = 0 \\ &\iff (-1)^x \mathsf{A}|\psi\rangle = -|\psi\rangle \\ &\iff \mathsf{A}|\psi\rangle = (-1)^{\tilde{x}} |\psi\rangle \end{aligned}$$

2. The squared length of $\mathsf{P}^{\mathsf{A}}_x|\psi\rangle$ is

$$\langle\psi|\mathsf{P}^{\mathsf{A}}_x\mathsf{P}^{\mathsf{A}}_x|\psi\rangle = \langle\psi|\mathsf{P}^{\mathsf{A}}_x|\psi\rangle$$

so, normalizing the vectors $\mathsf{P}^{\mathsf{A}}_x|\psi\rangle$, the state becomes

$$\sqrt{\langle\psi|\mathsf{P}^{\mathsf{A}}_0|\psi\rangle}|0\rangle\frac{\mathsf{P}^{\mathsf{A}}_0|\psi\rangle}{\sqrt{\langle\psi|\mathsf{P}^{\mathsf{A}}_0|\psi\rangle}} + \sqrt{\langle\psi|\mathsf{P}^{\mathsf{A}}_1|\psi\rangle}|1\rangle\frac{\mathsf{P}^{\mathsf{A}}_1|\psi\rangle}{\sqrt{\langle\psi|\mathsf{P}^{\mathsf{A}}_1|\psi\rangle}}.$$

The probability of getting result x is $\langle\psi|\mathsf{P}^{\mathsf{A}}_x|\psi\rangle$, leaving the data Qbits in state $\frac{\mathsf{P}^{\mathsf{A}}_x|\psi\rangle}{\sqrt{\langle\psi|\mathsf{P}^{\mathsf{A}}_x|\psi\rangle}}$. (Note – in the case where $\langle\psi|\mathsf{P}^{\mathsf{A}}_x|\psi\rangle = 0$, the probability is 0 so we can ignore that possible result and do not have to worry about the division by 0 in $\frac{\mathsf{P}^{\mathsf{A}}_x|\psi\rangle}{\sqrt{\langle\psi|\mathsf{P}^{\mathsf{A}}_x|\psi\rangle}}$.)