Exercises 11
Introduction to Quantum Error Correction

1. Verify some of the properties of projectors in the slides. If \( A \) is an involution, we define

\[
P^A_0 = \frac{1}{2} (1 + A), \quad P^A_1 = \frac{1}{2} (1 - A).
\]

Show that

\[
P^A_x P^A_y = \begin{cases} 
  P^A_x & \text{if } x = y \\
  0 & \text{if } x \neq y
\end{cases}
\]

\[
P^A_0 + P^A_1 = 1
\]

\[
A P^A_x = (-1)^x P^A_x
\]

\[
A |\psi\rangle = |\psi\rangle \iff P^A_0 |\psi\rangle = |\psi\rangle \iff P^A_1 |\psi\rangle = 0
\]

\[
A |\psi\rangle = -|\psi\rangle \iff P^A_1 |\psi\rangle = |\psi\rangle \iff P^A_0 |\psi\rangle = 0
\]

2. When measuring an involution \( A \), the state before measurement was \( |0\rangle P^A_0 |\psi\rangle + |1\rangle P^A_1 |\psi\rangle \). When the ancilla is measured, what are the probabilities of getting results 0 and 1? What are the resulting states for the data Qbits, as normalized vectors? (Remember that since the projectors \( P^A_x \) are not unitary, \( P^A_x |\psi\rangle \) need not have unit length.)
Solutions

1. 
\[ P^A_x P^A_y = \frac{1}{4} \left( 1 + (-1)^x A \right) \left( 1 + (-1)^y A \right) \]
\[ = \frac{1}{4} \left( \left( 1 + (-1)^{x+y} \right) 1 + ((-1)^x + (-1)^y) A \right) \]

If \( x \neq y \) this is 0, while if \( x = y \) it is
\[ \frac{1}{4} \left( (2) 1 + 2 (-1)^x A \right) = \frac{1}{2} \left( 1 + (-1)^x A \right) = P^A_x. \]

(Or you can check the cases separately.)

\[ P^A_0 + P^A_1 = \frac{1}{2} (1 + A) + \frac{1}{2} (1 - A) = 1 \]
\[ AP^A_2 = \frac{1}{2} A (1 + (-1)^x A) = \frac{1}{2} (A + (-1)^x 1) \]
\[ = (-1)^x \frac{1}{2} (1 + (-1)^x A) = (-1)^x P^A_x \]

\[ P^A_2 |\psi\rangle = |\psi\rangle \iff \frac{1}{2} (1 + (-1)^x A) |\psi\rangle = |\psi\rangle \]
\[ \iff |\psi\rangle + (-1)^x A|\psi\rangle = 2|\psi\rangle \]
\[ \iff (-1)^x A|\psi\rangle = |\psi\rangle \]
\[ \iff A|\psi\rangle = (-1)^x |\psi\rangle \]

\[ P^A_2 |\psi\rangle = 0 \iff \frac{1}{2} (1 + (-1)^x A) |\psi\rangle = 0 \]
\[ \iff |\psi\rangle + (-1)^x A|\psi\rangle = 0 \]
\[ \iff (-1)^x A|\psi\rangle = -|\psi\rangle \]
\[ \iff A|\psi\rangle = (-1)^x |\psi\rangle \]

2. The squared length of \( P^A_x |\psi\rangle \) is
\[ \langle \psi | P^A_x P^A_x |\psi\rangle = \langle \psi | P^A_x |\psi\rangle \]
so, normalizing the vectors \( P^A_x |\psi\rangle \), the state becomes
\[ \sqrt{\langle \psi | P^A_0 \rangle |0\rangle} \frac{P^A_0 |\psi\rangle}{\sqrt{\langle \psi | P^A_0 \rangle |0\rangle}} + \sqrt{\langle \psi | P^A_1 \rangle |1\rangle} \frac{P^A_1 |\psi\rangle}{\sqrt{\langle \psi | P^A_1 \rangle |1\rangle}}. \]

The probability of getting result \( x \) is \( \langle \psi | P^A_x |\psi\rangle \), leaving the data Qbits in state \( \frac{P^A_x |\psi\rangle}{\sqrt{\langle \psi | P^A_x \rangle}} \). (Note – in the case where \( \langle \psi | P^A_x \rangle = 0 \), the probability is 0 so we can ignore that possible result and do not have to worry about the division by 0 in \( \frac{P^A_x |\psi\rangle}{\sqrt{\langle \psi | P^A_x \rangle}} \).)