Exercises 2

Vectors and operators

1. Let kets $|\phi\rangle$ and $|\psi\rangle$ be defined as follows.

$$|\phi\rangle = (3 + i)|0\rangle + (1 - \sqrt{5}i)|1\rangle,$$
$$|\psi\rangle = -2|0\rangle + (3 + 4i)|1\rangle.$$  

Calculate $|\phi\rangle|\psi\rangle$ (i.e. $|\phi\rangle \otimes |\psi\rangle$), $\langle\phi|\psi\rangle$ and $\langle\phi|\phi\rangle$. Find a normalized form of $|\phi\rangle$.

I said in the lecture that any operator could be expressed as a linear combination of composites of 1-Qbit operators. The next two exercises show how to do this.

2. We define the number operator $n$ on 1-Qbit states by $n|x\rangle = x|x\rangle$ ($x = 0, 1$). We also define $\bar{n} = 1 - n$, so $\bar{n}|x\rangle = (1 - x)|x\rangle = \bar{x}|x\rangle$.

(a) Calculate their matrices.
(b) Show that $n^2 = n$, $\bar{n}^2 = \bar{n}$ and $n\bar{n} = \bar{n}n = 0$.
(c) Show that $Xn = \bar{n}X$ and $nX = X\bar{n}$.

3. Consider a system of $m$ Qbits. We then have operators $X_i$, $n_i$ and $\bar{n}_i$ for each $i$ ($0 \leq i \leq m - 1$).

(a) If $0 \leq x \leq 2^m - 1$, then we define an $m$-Qbit operator $n^{(x)}$ to be a product of $m$ operators, one for each $i$, where the $i$th operator is $n_i$ if the $i$th bit of $x$ is 1, or $\bar{n}_i$ if the $i$th bit of $x$ is 0. For example, if $m = 3$ then

$$n^{(0)} = \bar{n}_2\bar{n}_1\bar{n}_0$$
$$n^{(1)} = \bar{n}_2\bar{n}_1n_0$$
$$n^{(2)} = \bar{n}_2n_1\bar{n}_0$$
$$n^{(3)} = \bar{n}_2n_1n_0$$
$$n^{(4)} = n_2\bar{n}_1\bar{n}_0$$

etc.

Show that (for general $m$)

$$n^{(x)}|z\rangle_m = \begin{cases} |z\rangle_m & \text{if } z = x, \\ 0 & \text{otherwise.} \end{cases}$$

From that, work out the matrix for $n^{(x)}$.

(b) Next, for each $x$ we similarly define an $m$-Qbit operator $X^{(x)}$ to be a product of $m$ operators, one for each $i$, where the $i$th operator is $X_i$ if the $i$th bit of $x$ is 1, or $1_i$ if the $i$th bit of $x$ is 0. For example, if $m = 3$ then $X^{(0)} = 1, X^{(1)} = X_0, X^{(2)} = X_1, X^{(3)} = X_1X_0$, etc.

Show that (for general $m$)

$$X^{(x)}|z\rangle_m = |x \oplus z\rangle.$$  

(c) Deduce that

$$X^{(x \oplus y)}n^{(y)}|z\rangle_m = \begin{cases} |x\rangle_m & \text{if } y = z, \\ 0 & \text{otherwise} \end{cases}$$

and work out its matrix. Deduce that any operator is a linear combination of operators of the form $X^{(x)}n^{(y)}$, and hence is a linear combination of products of 1-Cbit operators $1_i$, $X_i$, $n_i$ and $\bar{n}_i$.

(d) Show that $n = \frac{1}{2}(1 - Z)$ and $\bar{n} = \frac{1}{2}(1 + Z)$, and deduce that any operator is a linear combination of products of 1-Cbit operators $1_i$, $X_i$ and $Z_i$.

4. The expressions obtained using the method of the previous question can often be simplified. Show that (for any number of Qbits)

$$C_{ij} = \bar{n}_i + X_jn_i,$$
$$S_{ij} = n_in_j + \bar{n}_i\bar{n}_j + X_iX_jn_i\bar{n}_j + X_iX_j\bar{n}_i\bar{n}_j.$$
1. \( |\phi\rangle \psi = ((3 + i)|0\rangle + (1 - \sqrt{5}i)|1\rangle) (-2|0\rangle + (3 + 4i)|1\rangle) \\
= (3 + i)(-2)|0\rangle + (3 + i)(3 + 4i)|0\rangle|1\rangle + (1 - \sqrt{5}i)(-2)|1\rangle|0\rangle + (1 - \sqrt{5}i)(3 + 4i)|1\rangle|1\rangle \\
= (-6 - 2i)|0\rangle_2 + (5 + 15i)|1\rangle_2 + (-2 + 2\sqrt{5}i)|2\rangle_2 + (3 + 4\sqrt{5} + 4 - 3\sqrt{5}i)|3\rangle_2 \\
\langle \phi | \psi = ((3 - i)|0\rangle + (1 + \sqrt{5}i)|1\rangle) (-2|0\rangle + (3 + 4i)|1\rangle) \\
= -6 + 2i + 3 - 4\sqrt{5} + 4i + 3\sqrt{5}i \\
= -3 - 4\sqrt{5} + (6 + 3\sqrt{5})i \\
\langle \phi | \phi = ((3 - i)|0\rangle + (1 + \sqrt{5}i)|1\rangle) ((3 + i)|0\rangle + (1 - \sqrt{5}i)|1\rangle) \\
= 9 + 1 + 1 + 5 = 16 \\

To normalize \( |\phi\rangle \) we divide it by \( \sqrt{\langle \phi | \phi \rangle} = 4 \), giving \( \frac{3 + i}{4}|0\rangle + \frac{1 - \sqrt{5}i}{4}|1\rangle \).

2. (a) We have \( n|0\rangle = 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) and \( n|1\rangle = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). Hence the matrix of \( n \) is \( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \). Similarly, the matrix of \( \bar{n} \) is \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \).

(b) 
\[
n^2|x\rangle = nx|x\rangle \text{ (by definition)} \\
= xn|x\rangle \text{ (by linearity)} \\
= x^2|x\rangle \text{ (by definition)} \\
= x|x\rangle \text{ (since } x^2 = x \text{ if } x \text{ is 0 or 1)} \\
= n|x\rangle.
\]

Since \( n^2 \) agrees with \( n \) on all the base vectors \( |x\rangle \), by linearity the agree on all vectors and so are equal as operators. Alternatively, it is easy to check by matrix multiplication that \( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \) sqaured is \( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \).

Then
\[
\bar{n}^2 = (1 - n)^2 = 1 - 2n + n^2 = 1 - n = \bar{n} \\
n\bar{n} = n(1 - n) = n - n^2 = 0 \\
\]
and similarly \( \bar{n}^2 = 0 \).

(c) \( Xn|x\rangle = xX|x\rangle = x|\bar{x}\rangle = \bar{n}|\bar{x}\rangle = \bar{n}X|x\rangle. \) So \( Xn = \bar{n}X \). Then \( nX = (1 - \bar{n})X = X - \bar{n}X = X - Xn = X(1 - n) = X\bar{n}. \)

3. (a) The only way \( n(x)|z\rangle_m \) can be non-zero is if all the \( n_i \)s and \( \bar{n}_i \)s making up \( n(x) \) are non-zero on their own Qbits, and this happens when all the bits of \( x \) match the bits of \( z \), i.e. \( z = x \). In that case all the \( n_i \)s and \( \bar{n}_i \)s leave their Qbits unchanged, so \( n(x)|z\rangle_m = |z\rangle_m \). If the matrix is \( A \), then
\[
A_{yz} = \begin{cases} 1 & \text{if } y = z = x \\ 0 & \text{otherwise.} \end{cases}
\]
(b) \( X^{(x)} \) flips those Qbits in the positions where \( x \) has bit 1, and that is the same as xoring with \( x \).

(c) \( X^{(x \oplus y)}n^{(y)}|z\rangle_m = \begin{cases} X^{(x \oplus y)}|y\rangle_m & \text{if } y = z, \\ 0 & \text{otherwise,} \end{cases} \)
\[
= \begin{cases} |x \oplus y \oplus y\rangle_m & \text{if } y = z, \\ 0 & \text{otherwise,} \end{cases} \)
\[
= \begin{cases} |x\rangle_m & \text{if } y = z, \\ 0 & \text{otherwise.} \end{cases}
\]

Its matrix has 1 in the \((x, y)\) position, 0 everywhere else. But then a general operator, with matrix \( A \), can be written as a linear combination \( \sum_{xy} A_{xy}X^{(x \oplus y)}n^{(y)} \).
(d) This is easily checked.

4. For notational simplicity, we assume that $i = 1$ and $j = 0$. The same reasoning applies to other values. We apply the operators to basis states $|z\rangle|x\rangle|y\rangle$ in which $|x\rangle$ and $|y\rangle$ are the single Qbits at indexes 1 and 0, and $|z\rangle$ is all the Qbits with indexes 2 and above. For $C_{10}$,

$$(\tilde{n}_1 + X_0 n_1)|z\rangle|0\rangle|y\rangle = |z\rangle|0\rangle|y\rangle$$

(using $n|0\rangle = 0$)

$$\tilde{n}_1 + X_0 n_1 |z\rangle|1\rangle|y\rangle = X_0 |z\rangle|1\rangle|y\rangle = |z\rangle|1\rangle|\tilde{y}\rangle$$

and in each case the answer agrees with $C_{10}|z\rangle|x\rangle|y\rangle$.

For $S_{10}$ the four terms $n_1 n_0$, $\tilde{n}_1 \tilde{n}_0$, $X_1 X_0 n_1 \tilde{n}_0$ and $X_1 X_0 \tilde{n}_1 n_0$ correspond to the 4 possible values of $|x\rangle|y\rangle$. For example, if $|x\rangle|y\rangle = |11\rangle$ then $n_1 n_0 |z\rangle|x\rangle|y\rangle = |z\rangle|x\rangle|y\rangle$, but the other three terms give 0. Summarizing these we see

$$n_1 n_0 + \tilde{n}_1 \tilde{n}_0 + X_1 X_0 n_1 \tilde{n}_0 + X_1 X_0 \tilde{n}_1 n_0 |z\rangle|x\rangle|y\rangle = \begin{cases} |z\rangle|x\rangle|y\rangle & \text{if } x = y \\ X_1 X_0 |z\rangle|x\rangle|y\rangle = |z\rangle|x\rangle|\tilde{y}\rangle & \text{if } x \neq y \end{cases}$$

and these agree with $S_{10}$. (If $x = y$ then the swap has no effect, while if $x \neq y$ it is the same as flipping both states.)