Exercise 3
Qbits

1. Consider the normalized vector

\[ |\psi\rangle = \frac{1}{\sqrt{6}} |00\rangle - \frac{i}{\sqrt{3}} |01\rangle + \frac{1}{2\sqrt{3}} |10\rangle - \frac{\sqrt{5}}{2\sqrt{3}} |11\rangle \]

(a) Check that it is normalized.
(b) If Qbit 0 (the right-hand Qbit) is measured, what are the probabilities of getting results 0 and 1?
(c) If Qbit 0 has been measured with result 1, what is the resulting state of Qbit 1? What is the probability that a measurement on Qbit 1 will now give result 1?

2. Suppose we have two 1-Qbit vectors

\[ |\phi\rangle = \beta_0 |0\rangle + \beta_1 |1\rangle, \quad |\psi\rangle = \gamma_0 |0\rangle + \gamma_1 |1\rangle. \]

Like any 2-Qbit vector, the unentangled state \(|\phi\rangle \otimes |\psi\rangle\) can be written in the form \(\sum_{x=0}^{3} \alpha_x |x\rangle\).

(a) Find the amplitudes \(\alpha_x\) (in terms of the \(\beta_x\) and \(\gamma_x\)).
(b) If \(|\phi\rangle\) and \(|\psi\rangle\) are both normalized, show that \(|\phi\rangle \otimes |\psi\rangle\) is also normalized.
(c) Show that \(\alpha_0 \alpha_3 = \alpha_1 \alpha_2\).
(d) Deduce that \(\frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)\) is entangled (i.e. not unentangled).

3. (This is the converse of part of question 2, and is rather harder.) Suppose a vector \(\sum_{x=0}^{3} \alpha_x |x\rangle\) is given for which \(\alpha_0 \alpha_3 = \alpha_1 \alpha_2\). Show that it can be written in the form \(|\phi\rangle \otimes |\psi\rangle\) for suitable \(|\phi\rangle\) and \(|\psi\rangle\).

Solutions

1. (a) The squared length is

\[ \left| \frac{1}{\sqrt{6}} \right|^2 + \left| \frac{i}{\sqrt{3}} \right|^2 + \left| \frac{1}{2\sqrt{3}} \right|^2 + \left| \frac{-\sqrt{5}}{2\sqrt{3}} \right|^2 = \frac{1}{6} + \frac{1}{3} + \frac{1}{12} + \frac{5}{12} = 1 \]

(b) \(\left|\psi\right\rangle = \left(\frac{1}{\sqrt{6}} |0\rangle + \frac{1}{2\sqrt{3}} |1\rangle\right) |0\rangle + \left(-\frac{i}{\sqrt{3}} |0\rangle - \frac{\sqrt{5}}{2\sqrt{3}} |1\rangle\right) |1\rangle = \frac{1}{2} \left(\frac{\sqrt{2}}{\sqrt{3}} |0\rangle + \frac{1}{\sqrt{3}} |1\rangle\right) |0\rangle - \frac{\sqrt{3}}{2} \left(\frac{2i}{3} |0\rangle + \frac{\sqrt{5}}{3} |1\rangle\right) |1\rangle\)

The probabilities of getting result 0 and 1 are \(\frac{1}{3}\) and \(\frac{2}{3}\) respectively.

(c) If the result for Qbit 0 is 1, then the resulting state for Qbit 1 is \(\frac{2i}{3} |0\rangle + \frac{\sqrt{5}}{3} |1\rangle\). The probability of result 1 on Qbit 1 is now \(\frac{5}{9}\).

2. (a) We have

\[ |\phi\rangle \otimes |\psi\rangle = (\beta_0 |0\rangle + \beta_1 |1\rangle) \otimes (\gamma_0 |0\rangle + \gamma_1 |1\rangle) = \beta_0 \gamma_0 |00\rangle + \beta_0 \gamma_1 |01\rangle + \beta_1 \gamma_0 |10\rangle + \beta_1 \gamma_1 |11\rangle \]

and so

\[ \alpha_0 = \beta_0 \gamma_0, \quad \alpha_1 = \beta_0 \gamma_1, \quad \alpha_2 = \beta_1 \gamma_0, \quad \alpha_3 = \beta_1 \gamma_1. \]
The sum of squared magnitudes is
\[ |β_0 γ_0|^2 + |β_1 γ_0|^2 + |β_1 γ_1|^2 + |β_0 γ_1|^2 \]
\[ = |β_0|^2 |γ_0|^2 + |β_1|^2 |γ_0|^2 + |β_1|^2 |γ_1|^2 + |β_0|^2 |γ_1|^2 \]
\[ = |β_0|^2 (|γ_0|^2 + |γ_1|^2) + |β_1|^2 (|γ_0|^2 + |γ_1|^2) \]
\[ = |β_0|^2 + |β_1|^2 = 1. \]

(b) \( α_0 α_3 = β_0 γ_0 β_1 γ_1 = β_0 γ_1 β_0 γ_0 = α_1 α_2. \)

(c) In \( \frac{1}{\sqrt{2}} (|00⟩ + |11⟩) \) we have \( α_0 = α_3 = \frac{1}{\sqrt{2}}, \ α_1 = α_2 = 0. \) Hence \( α_0 α_3 = \frac{1}{2} \neq 0 = α_1 α_2, \) so the state cannot be unentangled.

3. (a) We have
\[ H_1 |00⟩ = H |0⟩ ⊗ |0⟩ = \frac{1}{\sqrt{2}} (|0⟩ + |1⟩) ⊗ |0⟩ = \frac{1}{\sqrt{2}} |0⟩ + \frac{1}{\sqrt{2}} |2⟩ \]
\[ C_{10} H_1 |00⟩ = \frac{1}{\sqrt{2}} C_{10} (|00⟩ + |10⟩) = \frac{1}{\sqrt{2}} (|0⟩ + |1⟩) \]

(b) \( H_1 |00⟩ \) is obviously unentangled, since it is a tensor product. \( C_{10} H_1 |00⟩ \) is entangled – we proved that in the previous question.

(c)
\[ C_{10} H_1 |01⟩ = \frac{1}{\sqrt{2}} C_{10} (|0⟩ + |1⟩) ⊗ |1⟩ = \frac{1}{\sqrt{2}} (|0⟩ + |1⟩) \]
\[ C_{10} H_1 |10⟩ = \frac{1}{\sqrt{2}} C_{10} (|0⟩ - |1⟩) ⊗ |0⟩ = \frac{1}{\sqrt{2}} (|0⟩ - |1⟩) \]
\[ C_{10} H_1 |11⟩ = \frac{1}{\sqrt{2}} C_{10} (|0⟩ - |1⟩) ⊗ |1⟩ = \frac{1}{\sqrt{2}} (|0⟩ - |1⟩) \]

These are all entangled by the same method as before. In fact, the four states \( C_{10} H_1 |xy⟩ \) are a famous set of 4 entangled states that we shall see later.

4. At least one of the \( α_x \)'s is non-zero. Let us first suppose it is \( α_0. \) We start off by trying \( β_0 = 1 \) and \( γ_0 = α_0. \) It then follows that we need \( γ_1 = α_1 \) and \( β_1 α_0 = α_2, \) i.e. \( β_1 = \frac{α_2}{α_0}. \) We then get \( β_1 γ_1 = \frac{α_2 α_1}{α_0} = α_3 \) as required, because \( α_0 α_3 = α_1 α_2. \) Our attempt thus gives
\[ |φ⟩ = |0⟩ + \frac{α_2}{α_0} |1⟩ = \frac{1}{α_0} (α_0 |0⟩ + α_2 |1⟩) \]
\[ |ψ⟩ = α_0 |0⟩ + α_1 |1⟩. \]

However, these are not normalized. To normalize \( |ψ⟩, \) we divide by \( \sqrt{|α_0|^2 + |α_1|^2}. \) We show that \( |φ⟩ \) is normalized if we multiply it by the same number. The squared magnitude of \( \sqrt{|α_0|^2 + |α_1|^2} |φ⟩ \) is
\[ \frac{|α_0|^2 + |α_1|^2}{|α_0|^2} (|α_0|^2 + |α_2|^2) \]
\[ = \frac{|α_0|^2}{|α_0|^2} (|α_0|^2 + |α_1|^2 + |α_2|^2) + |α_1|^2 |α_2|^2 \]
\[ = \frac{|α_0|^2}{|α_0|^2} (|α_0|^2 + |α_1|^2 + |α_2|^2) + |α_1 α_2|^2 \]
\[ = \frac{|α_0|^2}{|α_0|^2} (|α_0|^2 + |α_1|^2 + |α_2|^2) + |α_0 α_3|^2 \]
\[ = \frac{|α_0|^2}{|α_0|^2} (|α_0|^2 + |α_1|^2 + |α_2|^2 + |α_3|^2) = 1. \]

(From line 3 to line 4 we use \( α_1 α_2 = α_0 α_3. \) Hence by these reciprocal scalings we get normalized \( |φ⟩ \) and \( |ψ⟩ \) whose tensor product gives the \( α_x. \)

We should also cover three other cases, for when \( α_1, α_2 \) or \( α_3 \) is non-zero. However, the arguments are essentially the same but with the variables permuted.