06-22457 *Quantum Computing and Cryptography* | The University of Birmingham
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Exercises 9
Quantum Fourier Transform

1. Consider quantum period finding for $n = 8$ Qbits, with the period $k = 15$. The integer multiples of $\frac{2^n}{k} = \frac{256}{15}$ give the following values.

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\frac{j2^n}{k}$	0	17.1	34.1	51.2	68.3	85.3	102.4	119.47	136.53	153.6	170.7	187.7	204.8	221.9	238.9

For $j = 0, 1, 2, 3, 7, 14$ and any other values you wish to try, take the nearest integer to the number in the bottom row of the table, evaluate the continued fraction for that integer divided by 256, and see whether this yields $\frac{j}{15}$.

2. Draw the circuit diagram for the 2-Qbit quantum Fourier transform, and calculate its action on $|x\rangle = |x_1 x_0\rangle$. Verify that this matches the general QFT definition.

Solutions

1. $j = 0$: This case is always unhelpful. Try again.

$j = 1$:

$$\frac{17}{256} = \frac{1}{15+}\frac{1}{17}$$

(Note the special notation for continued fractions. What this actually means is $\frac{1}{15+}\frac{1}{17} = \frac{1}{15+\frac{1}{17}}$.) If we ignore the $\frac{1}{17}$ we are left with $\frac{1}{15}$ and the denominator 15 shows the period.

$j = 2$:

$$\frac{34}{256} = \frac{17}{128} = \frac{1}{128/17} = \frac{1}{7+}\frac{9}{17} = \frac{1}{7+}\frac{1}{1+}\frac{8}{9} = \frac{1}{7+}\frac{1}{1+}\frac{1}{1+}\frac{1}{8}$$

Ignoring the $\frac{1}{8}$, we get

$$\frac{1}{7+}\frac{1}{1+}\frac{1}{1} = \frac{1}{7+}\frac{1}{2} = \frac{2}{15}.$$

Again, the denominator 15 shows the period.

$j = 3$:

$$\frac{51}{256} = \frac{1}{5+}\frac{5}{51} = \frac{1}{5+}\frac{1}{10+}\frac{1}{5}.$$

Here, the first approximation $\frac{1}{5}$ already gives us the fraction $\frac{3}{15}$, but because there is cancellation the denominator doesn't tell us 15. That is why if the denominator (5, here) turns out not to be the period, it is worth trying some small multiples of it.

$j = 7$:

$$\frac{119}{256} = \frac{1}{2+}\frac{18}{119} = \frac{1}{2+}\frac{1}{6+}\frac{11}{18} = \frac{1}{2+}\frac{1}{6+}\frac{1}{1+}\frac{7}{11} = \frac{1}{2+}\frac{1}{6+}\frac{1}{1+}\frac{1}{1+}\frac{4}{7}$$
$$= \frac{1}{2+}\frac{1}{6+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{3}{4} = \frac{1}{2+}\frac{1}{6+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{3}$$

Usually the fraction $\frac{j}{r}$ is found by ignoring the continues fraction from some large integer on. This suggests

$$\frac{1}{2+}\frac{1}{6+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1} = \frac{1}{2+}\frac{1}{6+}\frac{1}{1+}\frac{1}{1+}\frac{1}{2} = \frac{1}{2+}\frac{1}{6+}\frac{1}{1+}\frac{2}{3}$$
$$= \frac{1}{2+}\frac{1}{6+}\frac{3}{5} = \frac{33}{71}$$

which turns out to be wrong. However, stopping earlier gives

$$\frac{1}{2+}\frac{1}{6+}\frac{1}{1} = \frac{1}{2+}\frac{1}{7} = \frac{7}{15}$$

which is correct. Hence the continued fraction does not always need to be truncated at the obvious place.

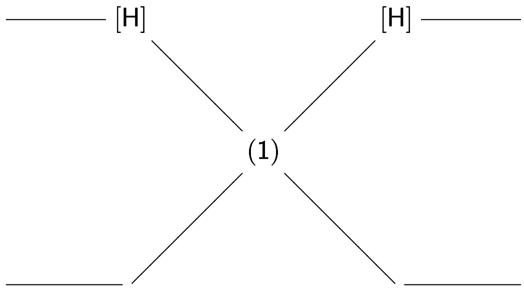
$j = 14:$

$$\frac{239}{256} = \frac{1}{1+}\frac{17}{239} = \frac{1}{1+}\frac{1}{14+}\frac{1}{17}.$$

Ignoring the $\frac{1}{17}$ gives

$$\frac{1}{1+}\frac{1}{14} = \frac{14}{15}.$$

2. The diagram is



where (1) denotes a SWAP S followed by a controlled gate $\mathsf{c}e^{i\pi\mathsf{n}/2}$ as described in the slides. We calculate

$$\begin{aligned}\mathsf{H}_1\mathsf{c}e^{i\pi\mathsf{n}/2}\mathsf{S}\mathsf{H}_1|x_1x_0\rangle &= \frac{1}{\sqrt{2}}\mathsf{H}_1\mathsf{c}e^{i\pi\mathsf{n}/2}\mathsf{S}\left(|0x_0\rangle + (-1)^{x_1}|1x_0\rangle\right)\\ &= \frac{1}{\sqrt{2}}\mathsf{H}_1\mathsf{c}e^{i\pi\mathsf{n}/2}\left(|x_00\rangle + (-1)^{x_1}|x_01\rangle\right)\end{aligned}$$

If $x_0 = 0$ this becomes

$$\frac{1}{\sqrt{2}}\mathsf{H}_1\left(|00\rangle + (-1)^{x_1}|01\rangle\right) = \frac{1}{2}\left(|00\rangle + |10\rangle + (-1)^{x_1}|01\rangle + (-1)^{x_1}|11\rangle\right)$$

This compares with the definition of the QFT as

$$\begin{aligned}\mathsf{U}_{FT}|x\rangle = \frac{1}{2}\sum_{y=0}^{3} e^{i\pi xy/2}|y\rangle &= \frac{1}{2}\sum_{y=0}^{3} e^{i\pi(2x_1+x_0)y/2}|y\rangle\\ = \frac{1}{2}\sum_{y=0}^{3} e^{i\pi x_1 y}|y\rangle &= \frac{1}{2}\left(|0\rangle + (-1)^{x_1}|1\rangle + |2\rangle + (-1)^{x_1}|3\rangle\right).\end{aligned}$$

If $x_0 = 1$, our expression becomes

$$\begin{aligned}\frac{1}{\sqrt{2}}\mathsf{H}_1\mathsf{c}e^{i\pi\mathsf{n}/2}\left(|10\rangle + (-1)^{x_1}|11\rangle\right) &= \frac{1}{\sqrt{2}}\mathsf{H}_1\left(e^{i\pi 0/2}|10\rangle + e^{i\pi 1/2}(-1)^{x_1}|11\rangle\right)\\ &= \frac{1}{\sqrt{2}}\mathsf{H}_1\left(|10\rangle + i(-1)^{x_1}|11\rangle\right)\\ &= \frac{1}{2}\left(|00\rangle - |10\rangle + i(-1)^{x_1}|01\rangle - i(-1)^{x_1}|11\rangle\right)\end{aligned}$$

The QFT definition gives

$$\begin{aligned}\frac{1}{2}\sum_{y=0}^{3} e^{i\pi(2x_1+1)y/2}|y\rangle &= \frac{1}{2}\sum_{y=0}^{3} i^y(-1)^{x_1 y}|y\rangle\\ &= \frac{1}{2}\left(|0\rangle + i(-1)^{x_1}|1\rangle - |2\rangle - i(-1)^{x_1}|3\rangle\right).\end{aligned}$$

which agrees.