Quantum Computing & Cryptography

Week 2

- Complex numbers
  - review notes

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Complex numbers

$$\underbrace{a}_{\text{real part}} + i\,\underbrace{b}_{\text{imaginary part}}$$

a real

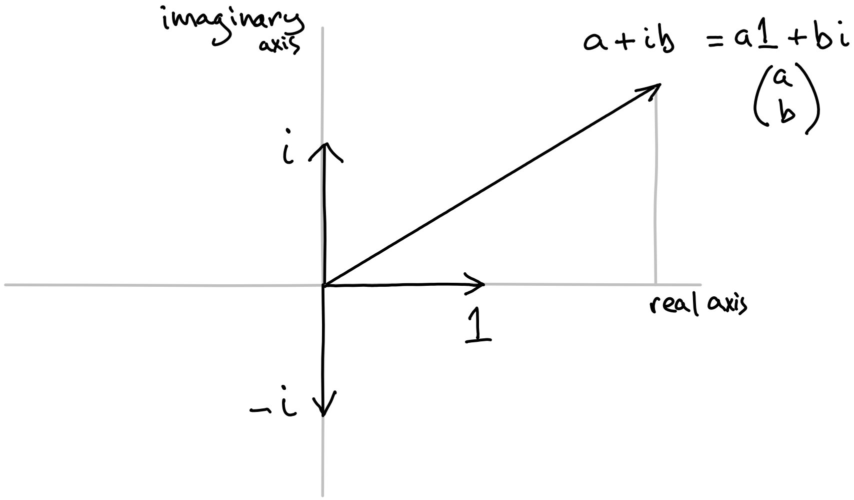
i: a square root of -1 (other one is $-i$)

b real

Complex conjugate is

$$(a+ib)^* = a - ib$$

Complex numbers as 2D-vectors



Complex arithmetic

$$(a+ib) + (a'+ib') = a+a' + i(b+b')$$

$$\begin{pmatrix} a \\ b \end{pmatrix} \quad \begin{pmatrix} a' \\ b' \end{pmatrix} \quad \begin{pmatrix} a+a' \\ b+b' \end{pmatrix}$$

same as vector addition

$$(a+i0)(a'+ib') = aa' + i\,ab'$$

$$a \quad \begin{pmatrix} a' \\ b' \end{pmatrix} \quad \begin{pmatrix} aa' \\ ab' \end{pmatrix}$$

same as scalar multiplication (a real)

$i^2 = -1$

$$(a+ib)(a'+ib') = aa' - bb' + i(ab' + ba')$$

not like vectors

$$(a+ib)^*(a+ib) = (a-ib)(a+ib) = a^2 + b^2$$

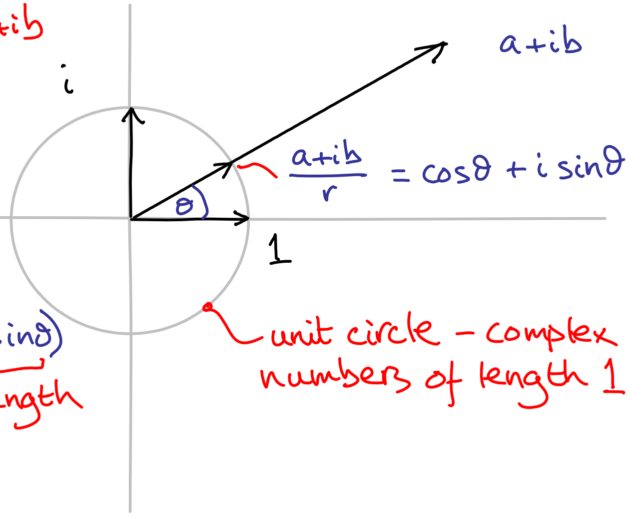
square of length of vector

Polar form

$r =$ length of $a+ib$

$= \sqrt{(a+ib)^*(a+ib)}$

$= |a+ib|$



$\frac{a+ib}{r} = \cos\theta + i\sin\theta$

unit circle – complex numbers of length 1

Polar form

$a+ib$

$= \underbrace{r}_{\text{real}} \underbrace{(\cos\theta + i\sin\theta)}_{\text{unit length}}$

De Moivre's theorem

$(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$

$= \cos\theta\cos\phi - \sin\theta\sin\phi$

$\quad + i(\cos\theta\sin\phi + \sin\theta\cos\phi)$

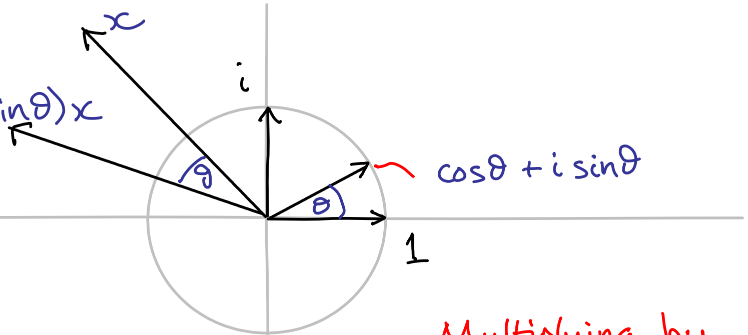
$= \cos(\theta+\phi) + i\sin(\theta+\phi)$

It follows that

multiplying any complex number x by $\cos\theta + i\sin\theta$

rotates x through an angle θ

Complex multiplication



Multiplying by $r(\cos\theta + i\sin\theta)$ would rotate θ and scalar multiply (by r).

Exponentiation

NB θ measured in radians

$\cos\theta + i\sin\theta = e^{i\theta}$

- shorter notation
- usual exponential laws still work
  - de Moivre says $e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$
- Polar form is $re^{i\theta}$

Mathematically → For real numbers x: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

(don't need to know this)

Replacing x by $i\theta$: get

$1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \quad = \cos\theta$

$+ i\left(\theta - \frac{\theta^3}{3!} + \cdots\right) \quad = i\sin\theta$

Unit circle is the $e^{i\theta}$s