Statistical Measurement of Information Leakage

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Overview

- Estimate information leakage statistically from trail runs of a real system.
  - Automatic tool to calculate information leakage.
- We work out bounds on the possible error.
- We present an ***if, and only if***, test for ***zero*** leakage.
- For actuate results you need more samples than the (no. of inputs) x (no. of observations).

Information Theory

Entropy: $H(X) = - \sum_x p(x).\log(p(x))$

the amount of uncertainty in X.

Conditional Entropy: $H(Y|X) = \sum_x p(x).H(Y|X=x)$

the amount of uncertain in Y if you know X

Mutual Information: $I(X;Y) = H(X) - H(X|Y)$

the reduce of uncertainty you get in X if you know Y.

Relative Entropy: $D(p||q) = \sum_x p(x).\log(p(x) / q(x))$

“distance” from one distribution to another.

Information Theory

- A **Channel,** has inputs X, outputs Y, and a probability transition matrix W(x|y).

- Information sent across the channel = I(X;Y),
  - We define I(Q,W) = I(Q;Y)

- Maximum rate is the **Channel Capacity**:

$$C(W) = \text{Max}_Q\ I(Q,W)$$

the most information you can send across a channel no matter how data is sent.

Information Leakage = Capacity(System)

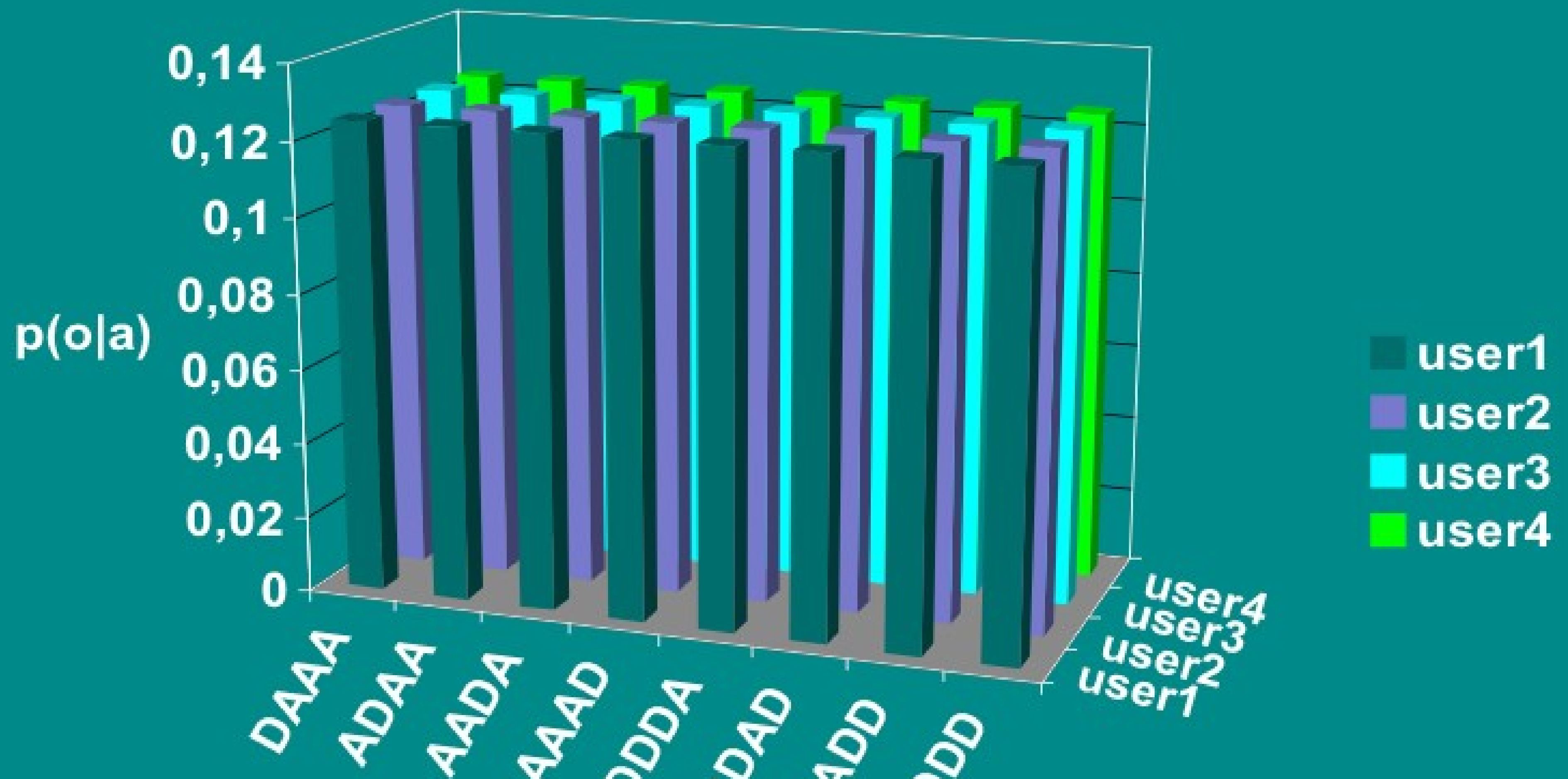
Following Chatzikokolakis et al.

- Think of the whole system as a channel.
  - The guilty user is the input to the “channel”.
  - The observable actions are the outputs from the “channel”.

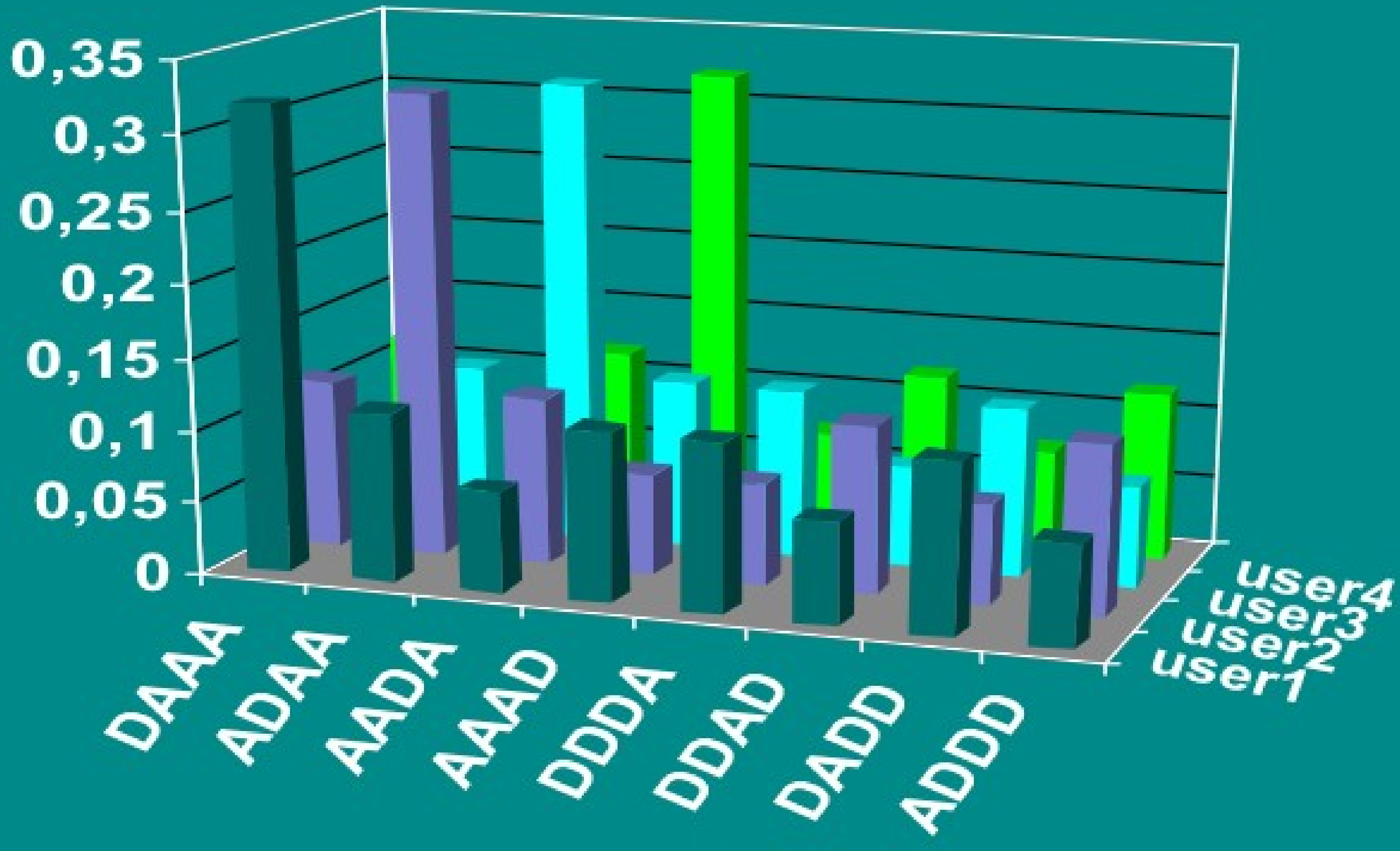
- Capacity tells us what we can learn about the users from the observable actions.

- Similar approach can be taken for Information Flow e.g. Millen, Clark et al. etc.

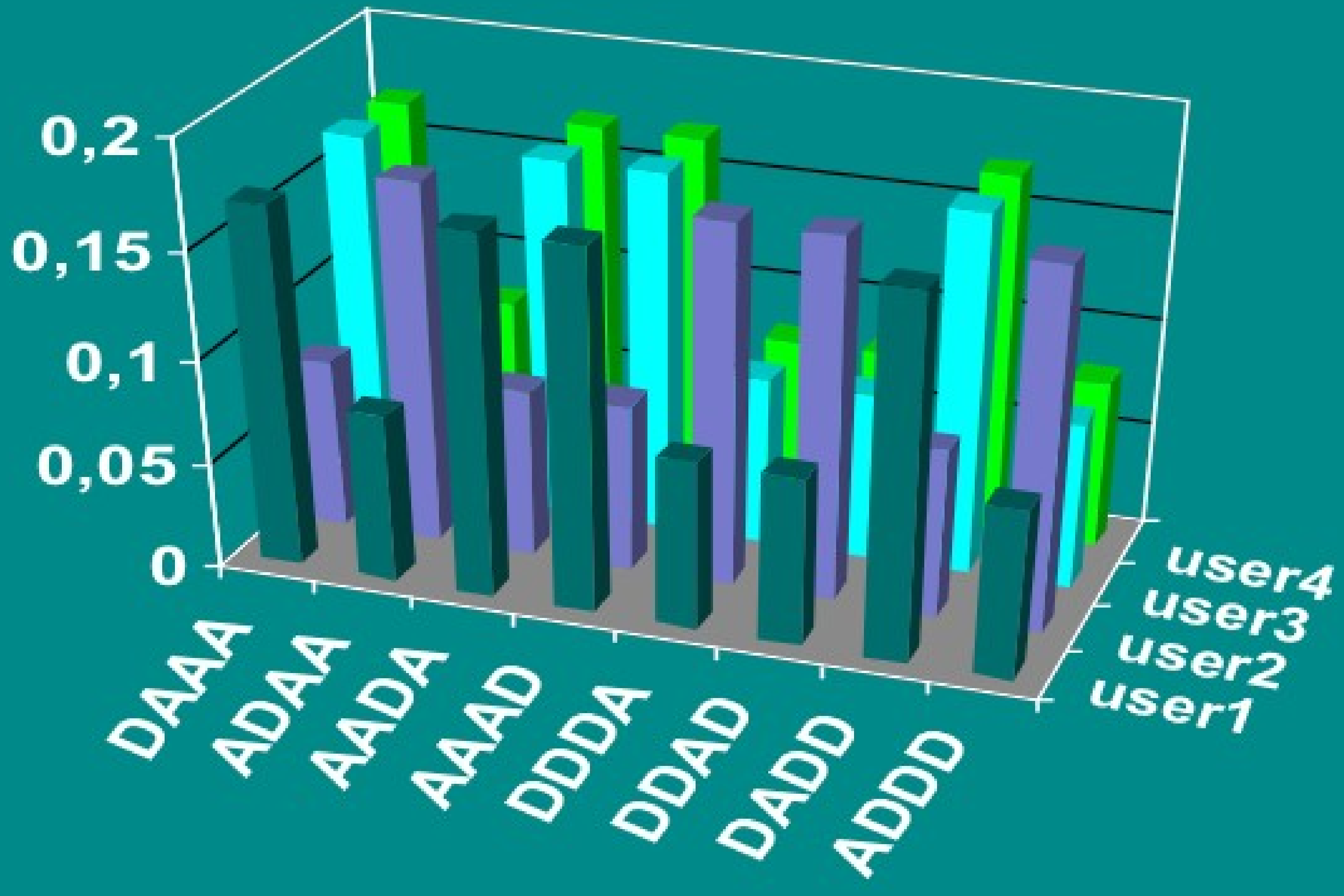
The Conditional Probabilities of the D.C. Protocol



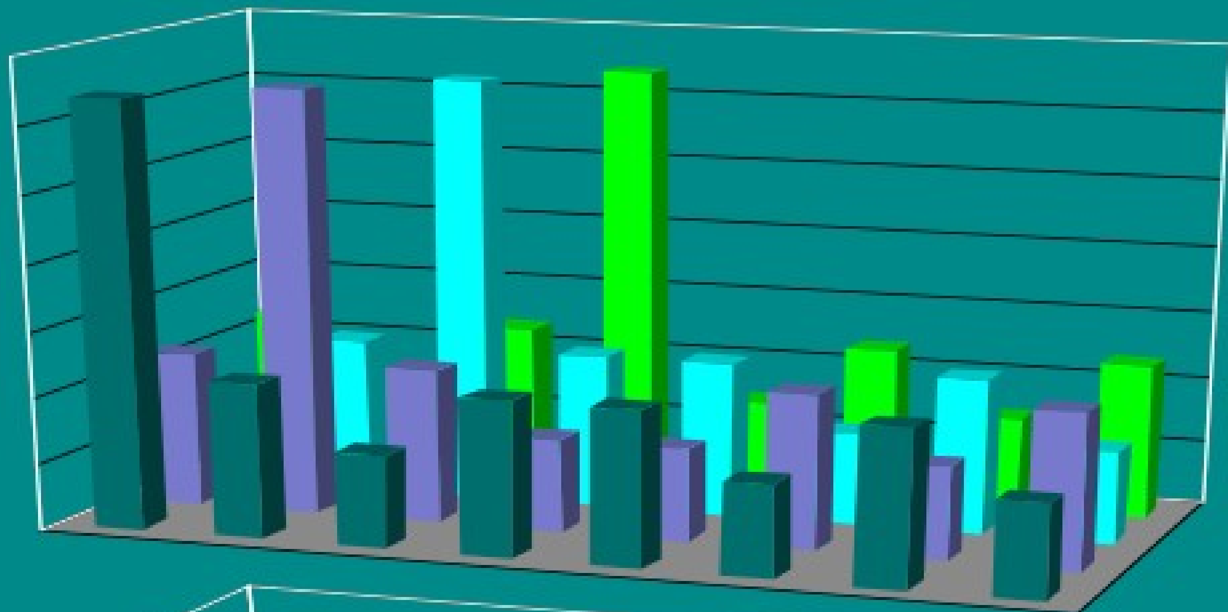
Biased Coins: if p(heads) = 0.25



2 Out of 4 p(heads) = 0.8

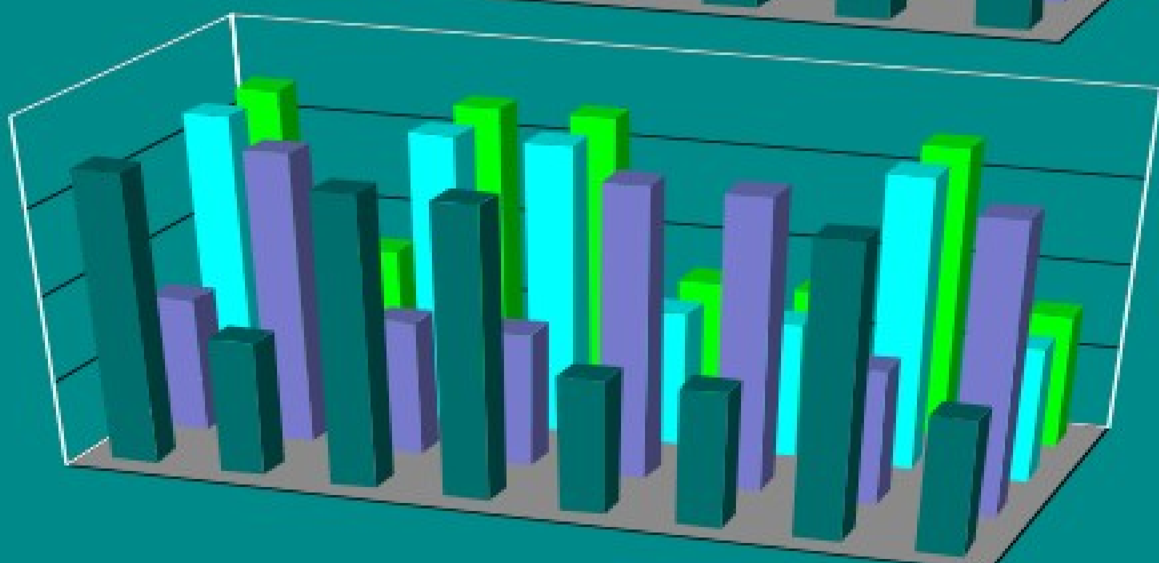


Which is Better?



All coins
P(heads) = 0.75

Anonymity
= 0.170536



2 coins
P(heads) = 0.8

Anonymity
= 0.095619

Applying this to Real Systems

How do we apply information theoretic measures of information leakage to real systems?

- Source code may be too complex to analyse directly or model / behave probabilistically.
- Leak may be caused by the implementation:
  - Time based attack on RSA, (Paul Kocher)
  - Bandwidth attack on Tor (Murdoch & Danezis)
  - CPU Heat attack on Tor (Murdoch)

Demo

- Examples of the kinds of sampled data we want to analyse

Blahut-Arimoto Algorithm

How do we find the maximising input distribution?

$I(X;Y) = H(X) - H(X|Y)$

$= \sum_{x,y} p(x)\, W(y|x) \log \left(W(y|x) \;/\; \sum_{x'} p(x')W(y|x)\right.$

$= \sum_x p(x).D(\; W(\_|x) \;\|\; \sum_{x'} p(x')W(\_|x') \;)$

$= \sum_x p(x).D_x(W\|pW)$

Distribution of y given x

Distribution of y

$\sum_x p(x).D_x(W\|pW) \leq C(W) \leq \mathrm{Max}_x\, D_x(W\|pW)$

Blahut-Arimoto Algorithm.

1) Guess an input distribution $p^0(a)$ e.g., uniform

2) Improve the guess, for all x:

$$p^{n+1}(x) = \frac{\exp(\ D_x(W \,||\, p^n W)\ )}{\sum_{x'} \exp(\ D_{x'}(W \,||\, p^n W)\ )}$$

3) Repeat until $I(p^n, W) - \mathrm{Max}_x\, D_x(W||p^n W) < e$

Can be tweaked for super linear convergence, conditional mutual information etc.

Other Ways to Find Capacity

- Special cases can be calculated directly
- A “gradient climb” e.g. Frank-Wolfe algorithm
- Kuhn-Tucker Theorem/Lagrange multipliers.

Method of Analysing Anonymity

- To analyse a system we define the inputs and outputs.
  - Some abstraction might be needed to make the number of observations manageable

- We run tests of the system for each input.

- From these tests we estimate a matrix.

- We estimate capacity, from the matrix.

Terms

W : the matrix of the true system.

$\hat{W}_n$: a matrix estimated from n samples.

Q : the input dist. that maximise M.I.

$\hat{Q}_m(\hat{W}_n)$: the B.A. algorithm applied to W_n.

$C = I(Q,W)$: the true system capacity.

$\hat{C}_{n,m} = I(\hat{Q}_m(\hat{W}_n),\hat{W}_n)$: estimate of capacity ??

Convergence

Theorem: $\hat{C}_{n,m}$ almost surely convergences to C as $n,m \rightarrow \infty$

i.e., for any p_e and error e there exists n‘ & m’ such that for n > n’ and m > m’:

$$p(|\, C - \hat{C}_{n,m} \,| > e\,) < p_e$$

The Distribution of Anonymity

We can get bounds on the error by ask what distribution $\hat{C}_{n,m}$ comes from.

Adapting a statistical method from Rao:

- We find the Taylor expansion of the $\hat{C}_{n,m}$
- We drop the terms smaller than sampleSize^{-2}
- We then calculate the mean and variance.
- We find the distribution using the central limit theory.

Estimated Value

As we can’t find the distribution for the maximising distribution we relate our estimate to $I(\hat{Q}_m(\hat{W}_n),W)$

Lemma: The estimate

- is less than or equal to the capacity,
- equals zero if, and only if, the capacity equals zero.

Expectation and Variance

To find a distribution we need to find the expectation:

E(X): the average value

And the variance:

$$Var\ (X) = E(mean - x)^2$$

What We Know

K_{ij} is the number of times the pair (i,j) shows up in our test.

Let the true prob: $p(i,j) = {}^{h}K_{ij}/n$

Then maximum likehood tells us that

- $E(K_{ij} - {}^{h}K_{ij}) = 0$
- $E((K_{ij} - {}^{h}K_{ij})^2) = p(i).\ W(j|i)(1-W(j|i))$
- $E((K_{ij} - {}^{h}K_{ij})^3) = K_{ij}(2W(j|i)^2-3W(j|i)+1)$ ...

Taylor's Theorem

To find the value of a function at x (near a):

$$f(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f''(a)(x-a)^3}{3!} + ...$$

We take I(X,_) as "f", W_n as "x" and W as "a" to give get an expression for the estimate in terms of the true value.

Taylor Expansion of Entropy

$I_n(X,Y) = H(X) + H_n(Y) - H_n(X,Y)$

$E(I_n(X,Y)) = E(H(X)) + E(H_n(Y)) - E(H_n(X,Y))$

$H(X,Y) \quad = - \sum_{x,y} p(x,y)\log(p(x,y))$

$H_n(X,Y) = - \sum_{x,y} K_{ij}/n.\log(K_{ij}/n)$

$H_n(X,Y) \quad = -\sum_{x,y} {}^hK_{ij}/n - 1/n.\sum_{x,y}(1+{}^hK_{ij}/n)$

$\quad -\sum_{x,y}(K_{ij}-{}^hK_{ij})^2/n.\,{}^hK_{ij})$

$\quad + \sum_{x,y}(K_{ij}+{}^hK_{ij})^3/6n.{}^hK_{ij}{}^2) + O(n^{-2})$

$E(H_n(X,Y)) \; = \; H(X,Y) - I(J-1)/2n + O(n^{-2})$

For Non-Zero Mutual Information

When the true value is not 0, an estimation of capacity is drawn from a normal distribution with:

Mean: $I(\hat{Q}_m(\hat{W}_n),W) + \frac{(I-1)(J-1)}{2n} + O(n^{-2})$

Variance: ...

Variance

$$\frac{1}{n} \sum_x Q(x).\left(\sum_y W(y|x).\left(\log\left(\frac{Q(x).W(y|x)}{\sum_{x'} Q(x')W(y|x')}\right)\right)^2 - \left(\sum_y W(y|x).\log\left(\frac{Q(x).W(y|x)}{\sum_{x'} Q(x')W(y|x')}\right)\right)^2\right)$$

$$+O(n^{-2})$$

When I = 0

- The $O(n^{-1})$ term disappears with X and Y are independent.

- In which case we need to find the $O(n^{-2})$ term.

- Following Rao, we observe when I = 0 the

$$\sum_{ij} \left((K_{ij} - E(K_{ij}))^2 / E(K_{ij}) \right) \sim \chi^2$$

and that this approximates mutual infromation.

Results for I = 0

When the true value is 0, an estimation of capacity (or mutual information) is drawn from the distribution:

$$2n.I \sim \chi^2((noOfInputs-1)(noOfOutputs-1))$$

Mean: (noOfInputs-1)(noOfOutputs-1)/2

Variance: (noOfInputs-1)(noOfOutputs-1)/2n^2

Upper Bound on the Variance

- In both cases var(C(W)) < I.J / n

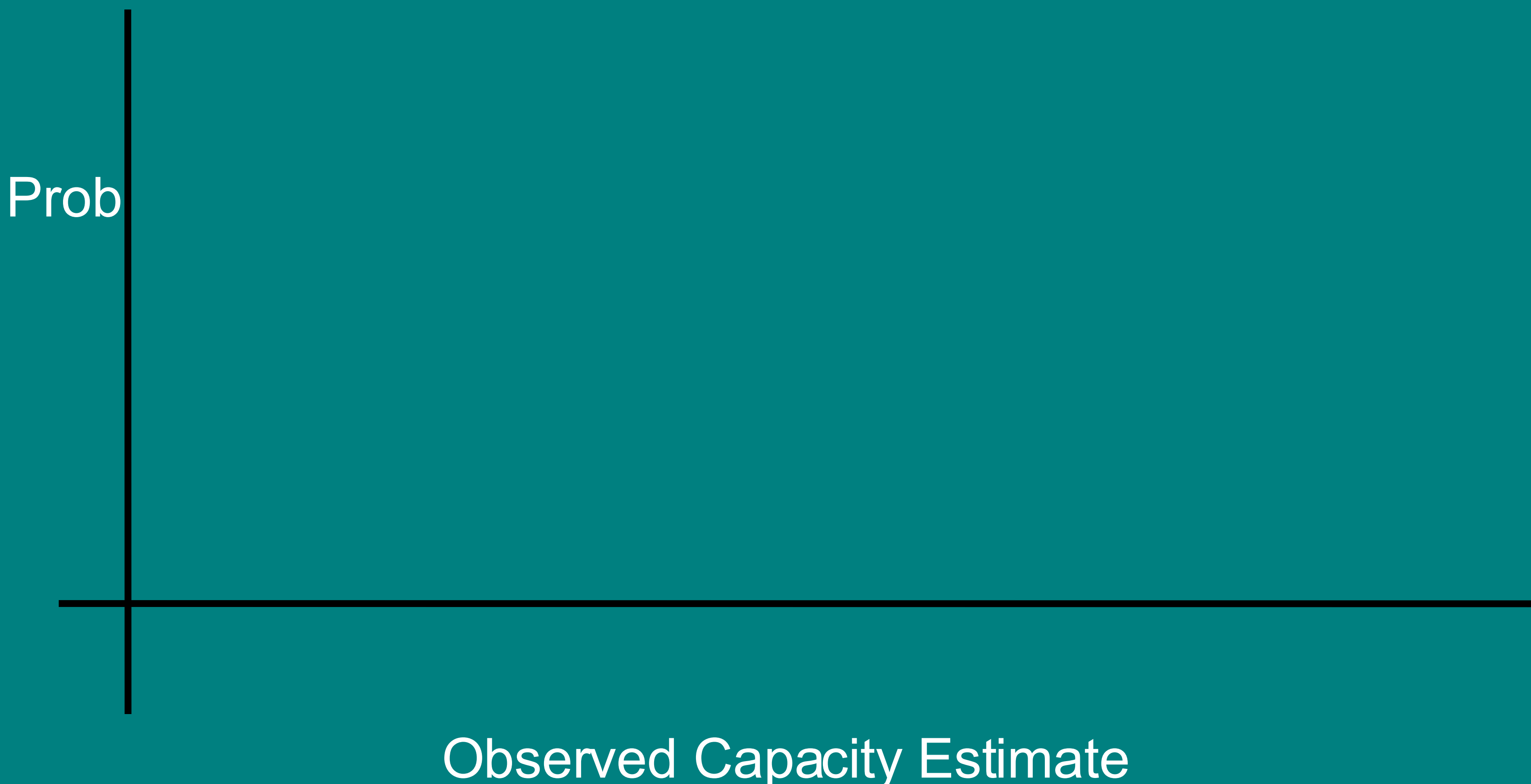
- Rule of thumb:
  - if I.J >> n the variance will be low and the results actuate.
  - If you can get this many samples then statically analysis is useful, otherwise not.

To Analyse a System.

- We define the inputs (I) and outputs (J).
- Run n tests of the system with n >> I.J
- Estimate the matrix and find $\hat{C} = I(\hat{Q}_m(\hat{W}_n),\hat{W}_n)$
- Point Estimate is:

  $$\mathrm{Max}\ (\ 0,\ I(\hat{Q}_m(\hat{W}_n),\hat{W}_n) - (I-1)(J-1)/2n\ )$$

Using the Distributions



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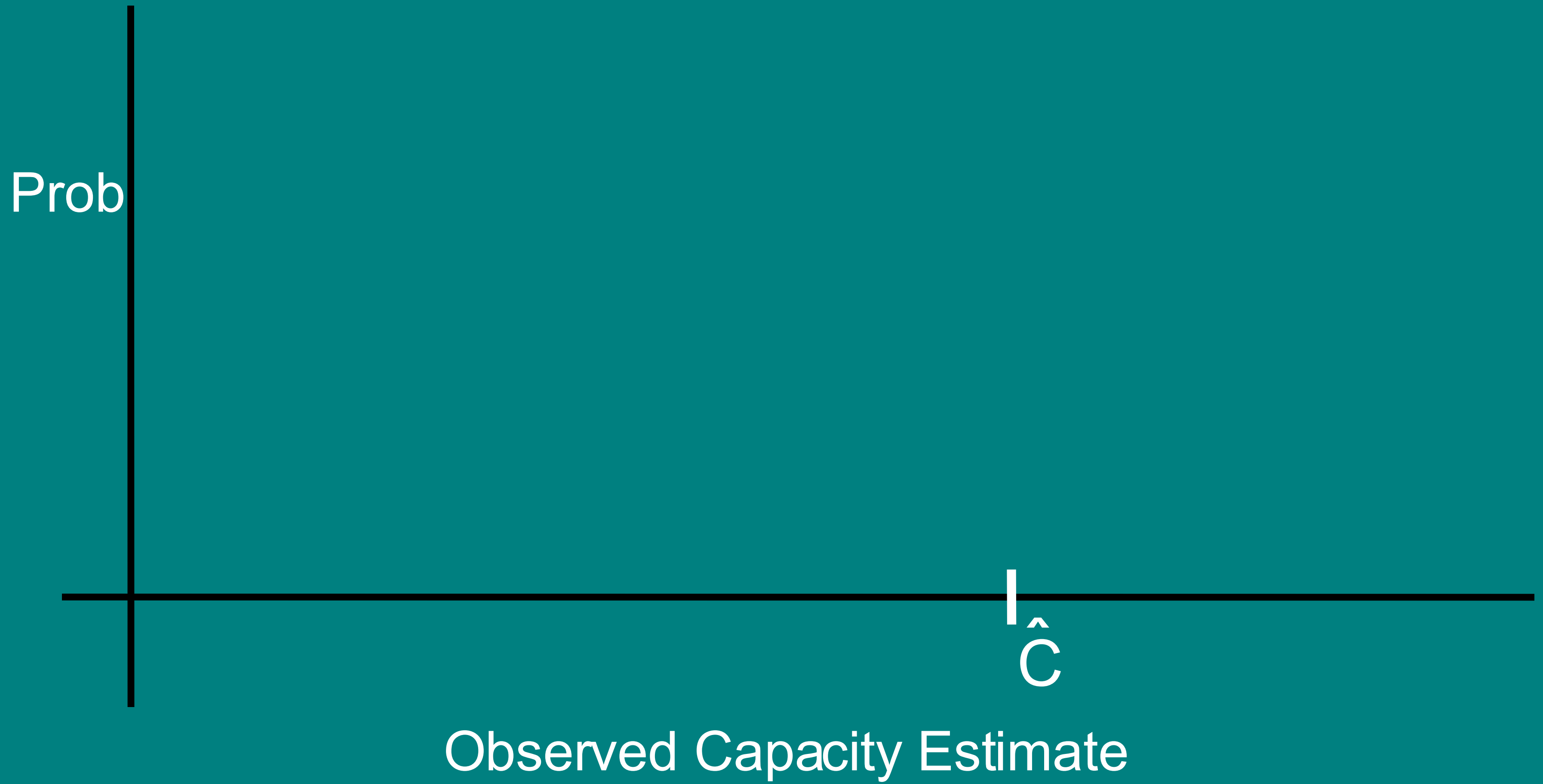

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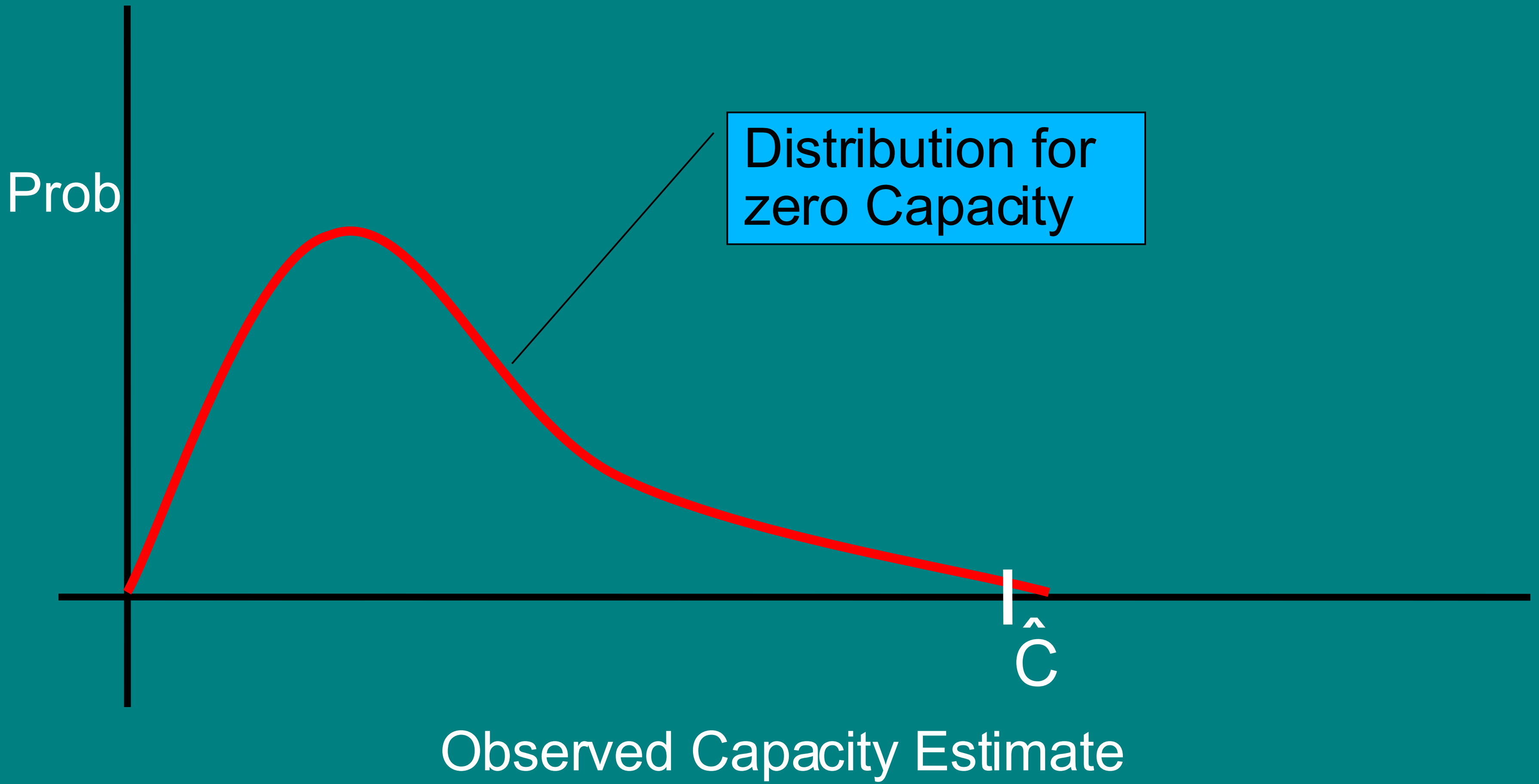

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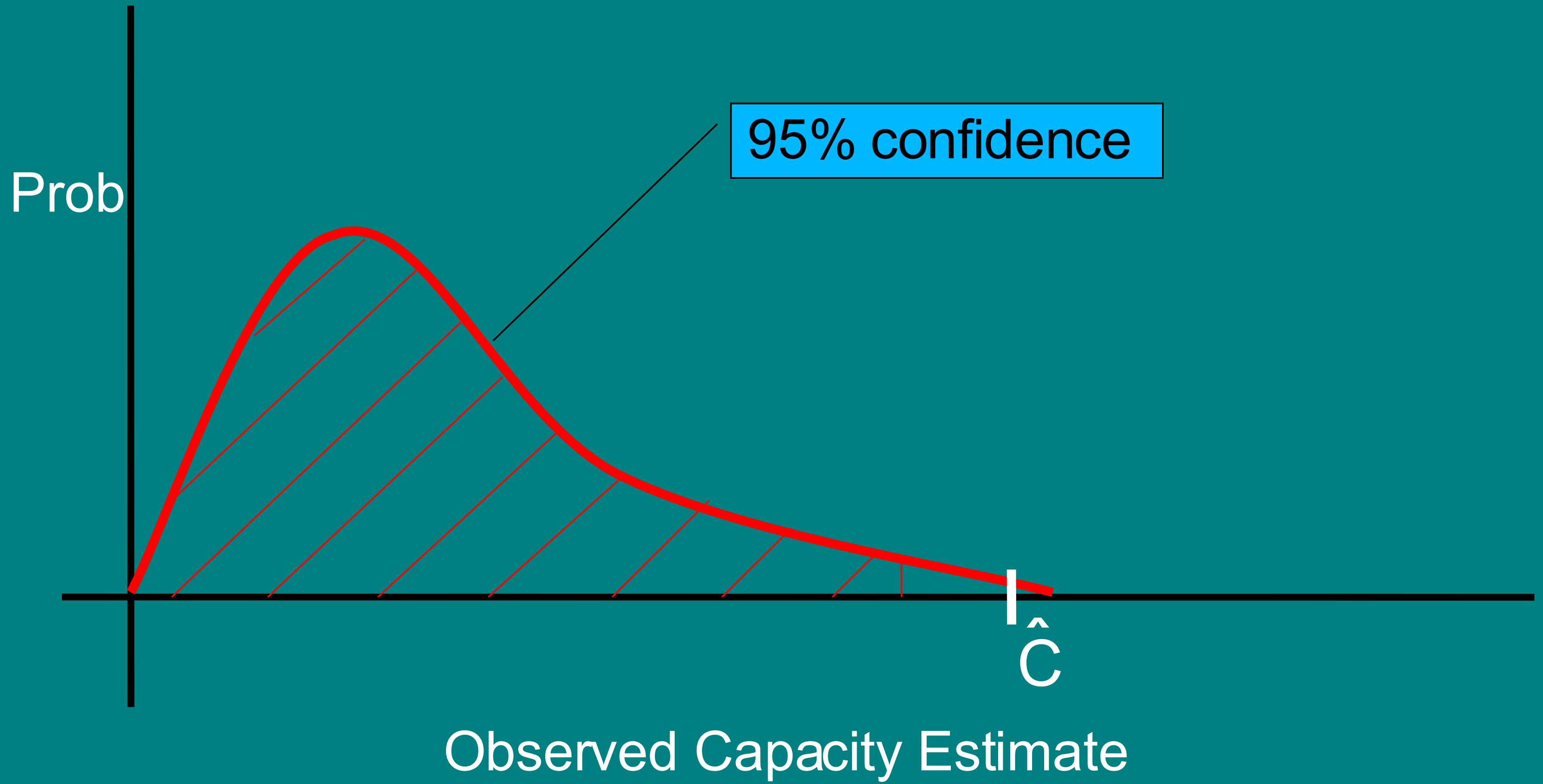

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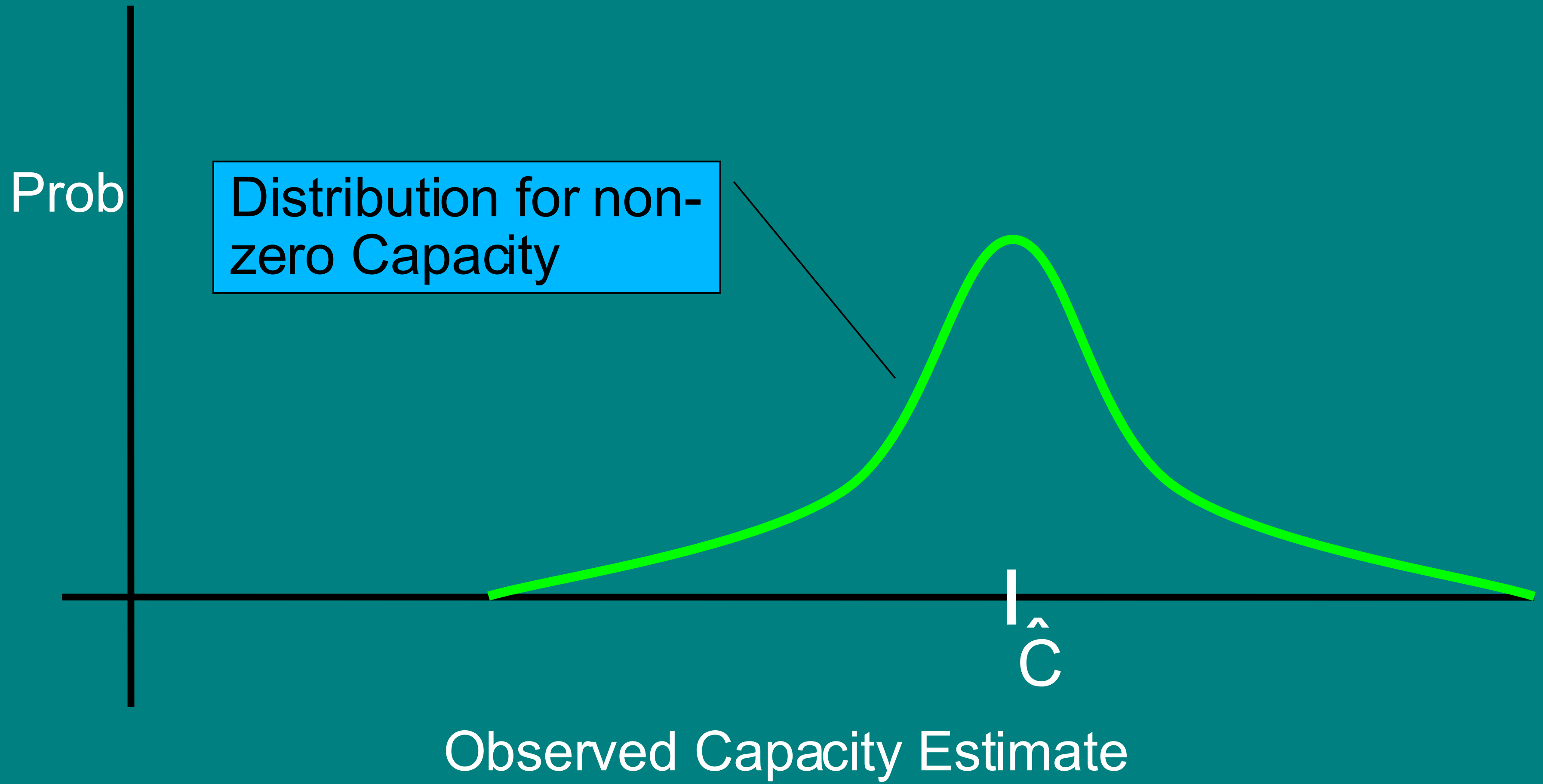

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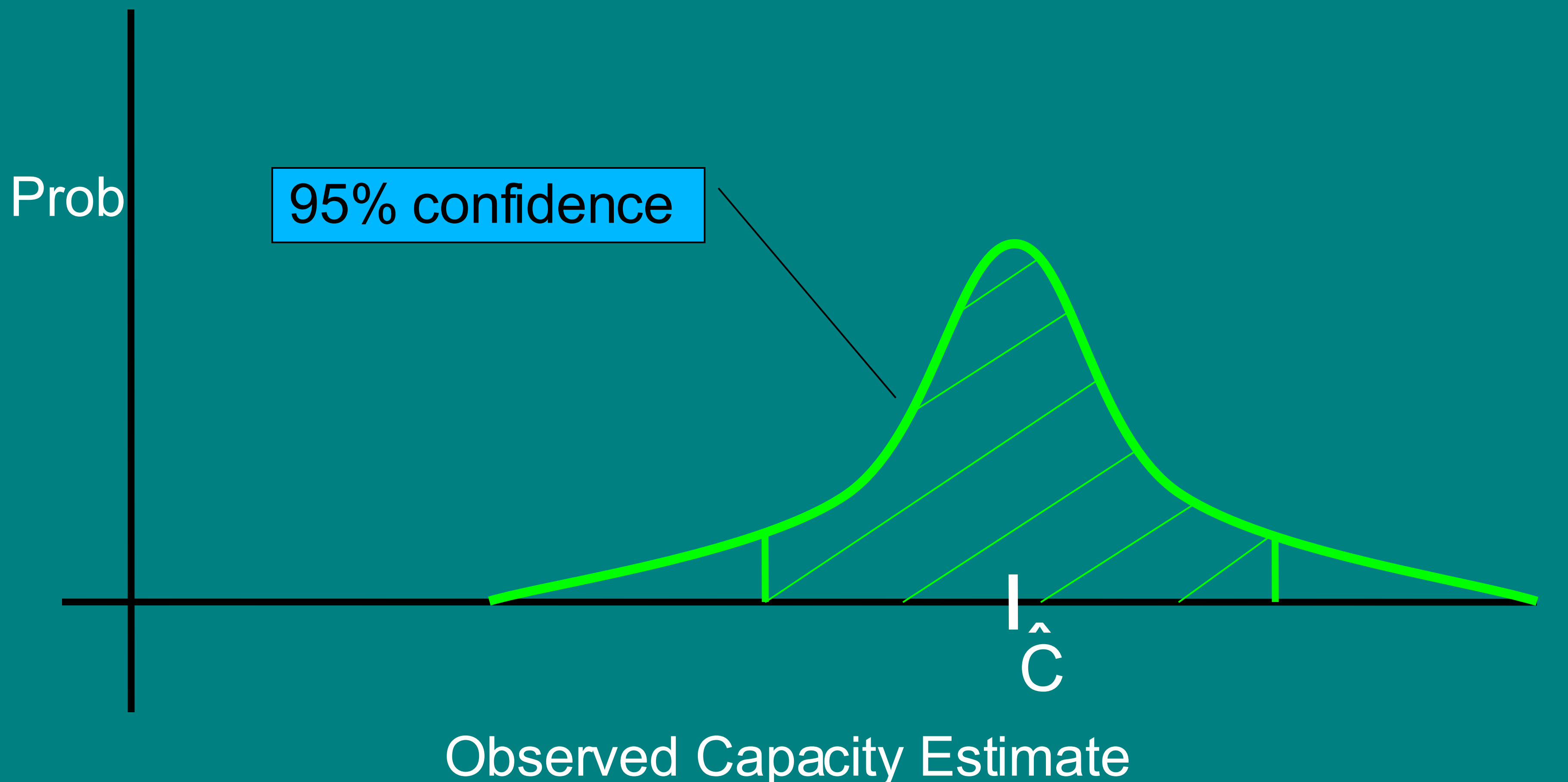

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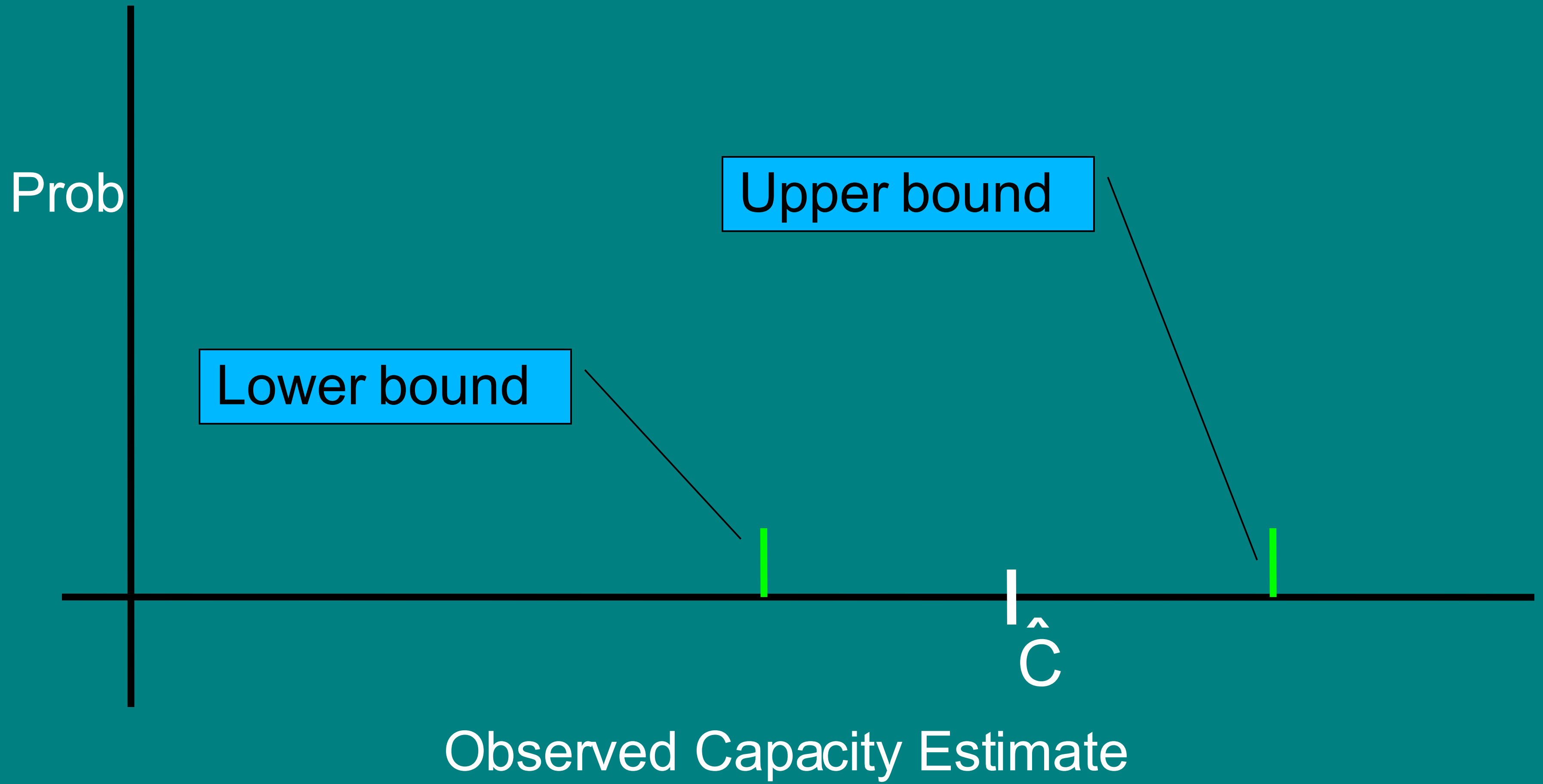

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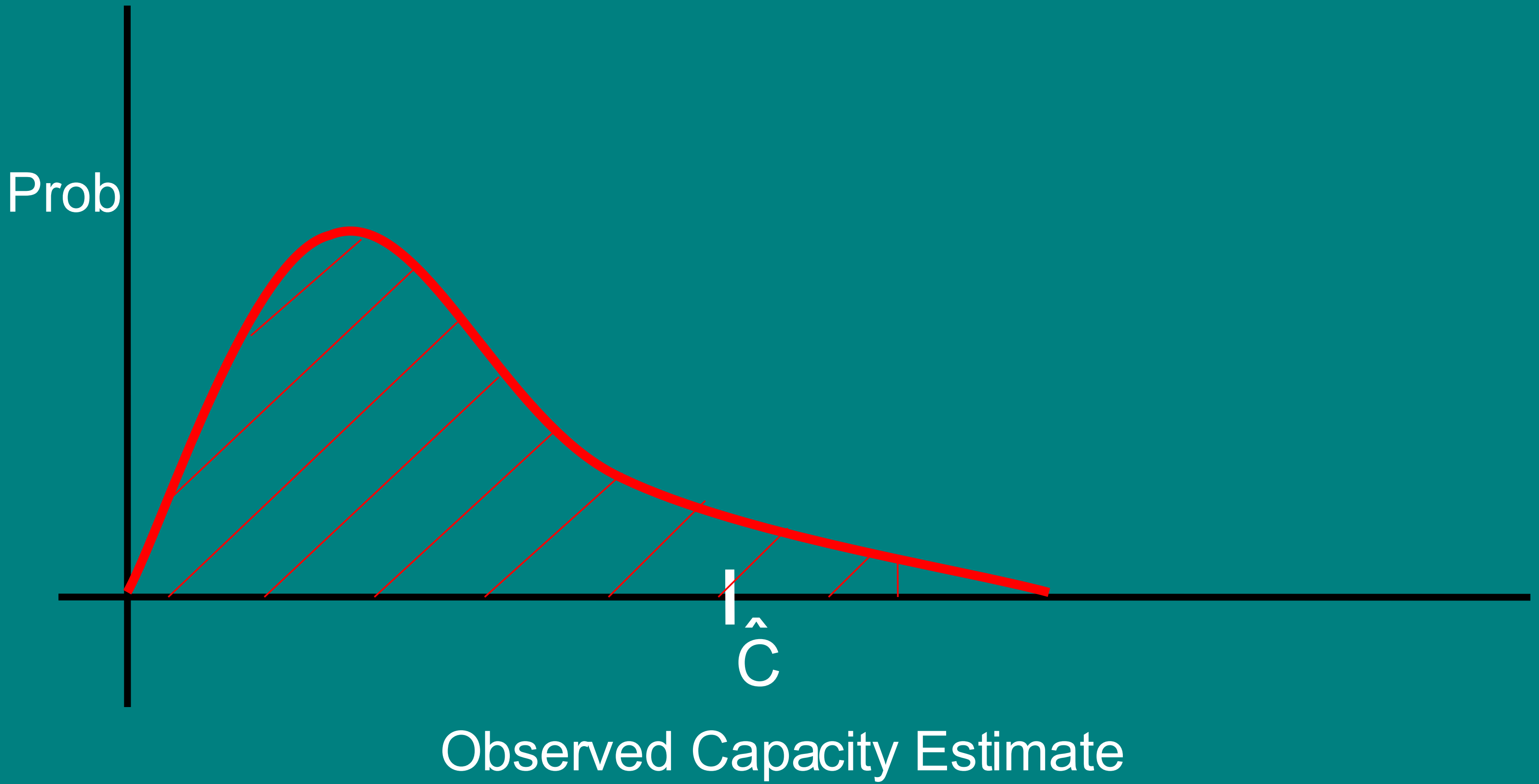

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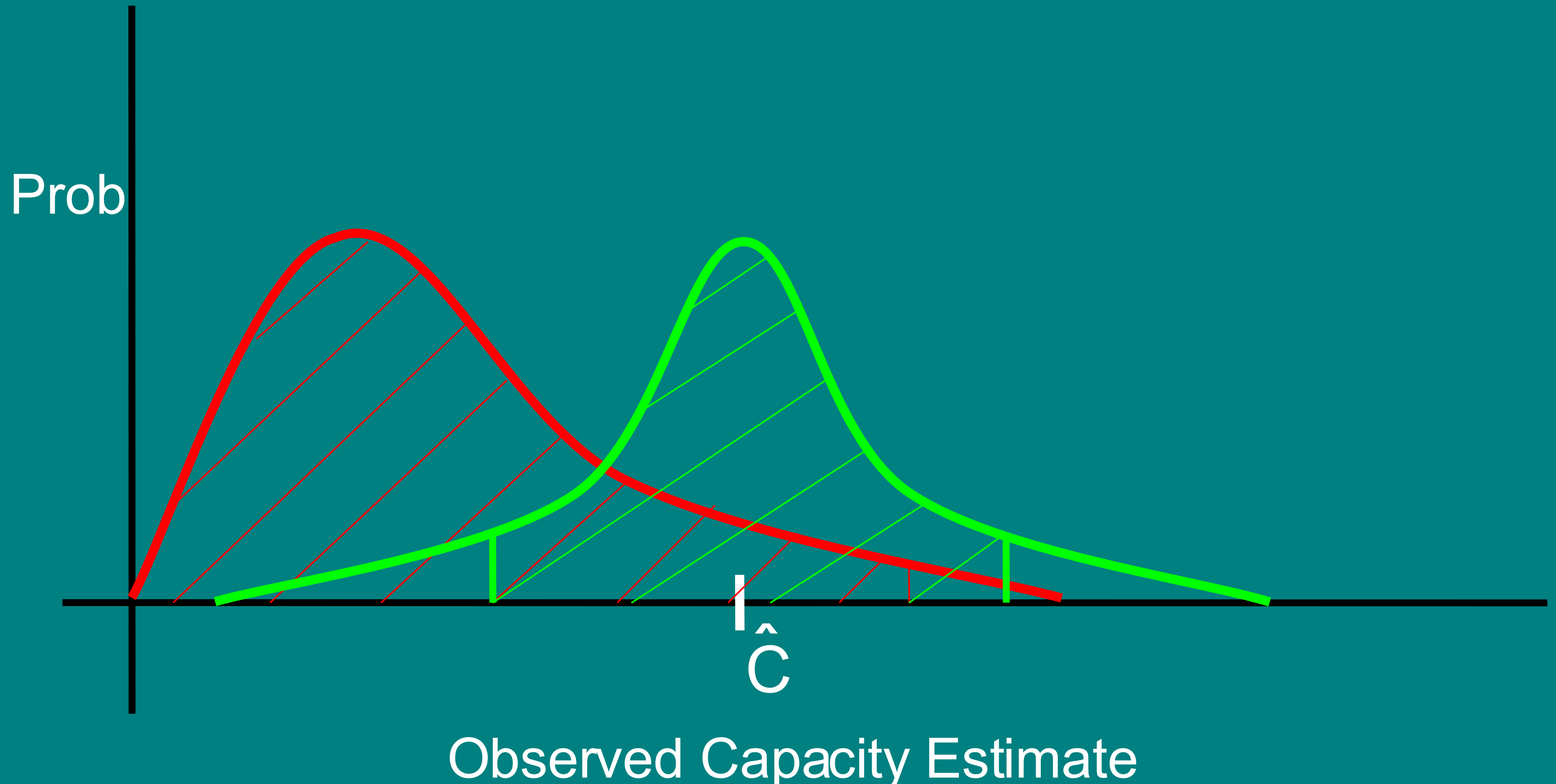

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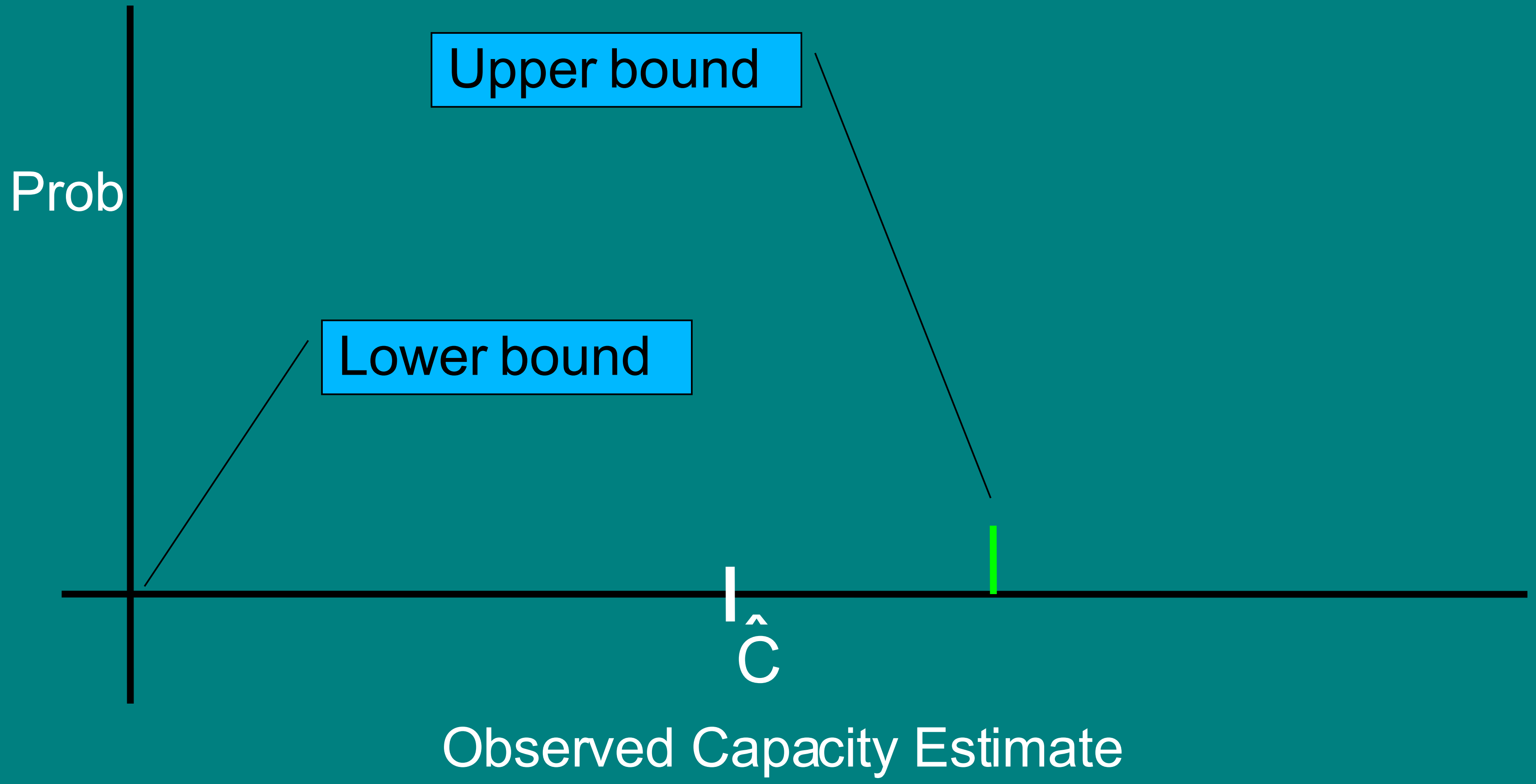

Test for Zero Leakage

- But what if we want to know if the leakage is really zero?
- What distinguishes the zero from the non-zero case is the variance:
  - $O(n^{-1})$ for non zero
  - $O(n^{-2})$ for zero.
- A large enough sample size will always tell these apart.

Test for Zero Leakage

- Run 100 tests of the system and calculate the observed variance “o” in the tests results.
- Test o against the predicated variance for zero and non-zero observations.
- If it matches the zero predication but not the non-zero we can conclude that there is zero leakage.
- If it only matches the non-zero predication then we can find the confidence interval for the results.
- If it matches both then increase the sample size.

Demo:

- Example of using analysing some data sampled form a toy Dining crypos implementation and a real Mixminion mix node.

Comparison with Bayesian Methods

- In our analysis we make no assumptions about how the system is used
  - says nothing about a single run of the system,
  - user actions can be depended on previous user action, Capacity is still the maximum amount of information that can be sent.

- Bayesian analysis makes assumption about the input distribution.
  - Can infer guilt of a user from the actions
  - Not valid if the assumptions aren’t correct or user actions depend on previous actions.

Extensions

- Calculate the distribution of conditional mutual information and the upper bound on capacity.
- Statefull channels, for more complex and interactive systems.
- Continuous mutual information and capacity for continuous data sets.

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  - Automatic tool to calculate information leakage.
- We work out bounds on the possible error.
- We present an ***if, and only if***, test for ***zero*** leakage.
- For actuate results you need more samples than the (no. of inputs) x (no. of observations).

Questions?