

When Parametricity implies Naturality (Notes)

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The issue of whether parametricity and naturality should be assumed separately or whether parametricity automatically implies naturality crops up in the parametricity semantics for Algol (O’Hearn and Tennent, 1995; O’Hearn and Reynolds, 1997). Intuitively, parametricity and naturality are trying to capture the same idea of “uniformity.” So, it is not immediately clear why one needs two theories of uniformity or whether one should find some other theory that subsumes both parametricity and naturality.

The following results are meant to shed some light on the issue. We use the framework of reflexive graphs from (O’Hearn and Tennent, 1995) and add suitable conditions which entail that parametricity implies naturality. Further, we show that the semantics of Algol can be given in such a framework.

1 Reflexive graphs

Recall the notion of reflexive graph (RG) from (O’Hearn and Tennent, 1995). We call reflexive graph *relational* if for any pair of vertex arrows $f: A \rightarrow B$, $f': A' \rightarrow B'$ and edges $R: A \leftrightarrow A'$, $S: B \leftrightarrow B'$, there is at most one edge morphism $\phi: R \rightarrow S$ such that $\delta_0(\phi) = f$ and $\delta_1(\phi) = f'$. Whenever such an edge morphism exists, we call the following diagram a *parametricity square*:

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \uparrow R & & \uparrow S \\
 A' & \xrightarrow{f'} & B'
 \end{array}$$

The terminology is motivated by the fact that edges R and S determine a relation $[R \rightarrow S] \subseteq \text{hom}(A, B) \times \text{hom}(A', B')$. We assume that all our reflexive graphs are relational.

A reflexive graph is *symmetric* if it is equipped with a family of bijections

$$(-)_{A,A'}^\smile : \text{edge}(A, A') \cong \text{edge}(A', A)$$

such that

1. $(I_A)^\smile = I_A$, and

2. the left hand diagram is a parametricity square iff the right hand diagram is a parametricity square:

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 R \uparrow & & \downarrow S \\
 A' & \xrightarrow{f'} & B'
 \end{array}
 \iff
 \begin{array}{ccc}
 A' & \xrightarrow{f'} & B' \\
 R \downarrow & & \downarrow S \\
 A & \xrightarrow{f} & B
 \end{array}$$

A *complete reflexive graph*¹ (CRG) is a symmetric reflexive graph equipped with a family of mappings $\langle - \rangle_{A,B}$ which associate to each arrow $f: A \rightarrow B$ an edge $\langle f \rangle: A \leftrightarrow B$ such that

1. $\langle \text{id}_X \rangle = I_X$, and
2. the left hand side diagram below is a parametricity square (edge morphism) iff the right hand side diagram is a commutative square:

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \langle g \rangle \uparrow & & \downarrow \langle h \rangle \\
 A' & \xrightarrow{f'} & B'
 \end{array}
 \iff
 \begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 g \downarrow & & \downarrow h \\
 A' & \xrightarrow{f'} & B'
 \end{array}
 \tag{1}$$

Note that $\langle - \rangle$ is injective. If $\langle g_1 \rangle = \langle g_2 \rangle: A \leftrightarrow A'$ then $\text{id}_A [\langle g_1 \rangle \rightarrow \langle g_2 \rangle] \text{id}_{A'}$ which implies $g_1 = g_2$ by (1).

Here are some examples.

- The reflexive graph **Set** of sets, functions and binary relations can be made complete by taking $\langle f \rangle$ to be graph of the function f . We refer to this as the “standard” edge mapping.
- The reflexive graph **Cpo** of ω -cpo’s, continuous functions and ω -complete relations can be made complete with the standard edge mapping. Similarly, the reflexive graph **Cpo_⊥** of pointed cpo’s, strict continuous functions and (strict) complete relations can be made complete. However the reflexive graph **pcpo** where the morphisms are *all* continuous functions cannot be made complete. Since a continuous function need not be strict, its graph is not a complete relation in general.
- The reflexive graph **Per** has per’s over a partial combinatory algebra D as vertices, per-morphisms as arrows, and “saturated” relations as edges (Bainbridge et al., 1990; Berlucci et al., 1995). A saturated relation $R: A \leftrightarrow B$ is a relation $R \subseteq D \times D$ such that $A; R; B = R$. Clearly, the graph of a morphism $f: A \rightarrow B$ is a saturated relation. This gives a CRG.
- This example is suggested by Tennent. Let **Pfn** be the reflexive graph whose vertices are sets, arrows are partial functions and edges are subsets $R \subseteq A + A' + (A \times A')$. A parametricity square $f [R \rightarrow S] f'$ exists iff the following conditions hold:
 - (for $x \in A$) $x \in R$ implies $f(x) \uparrow$ or $f(x) \in S$.
 - (for $x' \in A'$) $x' \in R$ implies $f'(x') \uparrow$ or $f'(x') \in S$.

¹Suggestions for better terminology?

– $(x, x') \in R$ implies either $f(x) \in S \wedge f'(x') \uparrow$, $f(x) \uparrow \wedge f'(x') \in S$ or $(f(x), f'(x')) \in S$.

The edge mapping is $\langle g \rangle = \{x \mid g(x) \uparrow\} \cup \{(x, g(x)) \mid g(x) \downarrow\}$. This gives a CRG. All this can be represented much more simply in the equivalent reflexive graph **Set**_• of pointed sets, pointed set morphisms and pointed relations with standard parametricity squares and edge mappings. The completeness of the reflexive graph can be seen much more directly in this alternative representation.

If **G** and **H** are CRG's, their product **G** × **H** is a CRG with the edge mapping $\langle (f, g) \rangle = (\langle f \rangle, \langle g \rangle)$. The dual reflexive graph **G**^{op} is a CRG with the edge mapping $\langle f^{\text{op}} \rangle = \langle f \rangle^\smile$.

Operators

If **G** and **H** are reflexive graphs, a RG-operator $F : \mathbf{G} \rightarrow \mathbf{H}$ maps vertices and edges of **G** to those of **H**, preserving the identity edges. A parametric transformation $\phi : F \rightarrow G$ between RG-operators $F, G : \mathbf{G} \rightarrow \mathbf{H}$ is a family of arrows $\{\phi_X\}_X$ indexed by vertices X of **G** such that for every edge $R : X \leftrightarrow X'$, the following is a parametricity square in **H**:

$$\begin{array}{ccc} FX & \xrightarrow{\phi_X} & GX \\ FR \downarrow & & \downarrow GR \\ FX' & \xrightarrow{\phi_{X'}} & GX' \end{array}$$

Note that RG-operators and parametric transformations completely ignore the arrows of **G**.

A CRG-functor $F : \mathbf{G} \rightarrow \mathbf{H}$ between CRG's is a RG-operator that also maps arrows and parametricity squares functorially and satisfies $F\langle f \rangle = \langle Ff \rangle$ for all arrows f . Since $\langle \rangle$ is injective, the action on arrows Ff is uniquely determined by the RG-operator. Hence, an RG-operator can be part of at most one CRG-functor.

Examples of CRG-functors include the usual product, coproduct and internal hom functors on the above mentioned CRG's. For the internal hom $\Rightarrow : \mathbf{G}^{\text{op}} \times \mathbf{G} \rightarrow \mathbf{G}$, note that $f : A' \rightarrow A$ and $g : B \rightarrow B'$ give an edge $\langle f \Rightarrow g \rangle : (A \Rightarrow B) \leftrightarrow (A' \Rightarrow B')$ which is equal to $\langle f \rangle^\smile \Rightarrow \langle g \rangle$.

Lemma 1 *If $F, G : \mathbf{G} \rightarrow \mathbf{H}$ are CRG-functors then any parametric transformation $\phi : F \rightarrow G$ is natural.*

Proof. If $f : X \rightarrow Y$ is an arrow in **G**, $\langle f \rangle : X \leftrightarrow Y$ is an edge in **G**. Then we have the parametricity square on the left which means the same as the commutative square on the right:

$$\begin{array}{ccc} FX & \xrightarrow{\phi_X} & GX \\ F\langle f \rangle \downarrow & & \downarrow G\langle f \rangle \\ FY & \xrightarrow{\phi_Y} & GY \end{array} \qquad \begin{array}{ccc} FX & \xrightarrow{\phi_X} & GX \\ Ff \downarrow & & \downarrow Gf \\ FY & \xrightarrow{\phi_Y} & GY \end{array}$$

□

2 Parametric types

For this section, we assume that \mathbf{W} is a small CRG that is a subgraph of \mathbf{Set} and consider RG-operators $\mathbf{W} \rightarrow \mathbf{Set}$. We write $f: A \leq B$ for the arrows of \mathbf{W} . For simplicity, we assume that the category part of \mathbf{W} is a preorder though the results also generalize to the general case.

If $F, G: \mathbf{W} \rightarrow \mathbf{Set}$ are RG-operators, $\forall X. FX \Rightarrow GX$ denotes the set of parametric transformations $F \rightarrow G$. Recall that all such transformations are natural whenever F and G are CRG-functors.

If W is a vertex of \mathbf{W} , then $\forall X \geq W. FX \Rightarrow GX$ denotes the set of families $\{\phi_X\}_{X \geq W}$ indexed by vertices $X \geq W$ satisfying the following condition: for all edges $R: X \leftrightarrow X'$ where $X \geq W$ and $X' \geq W$, if the diagram on the left is a parametricity square then so is the diagram on the right:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 W & \xrightarrow{\leq} & X \\
 I_W \uparrow & & \uparrow R \\
 W & \xrightarrow{\leq} & X'
 \end{array} & \Longrightarrow & \begin{array}{ccc}
 FX & \xrightarrow{\phi_X} & GX \\
 FR \uparrow & & \uparrow GR \\
 FX' & \xrightarrow{\phi_{X'}} & GX'
 \end{array}
 \end{array} \quad (2)$$

Theorem 2 *If $F, G: \mathbf{W} \rightarrow \mathbf{Set}$ are CRG-functors then every family $\phi \in \forall X \geq W. FX \Rightarrow GX$ is natural.*

Proof: Suppose $W \leq X \leq X'$ and $f: X \leq X'$ is the requisite unique arrow. Then, the diagram on the left commutes and is equivalent to the parametricity square on the right:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 W & \xrightarrow{\leq} & X \\
 \text{id}_W \downarrow & & \downarrow f \\
 W & \xrightarrow{\leq} & X'
 \end{array} & & \begin{array}{ccc}
 W & \xrightarrow{\leq} & X \\
 I_W \downarrow & & \downarrow \langle f \rangle \\
 W & \xrightarrow{\leq} & X'
 \end{array}
 \end{array}$$

Then the parametricity condition on ϕ gives the parametricity square on the left with the effect of the commutative square on the right:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 FX & \xrightarrow{\phi_X} & GX \\
 F\langle f \rangle \downarrow & & \downarrow G\langle f \rangle \\
 FX' & \xrightarrow{\phi_{X'}} & GX'
 \end{array} & & \begin{array}{ccc}
 FX & \xrightarrow{\phi_X} & GX \\
 Ff \downarrow & & \downarrow Gf \\
 FX' & \xrightarrow{\phi_{X'}} & GX'
 \end{array}
 \end{array}$$

□

Corollary 3 *If $F, G: \mathbf{W} \rightarrow \mathbf{Set}$ are CRG-functors then every family $\phi \in \forall X \leq W. FX \Rightarrow GX$ is natural.*

The expression $\forall X \geq A. FX \Rightarrow GX$ can be turned into an RG-operator in A . The action on an edge $R: A \leftrightarrow B$ is the relation

$$[\forall S \geq R. FS \Rightarrow GS] : [\forall X \geq A. FX \Rightarrow GX] \leftrightarrow [\forall X \geq B. FX \Rightarrow GX]$$

which relates families $\{\phi_X\}_{X \geq A}$ and $\{\psi_X\}_{X \geq B}$ as follows: for all relations $S: X \leftrightarrow X'$ if the diagram on the left is a parametricity square then so is the diagram on the right:

$$\begin{array}{ccc}
 A & \xrightarrow{\leq} & X \\
 R \uparrow & & \uparrow S \\
 B & \xrightarrow{\leq} & X'
 \end{array}
 \implies
 \begin{array}{ccc}
 FX & \xrightarrow{\phi_X} & GX \\
 FS \uparrow & & \uparrow GS \\
 FX' & \xrightarrow{\psi_{X'}} & GX'
 \end{array}$$

Theorem 4 *The RG-operator $HA = \forall X \geq A. FX \Rightarrow GX$ uniquely extends to a CRG-functor. Its action on an arrow $f: A \leq A'$ is given by*

$$Hf(\phi) = \Lambda X \geq A'. \phi_X$$

Proof: Suppose $f: A \leq A'$. Then, for any $X \geq A'$, we have the commutative square and parametricity square:

$$\begin{array}{ccc}
 A & \xrightarrow{\leq} & X \\
 f \downarrow & & \downarrow \text{id}_X \\
 A' & \xrightarrow{\leq} & X
 \end{array}
 \quad
 \begin{array}{ccc}
 A & \xrightarrow{\leq} & X \\
 \langle f \rangle \downarrow & & \downarrow I_X \\
 A' & \xrightarrow{\leq} & X
 \end{array}$$

Consider the relation $H\langle f \rangle: HA \leftrightarrow HA'$. If $\phi H\langle f \rangle \psi$ then, for all $X \geq A'$, we have

$$\begin{array}{ccc}
 FX & \xrightarrow{\phi_X} & GX \\
 FI_X \uparrow & & \uparrow GI_X \\
 FX & \xrightarrow{\psi_X} & GX
 \end{array}$$

Since $FI_X = I_{FX} = \langle \text{id}_{FX} \rangle$ and $GI_X = I_{GX} = \langle \text{id}_{GX} \rangle$, we conclude that $\phi_X = \psi_X$. So, $\psi = Hf(\phi)$ as defined in the theorem, and $H\langle f \rangle = \langle Hf \rangle$. \square

Note that $\forall X \geq A. FX \Rightarrow GX$ is a CRG-functor in A even if F and G are not functors. However, the naturality of the family requires F and G to be CRG-functors.

Similarly, $HA = \forall X \leq A. FX \Rightarrow GX$ is a *contravariant* CRG-functor in A . The relation $HR = \forall S \leq R. FS \Rightarrow GS$ means that the parametricity square on the left implies that on the right:

$$\begin{array}{ccc}
 X & \xrightarrow{\leq} & A \\
 S \uparrow & & \uparrow R \\
 X' & \xrightarrow{\leq} & B
 \end{array}
 \quad
 \begin{array}{ccc}
 FX & \xrightarrow{\phi_X} & GX \\
 FS \uparrow & & \uparrow GS \\
 FX' & \xrightarrow{\psi_{X'}} & GX'
 \end{array}$$

Corollary 5 *The RG-operator $HA = \forall X \leq A. FX \Rightarrow GX$ uniquely extends to a contravariant CRG-functor. Its action on an arrow $f: A' \leq A$ is given by*

$$Hf(\phi) = \Lambda X \geq A'. \phi_X$$

Generalizations It is easy enough to relax the condition that \mathbf{W} is a preorder. If it is not then an element $\phi \in \forall X \geq W. FX \Rightarrow GX$ is a family $\{\phi_e\}_{e:W \leq X}$ indexed by arrows $e:W \leq X$ with X ranging over the vertices of \mathbf{W} . The condition (2) is modified to:

$$\begin{array}{ccc}
 W & \xrightarrow{e} & X \\
 I_W \uparrow & & \uparrow R \\
 W & \xrightarrow{e'} & X'
 \end{array}
 \quad \Longrightarrow \quad
 \begin{array}{ccc}
 FX & \xrightarrow{\phi_e} & GX \\
 FR \uparrow & & \uparrow GR \\
 FX' & \xrightarrow{\phi_{e'}} & GX'
 \end{array}
 \quad (3)$$

All the results of this section extend to this case.

The case of \mathbf{Cpo}_\perp runs parallel to the case of \mathbf{Set} . The type $\forall X \geq W. FX \Rightarrow GX$ is made into a pointed cpo by pointwise order.

The case of \mathbf{Per} is more complicated, as is the general categorical case.

3 Interpreting ALGOL

If X is a set, let $T(X)$ denote the monoid of total transformations $[X \Rightarrow X]$. The multiplication is composition “;” and the unit is the identity transformation. A monoid homomorphism $\tau: T(X) \rightarrow T(Y)$ preserves the composition and the identity. A monoid relation $R: T(X) \leftrightarrow T(X')$ is a relation satisfying:

- $\text{id}_X R \text{id}_{X'}$, and
- $a R a'$ and $b R b'$ implies $(a; b) R (a'; b')$.

All our transformation monoids are equipped with an assignment operation given by

$$\begin{aligned}
 \text{asgn}_X^\delta &: (\delta \rightarrow T(X)) \times (X \Rightarrow \delta) \rightarrow T(X) \\
 \text{asgn}_X^\delta(a, e) &= \lambda x. a(e x) x
 \end{aligned}$$

We define a complete reflexive graph **State** as follows:

- The objects are sets.
- The arrows $f: X \leq W$ are pairs $f = (\phi_f, \tau_f)$ where $\phi_f: X \rightarrow W$ is a function and $\tau_f: T(W) \rightarrow T(X)$ is monoid homomorphism such that the following squares commute for all $a \in T(W)$:

$$\begin{array}{ccc}
 X & \xrightarrow{\tau_f(a)} & X \\
 \phi_f \downarrow & & \downarrow \phi_f \\
 W & \xrightarrow{a} & W
 \end{array}
 \quad
 \begin{array}{ccc}
 (\delta \Rightarrow T(X)) \times (X \Rightarrow \delta) & \xrightarrow{\text{asgn}_X} & T(X) \\
 \uparrow (\text{id}_\delta \Rightarrow \tau_f) \times (\phi_f \Rightarrow \text{id}_\delta) & & \uparrow \tau_f \\
 (\delta \Rightarrow T(W)) \times (W \Rightarrow \delta) & \xrightarrow{\text{asgn}_W} & T(W)
 \end{array}
 \quad (4)$$

Intuitively, ϕ_f projects a small state from a large one, and τ_f expands a transformation of a small state to a large one. The commutative square on the left says that the expansion of a transformation should preserve its action on the small state while the square on the right says that the assignment operation is preserved.

- The edges are pairs $R = (R_S, R_T) : W \leftrightarrow W'$ where $R_S : W \leftrightarrow W'$ is a relation and $R_T : T(W) \leftrightarrow T(W')$ is a monoid relation such that

- $a R_T a' \implies a [R_S \rightarrow R_S] a'$.
- $a [\Delta_\delta \Rightarrow R_T] a' \wedge e [R_S \Rightarrow \Delta_\delta] e' \implies \text{asgn}_W(a, e) R_T \text{asgn}_{W'}(a', e')$

The identity edges are $I_W = (\Delta_W, \Delta_{T(W)})$. There is an edge morphism $(\phi, \tau) [S \leq R] (\phi', \tau')$ iff the following diagrams are parametricity squares in **Set**:

$$\begin{array}{ccc}
 X & \xrightarrow{\phi} & W \\
 S_S \uparrow & & \downarrow R_S \\
 X' & \xrightarrow{\phi'} & W'
 \end{array}
 \qquad
 \begin{array}{ccc}
 T(W) & \xrightarrow{\tau} & T(X) \\
 R_T \uparrow & & \downarrow S_T \\
 T(W') & \xrightarrow{\tau'} & T(X')
 \end{array}$$

- The edge mapping is $\langle f : X \leq W \rangle = (\langle \phi_f \rangle, \langle \tau_f \rangle^\smile)$, a pair consisting of the graphs of the projection and expansion functions. This makes **State** a CRG.

A useful subcategory of **State** is **W**, whose objects are record types (indexed products) over a countable set of labels $\{l_0, l_1, \dots\}$. These objects are partially ordered by $X \leq W$ iff X is a record extension of W . The unique morphisms $f : X \leq W$ are given by:

- $\phi_f : X \rightarrow W$ which projects the W -components from an X -record, and
- $\tau_f : T(W) \rightarrow T(X)$ which expands a transformation to a larger record by leaving the new components unchanged.

We can now interpret ALGOL types as CRG-operators $\mathbf{State} \rightarrow \mathbf{Set}$:

$$\begin{array}{ll}
 \mathbf{comm}(W) & = \forall X \leq W. T(X) & \mathbf{comm}(R) & = \forall S \leq R. S_T \\
 \mathbf{exp}_\delta(W) & = \forall X \leq W. X \Rightarrow \delta & \mathbf{exp}_\delta(R) & = \forall S \leq R. S_S \Rightarrow \Delta_\delta \\
 (\theta_1 \times \theta_2)(W) & = \theta_1(W) \times \theta_2(W) & (\theta_1 \times \theta_2)(R) & = \theta_1(R) \times \theta_2(R) \\
 (\theta_1 \rightarrow \theta_2)(W) & = \forall X \leq W. \theta_1(X) \Rightarrow \theta_2(X) & (\theta_1 \rightarrow \theta_2)(R) & = \forall S \leq R. \theta_1(S) \Rightarrow \theta_2(S)
 \end{array}$$

By Corollary 5, all these operators uniquely extend to contravariant CRG-functors, with the action on an arrow $f : W' \leq W$ given by:

$$\begin{array}{ll}
 \mathbf{comm}(f) & = \lambda a. \Lambda X \leq W'. a[X] \\
 \mathbf{exp}_\delta(f) & = \lambda e. \Lambda X \leq W'. e[X] \\
 (\theta_1 \times \theta_2)(f) & = \theta_1(f) \times \theta_2(f) \\
 (\theta_1 \rightarrow \theta_2)(f) & = \lambda p. \Lambda X \leq W'. p[W']
 \end{array}$$

Lemma 1 ensures that all parametric transformations on them are natural.

The fact that $\mathbf{comm}(W)$ has enough elements is shown by the following result:

Lemma 6 *There is a parametric isomorphism:*

$$\pi_W : [\forall X \leq W. T(X)] \cong T(W)$$

Proof. Let $p \in \forall X \leq W. T(X)$. For any edge $R \leq I_W$ where $R : X \leftrightarrow X'$,

$$\begin{array}{ccc} X & \xrightarrow{\leq} & W \\ R \uparrow & & \uparrow I_W \\ X' & \xrightarrow{\leq} & W \end{array}$$

the components $p[X]$ and $p[X']$ must be related by R_T . In particular, we have the parametricity square on the left which comes from the commutative square on the right:

$$\begin{array}{ccc} X & \xrightarrow{f} & W \\ \langle f \rangle \uparrow & & \uparrow I_W \\ W & \xrightarrow{\text{id}_W} & W \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{f} & W \\ f \downarrow & & \downarrow \text{id}_W \\ W & \xrightarrow{\text{id}_W} & W \end{array}$$

Hence, $p[X] = \tau_f(p[W])$ and p is uniquely determined by $p[W]$. Take $\pi_W(p) = p[W]$.

Let $R : W \leftrightarrow W'$ be a relation such that $p \in \forall X \leq W. T(X)$ and $p' \in \forall X \leq W'. T(X)$ are related by $\forall S \leq R. S_T$. Since $R \leq R$, $p[W] R_T p'[W']$. Conversely, if $p[W] R_T p'[W']$ then the families $\{\tau_f(p[W])\}_{f.X \leq W}$ and $\{\tau_{f'}(p'[W'])\}_{f'.X \leq W'}$ are related by $\forall S \leq R. S_T$. If we have a parametricity square such as the one on the left, its meaning includes the parametricity square on the right, which gives the conclusion:

$$\begin{array}{ccc} X & \xrightarrow{f} & W \\ S \uparrow & & \uparrow R \\ X' & \xrightarrow{f'} & W' \end{array} \qquad \begin{array}{ccc} T(W) & \xrightarrow{\tau_f} & T(X) \\ R_T \uparrow & & \uparrow S_T \\ T(W') & \xrightarrow{\tau_{f'}} & T(X') \end{array}$$

□

Let us look at an example that illustrates how our translation achieves implicit naturality while the O'Hearn-Tennent translation doesn't. Consider the type

$$(\mathbf{comm} \rightarrow \mathbf{comm})(W) = \forall X \leq W. (X \Rightarrow X) \Rightarrow (X \Rightarrow X)$$

In (O'Hearn and Tennent, 1995) is exhibited a parametric family of this type which works as follows:

$$p_W = \Lambda X \leq W. \lambda a : X \Rightarrow X. \lambda x : X. \pi_{W,X}(x) \oplus \pi'_{W,X}(a(x))$$

Here $\pi_{W,X}$ projects the W part of an X -record, $\pi'_{W,X}$ projects all the components of X that are not in W , and \oplus concatenates records with disjoint fields. The effect of p is to run the input command once and to reset the local part of the state to the original.

The family p is not natural (regarded as a transformation of type $1 \rightarrow [\mathbf{comm} \rightarrow \mathbf{comm}]$). If $f : W' \leq W$ then the functor action on f maps p_W to:

$$\Lambda X \leq W'. \lambda a : X \Rightarrow X. \lambda x : X. \pi_{W,X}(x) \oplus \pi'_{W,X}(a(x))$$

whereas $p_{W'}$ is:

$$\Lambda X \leq W'. \lambda a : X \Rightarrow X. \lambda x : X. \pi_{W',X}(x) \oplus \pi'_{W',X}(a(x))$$

However, the family p is parametric in the setting of (O’Hearn and Tennent, 1995).

In our setting, the family p is not parametric. By Corollary 5, parametric families are necessarily natural whenever they operate on CRG-functors. Specifically, since we have the parametricity square

$$\begin{array}{ccc}
 X & \xrightarrow{(\phi', \tau')} & W' \\
 I_X \updownarrow & & \updownarrow \langle f \rangle \\
 X & \xrightarrow{(\phi, \tau)} & W
 \end{array} \tag{5}$$

parametricity of p requires $p_{W'}[X] = p_W[X]$. This is not the case.

This example also shows why we need the T components in the edges of **State**. Part of the meaning of (5) is the following parametricity square in **Set**:

$$\begin{array}{ccc}
 T(W') & \xrightarrow{\tau'} & T(X) \\
 \langle \tau_f \rangle^\smile \updownarrow & & \updownarrow \Delta_{T(X)} \\
 T(W) & \xrightarrow{\tau} & T(X)
 \end{array}$$

If we had only S components and no T components in the edges, then we would be forced to use $[\langle \phi_f \rangle \rightarrow \langle \phi_f \rangle]$ in place of $\langle \tau_f \rangle^\smile$. If $a' \in T(W')$ and $a \in T(W)$ are related by $[\langle \phi_f \rangle \rightarrow \langle \phi_f \rangle]$ that means the action of a and a' on the W part of a large state is the same. It can however change the new components in arbitrarily different ways. The expansion of a and a' to X then have no reason to be identical. Thus, a single relation for the edges does not suffice for obtaining a complete reflexive graph.

These ideas also give a new semantics to the positive subtyping calculus (?) with better properties. For instance, the following type of a point move function was suggested in (Cardelli and Wegner, 1986):

$$move : \forall X \leq \{x, y : Int\}. X \rightarrow X$$

However, in the standard per models this type is trivial (has only the identity function). The following type is recommended in (Plotkin et al., 1994):

$$move : \forall Z \uparrow \{x, y\}. \{x, y : Int, Z\} \rightarrow \{x, y : Int, Z\}$$

But, this treatment is essentially like that of (O’Hearn and Tennent, 1995) which is known not to possess the requisite naturality properties. Our semantics gives a satisfactory interpretation to the original Cardelli-Wegner type with its implicit naturality properties.

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