The University of Birmingham
School of Computer Science

Exercise Sheet 2

Attempt as many questions as you can during the exercise class, and work on the remainder at home. Hand in your solutions to the tutor in the Friday exercise class, on 5 Feb, 2010.

Exercise 1: Free variables and substitution

a. List the free variables of the following lambda terms. (Note that everything other than the λ symbol is a "variable" in the pure lambda calculus.)

   i. $\lambda x.\, \lambda y.\, x\ z$

   ii. $\lambda r.\, f\ r\ 1$

   iii. $\lambda y.\, *\ y\ 2$

   iv. $\lambda y.\, \lambda z.\, g\ (x\ z)\ (y\ z)$.

   v. $\lambda x.\, \lambda x.\, x$

b. Write down the result of the following substitution operations.

   i. $(\lambda y.\, \lambda z.\, g\ (x\ z)\ (y\ z))\ [g := (\lambda r.\, f\ r\ 1)]$

   ii. $(\lambda x.\, \lambda y.\, x\ z)\ [z := (\lambda y.\, 2)]$

   iii. $(\lambda x.\, \lambda x.\, x)\ [x := (\lambda y.\, *\ y\ 2)]$

   iv. $(\lambda x.\, \lambda y.\, x\ z)\ [z := (y\ 2)]$

c. A substitution of the form $M[x := N]$ is said to be *valid* if all the free variables of N continue to be free after N is substituted for x in M. For each of the above substitutions, mention if it is valid or not.

d. Explain in a few sentences why invalid substitutions can be problematic.

Exercise 2: Types for lambda terms

Write down possible types for the following lambda terms. You can use A, B, ... etc. to stand for arbitrary types. Assume that 2 is of type **int** and $*$ is of type **int** $\rightarrow$ **int** $\rightarrow$ **int**.

a. $\lambda x.\, \lambda y.\, y\ x$

b. $\lambda y.\, *\ y\ 2$

c. $\lambda f.\, \lambda y.\, f\ y\ y$

d. $\lambda g.\, \lambda x.\, \lambda y.\, \lambda z.\, g\ (x\ z)\ (y\ z)$

Exercise 3: beta redexes

Given below is a term M:

$$M = \lambda x.\, ((\lambda z.\, z\ x)((\lambda r.\, \lambda s.\, s\ r)\ y\ f))$$

a. Identify two separate beta redexes in the term M.

b. Show the results of reducing M at each of these redexes. Call the results of these reductions N_1 and N_2.

c. Reduce N_1 and N_2 to normal forms and check if they reduce to the same normal form.