Choosing Semantics for Linear Logic

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Objectives

- Tools/criteria for choosing between alternative semantics for (linear) logic.

- Favourite Tool:
  Categorical Proof Theory.

- Models:
  algebraic, categorical, domain-theoretical, Kripke-style, games, etc

- Plus: petri nets, complexity of computation, topological spaces, hopf algebras, quantum stuff, etc

Outline

- Algebraic Semantics
- Algebraic Semantics for Linear Logic
- Categorical Proof Theory
- Categorical Semantics of Linear Logic
- Other models

Algebraic Semantics

- A set with a partial order (this models entailment) and operations on its elements modelling the logical connectives

- An interpretation of formulae, built from atomic components, into the set.

- Boolean algebra: propositional classical logic

- Heyting algebra: propositional intuitionistic logic

- Just syntax in disguise?

Why Algebraic Semantics?

- Provide some intuitions on meaning of connectives.
- Used to establish soundness and completeness
  (via construction of Lindenbaum algebras).
- Hence, needed for minimal adequacy
  (No algebraic semantics, no logic).
- Easier to calculate with (sometimes)
- But: proliferation of algebraic semantics, even for propositional logic, and especially for linear logic . . .

Algebraic Semantics of Linear Logic

- Girard's Phase spaces

- Troelstra / Ono's IL(CL)-algebras

- Quantales, *lineales*

- Mitchell and Simmons IE-models (notes)

- Sambin's pretopologies

- Ursini's 10 colourful structures, etc..

Phase Spaces: Girard'87,95

- A phase space is a pair $(M, \perp)$ where M is a commutative monoid $(M, \circ, 1)$ and $\perp$ is a subset of M.

  Given X and Y subsets of M, define:
  $X \multimap Y = \{m \in M | \forall n \in X, m \circ n \in Y\}$ and $X^{\perp} = X \multimap \perp$.

- A *fact* is any subset of M equal to its double dual $X^{\perp\perp}$.

- Interpret LL connectives by operations on facts, eg
  $X \otimes Y = \{m \circ n | m \in X$ and $n \in Y\}$ and $!X = \{(X \cap I)^{\perp\perp}\}$
  where I is the set of idempotents of M, which belong to 1.

- Girard soundness and completeness, using Lindenbaum algebra
  a phase space not defined as algebraic structure satisfying the laws of linear logic.

Troelstra / Ono's IL(CL)-algebras

- An IL-algebra $(X, \wedge, \vee, 0, \multimap, \circ, 1)$ consists of:
  — $(X, \wedge, \vee, 0)$ a lattice with smallest element 0
  — $(X, \circ, 1)$ a commutative monoid with unit 1

  Satisfying:
  — If $x \leq x'$ and $y \leq y'$ then $x \circ y \leq y \circ y'$ and $x' \multimap y \leq x \multimap y'$
  — $x \circ y \leq z$ if and only if $x \leq y \multimap z$

- To cope with negation add constant $\perp$, say dual $x^{\perp} = x \multimap \perp$ and say CL-algebra one such that $x^{\perp\perp} = x$.

- Easier to prove soundness and completeness

- Easier to show relationship to other algebraic structures

Choosing a Semantics

- Explanation of connectives
- Relation to (behaviour of connectives) in proof theory
- Ease of calculation
- Generality
- Stepwise fragments (modularity)
- Showing equivalences

Crazy Idea: Use categorical semantics

- Explanation of connectives
- Relation to (behaviour of connectives) in proof theory
- Ease of calculation: well...
- Generality
- Stepwise fragments (modularity):
- Showing equivalences

Categorical Choice of Algebraic Semantics

- Every category can be collapsed to a poset, forget the morphisms and you have a algebraic structure
- Forgetting structure is much easier than creating it: algebraic semantics from categorical semantics a no-brainer
- Then a symm monoidal closed cat becomes a lineale
  Products and coproducts become meets and joins
- The monoidal comonad ! becomes a closure operator satisfying some inequations
  some of the complications go away too.
- Obtain all soundness and completeness theorems by fiat, almost.

- Generalisation of algebra:
  Structures come with structure-preserving maps, e.g.

sets	with	*functions*
groups	with	*homomorphisms*
topological spaces	with	*continuous maps*
vector spaces	with	*linear transformations*.

  Maps that are not structure-preserving shouldn't be considered.
  Relations (maps) between objects matter more than objects.

- A category $\mathcal{C}$ is a collection of objects $(A, B, \ldots)$ and maps M between them $(f : A \to B, \ldots)$, such that
  (a) there exists an associative composition of maps, $\circ$
  $f : A \to B$, $g : B \to C$ then $g \circ f = h : A \to C$
  (maps are closed under composition: h is in M)
  (b) each object A has an identity map $id_A : A \to A$ such that
  for any $f : A \to B$, $\quad id_A \circ f = f = f \circ id_B$

Basics of Categorical Semantics

- Model theory using categories, instead of sets or posets.

- Two main kinds of "categorical semantics":
  categorical proof theory, and categorical model theory

- Categorical proof theory models derivations (proofs) not simply whether theorems are true or not

- Back to main theme of this course: pay attention to proofs

Categorical Semantics: Extension of Curry-Howard

- Curry-Howard Isomorphism:

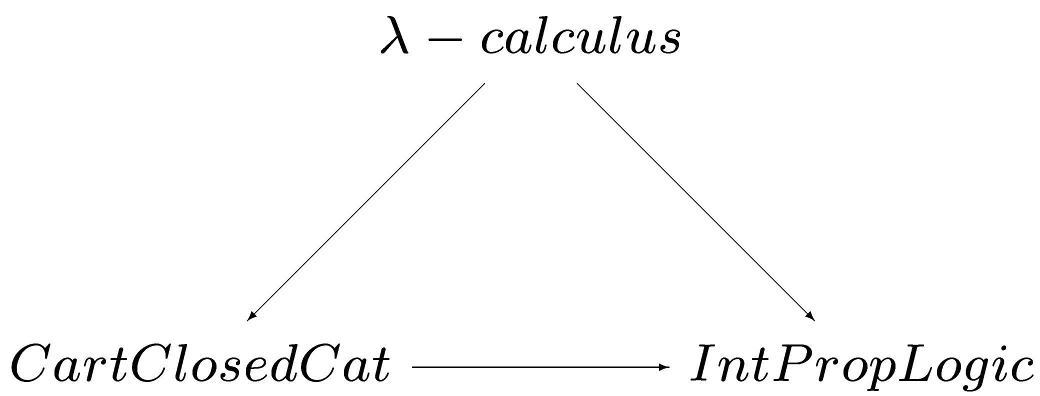
  Natural deduction (ND) proofs $\Leftrightarrow$ λ-terms
  Normalization of ND proof $\Leftrightarrow$ reduction in λ-calculus.

- Cat Proof Theory:
  λ-calculus types $\Leftrightarrow$ objects in (appropriate) category
  λ-calculus terms $\Leftrightarrow_{1-1}$ morphisms in category
  Cat. structure models logical connectives (type operators).

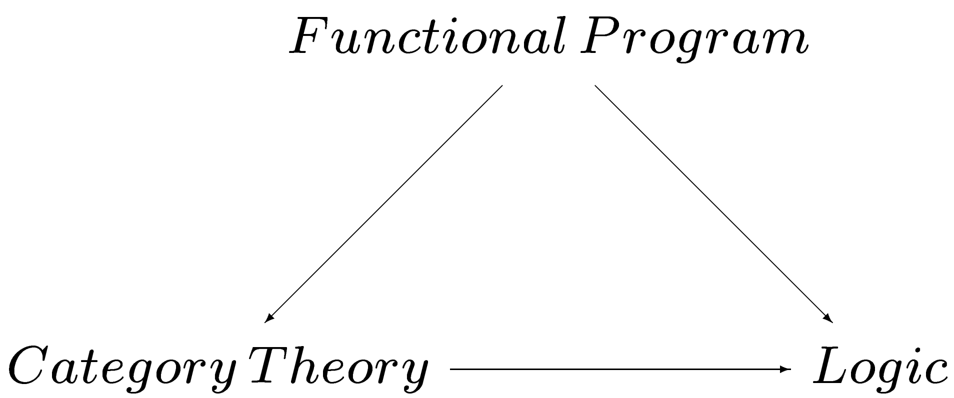
- Isomorphism:
  transfers results/tools from one side to the other

- λ-calculus basis of functional programming:
  logical view of programming

Categorical Semantics: a picture



Framework connecting



lots of important stuff elided

Categorical Semantics: some gains

- Functional Programming: optimizations that do not compromise semantic foundations (e.g. referential transparency)

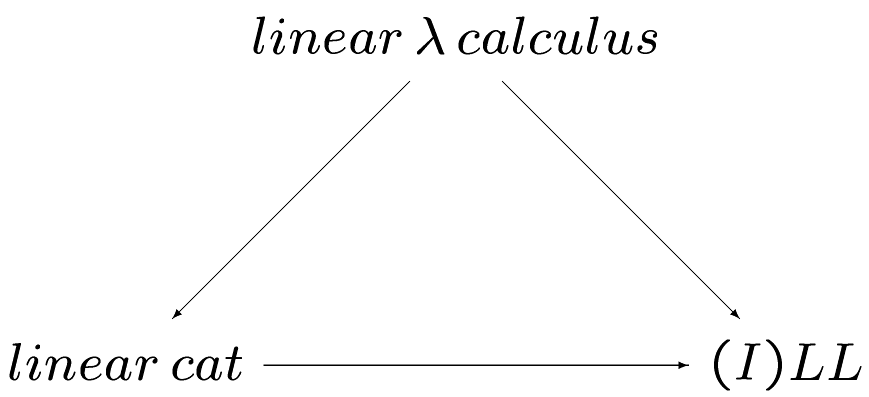
- Logic: new applications of old theorems (e.g. normalization)

- Category Theory: new concepts not from maths

- Philosophy: new criteria for identities of proofs

Categorical Semantics of (Intuitionistic) Linear Logic

Want linear version of Extended Curry-Howard Isomorphism



- Logic intuitionistic

- Already have Intuitionistic Linear Logic (ILL),
  — must find other two sides of triangle...

- Problem with linear lambda-calculus, and
  Problem with linear category

Linear λ-calculus: which?

- Avron(88), Abramsky(90):
  wrong calculus, no substitution in general
- Benton et al(93), Brauner(93):
  ILL, verbose calculus, comm conversions, CHI ok
- Barber&Plotkin DILL(97), Benton's LNL(95):
  less verbose, comm conv, CHI ok
- MILL, Plotkin, Wadler:
  even less verbose, comm conv, best for implementation, CHI?
- others: Troelstra, Mints, Roversi, etc..

Linear categories

- multiplicative-additive fragment easy
  - multiplicative fragment: a symmetric monoidal closed category (SMCC)
  - additive fragment: categorical products and coproducts
- modalities (exponentials): the problem...
- several solutions, depending on linear λ-calculus chosen...

Cartesian Closed Categories

- A category $\mathcal{C}$ is a cartesian closed category if
  - it has finite (categorical) products $A \times B$ for all objects A,B of $\mathcal{C}$
  - for each object B of $\mathcal{C}$, the product functor $\_ \times B$ has a right-adjoint written, as $(B \Rightarrow \_)$.

    This gives an isomorphism: for any map $(A \times B) \to C$ in $\mathcal{C}$ there is an isomorphic map $A \to (B \Rightarrow C)$ in $\mathcal{C}$.

- Products, $\times$, model conjunction and function spaces, $\Rightarrow$, model implication

- To make this definition resource-conscious, we need to make categorical product $\times$ a tensor product $\otimes$.

Symmetric Monoidal Closed Categories (SMCC)

- A category $\mathcal{L}$ is a symmetric monoidal closed category if
  - it has a tensor product $A \otimes B$ for all object A,B of $\mathcal{C}$.
  - for each object B of $\mathcal{C}$, the tensor-product functor $\_ \otimes B$ has a right-adjoint, written $B \multimap \_$
    (ie have isomorphism from maps $(A \otimes B) \to C$ to $A \to (B \multimap C)$)

	Categorical product $A \times B$	Tensor product $A \otimes B$
1	Projections onto A and B	Not necessarily
2	For any object C with projections to A and B, there is a *unique* map from C to $A \times B$	Not necessarily

- Objects A in a CCC have duplication $A \to A \times A$ and erasing $A \to 1$ (satisfying equations of a comonoid).

- Objects A in a SMCC have no duplication $A \to A \otimes A$ nor erasing $A \to I$, in general. Neither have they projections $A \otimes B \to A$.

- But objects of the form $!A$ should have duplication and erasing. Moreover they must satisfy S4-Box introduction and elimination rules.

- S4-axioms give you proofs $!A \to A$ and $!A \to !!A$, uniformly for any A object A in $\mathcal{C}$. Hence want a functor (unary operator) on category, $! \colon \mathcal{C} \to \mathcal{C}$ such that there are such natural transformations. Such a structure already in category theory, a comonad.

Modelling the Modality !

- An object $!A$ must have four maps:

$$\epsilon\colon !A \to A,\quad \delta\colon !A \to !!A,\quad \mathsf{er}\colon !A \to I,\quad \mathsf{dupl}\colon !A \to !A \otimes !A$$

  To make identity of proofs sensible, must have lots of equations.

- Equations for duplicating and erasing say $!A$ is a *comonoid*; equations for S4-axioms say ! is a *comonad*.

- How do comonad and comonoid structures interact?

- First idea (Seely'87): From logical equivalence $!(A\&B) \equiv !A \otimes !B$ Introduce a comonad $(!, \delta, \epsilon)$ taking the comonoid structure $(A,\ A \to A \times A,\ A \to 1)$ to $(!A,\ \mathsf{dupl}\colon !A \to !A \otimes !A,\ \mathsf{er}\colon !A \to I)$.

Categorical Semantics of ILL: problems

- Using the isomorphism $!(A\&B) \cong !A \otimes !B$ can't model multiplicatives & modalities *without* additives.

- Moreover, this semantics is not *sound*:
  There are proofs π_1 and π_2, equivalent under cut-elimination, which are not equal morphisms in the category.
  see Bierman "What is a model of ILL?"

- Better idea: ask for a *monoidal* comonad

$$m_{A,B}\colon\ !(A \otimes B) \to !A \otimes !B \text{ and } m_I\colon I \to !I$$

  satisfying conditions relating comonad to comonoid structure.
  which conditions?

Categorical Semantics of ILL: first solution

A *linear category* comprises
— an SMCC $\mathcal{C}$, (with products and coproducts),
— a symmetric monoidal comonad $(!, \epsilon, \delta, m_I, m_{A,B})$
such that:
(a) For every free co-algebra $(!A, \delta)$ there are nat. transfns.
$\mathsf{er}_A\colon !A \to I$ and $\mathsf{dupl}_A\colon !A \to !A \otimes !A$, forming a commutative comonoid, which are coalgebra maps.
(b) Every map of free coalgebras is also a map of comonoids.
Bierman 1995, based on Benton et al'93

- long definition, lots of diagrams to check..
- Prove that $A \Rightarrow B \cong !A \multimap B$ by constructing CCC out of SMCC plus comonad !.
- Extended Curry-Howard works!! but complicated.

Categorical Semantics of ILL: second solution

- (Benton 1995) A model of LNL (Linear/NonLinear Logic) consists of a symm monoidal closed category $\mathbf{L}$, a cartesian closed category $\mathbf{C}$, and a *monoidal adjunction* between these categories.

- Modelling two logics, intuitionistic and linear at the same time. Logics on equal footing.

- First solution builds the cartesian closed category out of the monoidal comonad; this definition asks for the adjunction directly

- Easier to remember, same diagrams to check.

- Is it the *same* logic?

Categorical Semantics of ILL: third solution

- (Barber 1996) A model of DILL (Dual Intuitionistic and Linear Logic) consists of a symm monoidal closed category $\mathbf{L}$, a cartesian (not necessarily closed) category $\mathbf{C}$, and a *monoidal adjunction* between these categories. small simplification of semantics

- Extended Curry-Howard works, Barber's thesis.
  Linear λ-calculus much nicer as PL

- DILL sound and complete for adjunctions above, but not *internal language*. Does it matter?

- Each model of DILL in bijection with models of ILL. But looking at categories of models have **DILL** not equivalent to **LIN**, only a retraction ???!!!

Kripke-style Semantics for LL

- Some work done
  - Hodas/Miller and Ambler (only fragment of ILL)
  - Allwein/Dunn
    (three-valued, off-shelf: best use of Kripke models, counterexamples, lost? too many acessibility relations?)
  - B. Mitchell's
  - Ursini
- how to choose from? Use categorical semantics plus duality, I say.
- NOT DONE. want to lend a hand?

Conclusions

- discussed several alternatives for algebraic and categorical semantics of LL

- Is this plurality good or bad?

  - similar to IL, various ways to approach the essence

Some References

- Categorical Proof Theory and Linear Logic –preliminary notes for Summer School
- Lineales: algebras and categories in the semantics of Linear Logic
- G. Bierman, What's a Model of ILL? TLCA95.
- P. Scott, Category Theory for Linear Logicians
- A. Troelstra Lectures on Linear Logic
- Girard, Lafont and Regnier Advances in Linear Logic
- Relating Categorical Semantics for Intuitionistic Linear Logic (with M. Maietti, P. Maneggia and E. Ritter), available from webpage