

Choosing Semantics for Linear Logic

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Objectives

- Tools/criteria for choosing between alternative semantics for (linear) logic.
- Favourite Tool:
Categorical Proof Theory.
- Models:
algebraic, categorical, domain-theoretical, Kripke-style, games, etc
- Plus: petri nets, complexity of computation, topological spaces, hopf algebras, quantum stuff, etc

Outline

- Algebraic Semantics
- Algebraic Semantics for Linear Logic
- Categorical Proof Theory
- Categorical Semantics of Linear Logic
- Other models

Algebraic Semantics

- A set with a partial order (this models entailment) and operations on its elements modelling the logical connectives
- An interpretation of formulae, built from atomic components, into the set.
- Boolean algebra: propositional classical logic
- Heyting algebra: propositional intuitionistic logic
- Just syntax in disguise?

Why Algebraic Semantics?

- Provide some intuitions on meaning of connectives.
- Used to establish soundness and completeness (via construction of Lindenbaum algebras).
- Hence, needed for minimal adequacy (No algebraic semantics, no logic).
- Easier to calculate with (sometimes)
- But: proliferation of algebraic semantics, even for propositional logic, and especially for linear logic . . .

Algebraic Semantics of Linear Logic

- Girard's Phase spaces
- Troelstra / Ono's IL(CL)-algebras
- Quantales, *lineales*
- Mitchell and Simmons IE-models (notes)
- Sambin's pretopologies
- Ursini's 10 colourful structures, etc..

Phase Spaces: Girard'87,95

- A phase space is a pair (M, \perp) where M is a commutative monoid $(M, \circ, 1)$ and \perp is a subset of M .

Given X and Y subsets of M , define:

$$X \multimap Y = \{m \in M \mid \forall n \in X, m \circ n \in Y\} \quad \text{and} \quad X^\perp = X \multimap \perp.$$

- A *fact* is any subset of M equal to its double dual $X^{\perp\perp}$.
- Interpret LL connectives by operations on facts, eg
 $X \otimes Y = \{m \circ n \mid m \in X \text{ and } n \in Y\}$ and $!X = \{(X \cap I)^{\perp\perp}\}$
where I is the set of idempotents of M , which belong to 1 .
- Girard soundness and completeness, using Lindenbaum algebra
a phase space not defined as algebraic structure satisfying the laws of linear logic.

- An IL-algebra $(X, \wedge, \vee, 0, \neg, \circ, 1)$ consists of:
 - $(X, \wedge, \vee, 0)$ a lattice with smallest element 0
 - $(X, \circ, 1)$ a commutative monoid with unit 1

Satisfying:

- If $x \leq x'$ and $y \leq y'$ then $x \circ y \leq y \circ y'$ and $x' \neg y \leq x \neg y'$
- $x \circ y \leq z$ if and only if $x \leq y \neg z$
- To cope with negation add constant \perp , say dual $x^\perp = x \neg \perp$ and say CL-algebra one such that $x^{\perp\perp} = x$.
- Easier to prove soundness and completeness
- Easier to show relationship to other algebraic structures

Choosing a Semantics

- Explanation of connectives
- Relation to (behaviour of connectives) in proof theory
- Ease of calculation
- Generality
- Stepwise fragments (modularity)
- Showing equivalences

Crazy Idea: Use categorical semantics

- Explanation of connectives
- Relation to (behaviour of connectives) in proof theory
- Ease of calculation: well...
- Generality
- Stepwise fragments (modularity):
- Showing equivalences

Categorical Choice of Algebraic Semantics

- Every category can be collapsed to a poset, forget the morphisms and you have an algebraic structure
- Forgetting structure is much easier than creating it: algebraic semantics from categorical semantics a no-brainer
- Then a symmetric monoidal closed category becomes a lineale
Products and coproducts become meets and joins
- The monoidal comonad ! becomes a closure operator satisfying some inequations
some of the complications go away too.
- Obtain all soundness and completeness theorems by fiat, almost.

Basics of Category Theory

- Generalisation of algebra:

Structures come with structure-preserving maps, e.g.

sets with *functions*

groups with *homomorphisms*

topological spaces with *continuous maps*

vector spaces with *linear transformations*.

Maps that are not structure-preserving shouldn't be considered.

Relations (maps) between objects matter more than objects.

- A category \mathcal{C} is a collection of objects (A, B, \dots) and maps M between them $(f : A \rightarrow B, \dots)$, such that

(a) there exists an associative composition of maps, \circ

$f : A \rightarrow B, g : B \rightarrow C$ then $g \circ f = h : A \rightarrow C$

(maps are closed under composition: h is in M)

(b) each object A has an identity map $id_A : A \rightarrow A$ such that

for any $f : A \rightarrow B, id_A \circ f = f = f \circ id_B$

Basics of Categorical Semantics

- Model theory using categories, instead of sets or posets.
- Two main kinds of “categorical semantics” :
categorical proof theory, and categorical model theory
- Categorical proof theory models derivations (proofs) not simply whether theorems are true or not
- Back to main theme of this course: pay attention to proofs

Categorical Semantics: Extension of Curry-Howard

- Curry-Howard Isomorphism:

Natural deduction (ND) proofs \Leftrightarrow λ -terms

Normalization of ND proof \Leftrightarrow reduction in λ -calculus.

- Cat Proof Theory:

λ -calculus types \Leftrightarrow objects in (appropriate) category

λ -calculus terms \Leftrightarrow_{1-1} morphisms in category

Cat. structure models logical connectives (type operators).

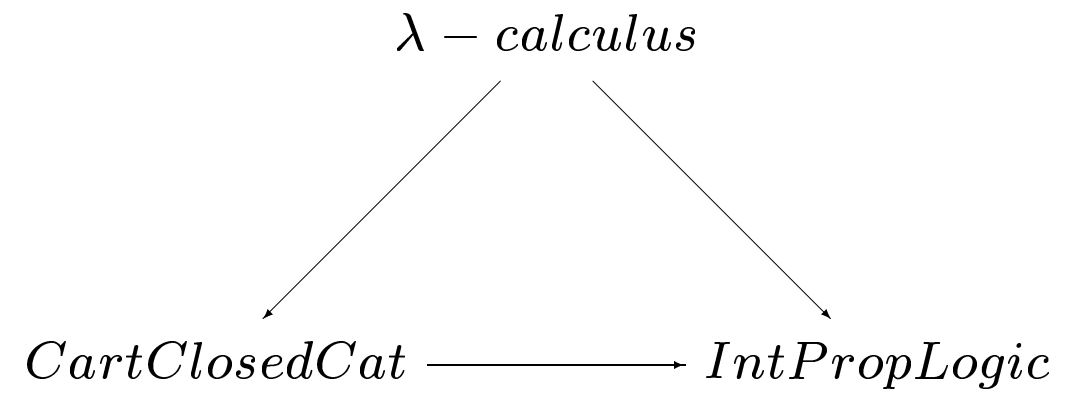
- Isomorphism:

transfers results/tools from one side to the other

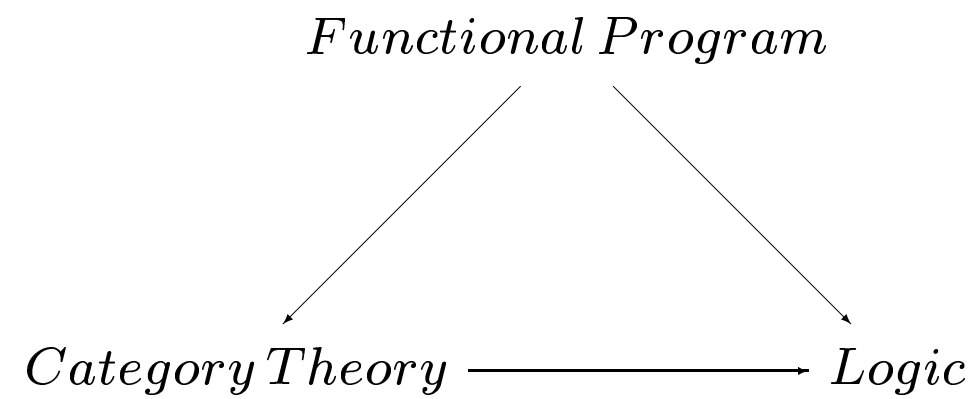
- λ -calculus basis of functional programming:

logical view of programming

Categorical Semantics: a picture



Framework connecting



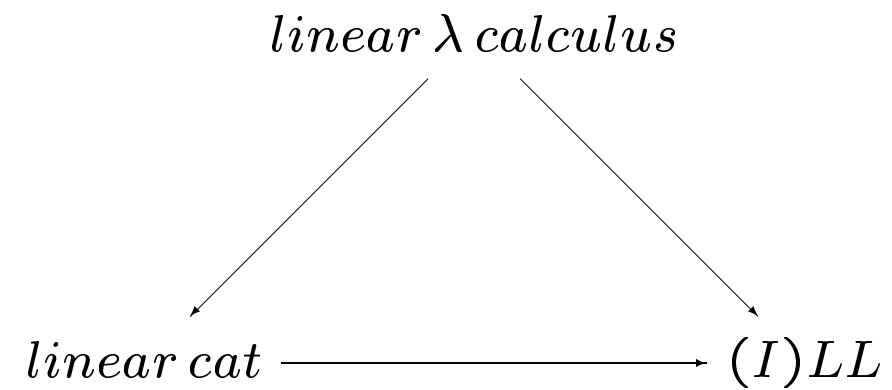
lots of important stuff elided

Categorical Semantics: some gains

- Functional Programming: optimizations that do not compromise semantic foundations (e.g. referential transparency)
- Logic: new applications of old theorems (e.g. normalization)
- Category Theory: new concepts not from maths
- Philosophy: new criteria for identities of proofs

Categorical Semantics of (Intuitionistic) Linear Logic

Want linear version of Extended Curry-Howard Isomorphism



- Logic intuitionistic
- Already have Intuitionistic Linear Logic (ILL),
— must find other two sides of triangle...
- Problem with linear lambda-calculus, and
Problem with linear category

Linear λ -calculus: which?

- Avron(88), Abramsky(90):
wrong calculus, no substitution in general
- Benton et al(93), Brauner(93):
ILL, verbose calculus, comm conversions, CHI ok
- Barber&Plotkin DILL(97), Benton's LNL(95):
less verbose, comm conv, CHI ok
- MILL, Plotkin, Wadler:
even less verbose, comm conv, best for implementation, CHI?
- others: Troelstra, Mints, Roversi, etc..

Linear categories

- multiplicative-additive fragment easy
 - multiplicative fragment: a symmetric monoidal closed category (SMCC)
 - additive fragment: categorical products and coproducts
- modalities (exponentials): the problem...
- several solutions, depending on linear λ -calculus chosen...

Cartesian Closed Categories

- A category \mathcal{C} is a cartesian closed category if
 - it has finite (categorical) products $A \times B$ for all objects A, B of \mathcal{C}
 - for each object B of \mathcal{C} , the product functor $_ \times B$ has a right-adjoint written, as $(B \Rightarrow _)$.

This gives an isomorphism: for any map $(A \times B) \rightarrow C$ in \mathcal{C} there is an isomorphic map $A \rightarrow (B \Rightarrow C)$ in \mathcal{C} .

- Products, \times , model conjunction and function spaces, \Rightarrow , model implication
- To make this definition resource-conscious, we need to make categorical product \times a tensor product \otimes .

Symmetric Monoidal Closed Categories (SMCC)

- A category \mathcal{C} is a symmetric monoidal closed category if
 - it has a tensor product $A \otimes B$ for all object A, B of \mathcal{C} .
 - for each object B of \mathcal{C} , the tensor-product functor $-\otimes B$ has a right-adjoint, written $B \multimap -$
 (ie have isomorphism from maps $(A \otimes B) \rightarrow C$ to $A \rightarrow (B \multimap C)$)

	Categorical product $A \times B$	Tensor product $A \otimes B$
1	Projections onto A and B	Not necessarily
2	For any object C with projections to A and B , there is a <i>unique</i> map from C to $A \times B$	Not necessarily

Modelling the Modality !

- Objects A in a CCC have duplication $A \rightarrow A \times A$ and erasing $A \rightarrow 1$ (satisfying equations of a comonoid).
- Objects A in a SMCC have no duplication $A \rightarrow A \otimes A$ nor erasing $A \rightarrow I$, in general. Neither have they projections $A \otimes B \rightarrow A$.
- But objects of the form $!A$ should have duplication and erasing. Moreover they must satisfy S4-Box introduction and elimination rules.
- S4-axioms give you proofs $!A \rightarrow A$ and $!A \rightarrow !!A$, uniformly for any A object A in \mathcal{C} . Hence want a functor (unary operator) on category, $!:\mathcal{C} \rightarrow \mathcal{C}$ such that there are such natural transformations. Such a structure already in category theory, a comonad.

Modelling the Modality !

- An object $!A$ must have four maps:

$$\epsilon: !A \rightarrow A, \quad \delta: !A \rightarrow !!A, \quad \text{er}: !A \rightarrow I, \quad \text{dupl}: !A \rightarrow !A \otimes !A$$

To make identity of proofs sensible, must have lots of equations.

- Equations for duplicating and erasing say $!A$ is a *comonoid*; equations for S4-axioms say $!$ is a *comonad*.
- How do comonad and comonoid structures interact?
- First idea (Seely'87): From logical equivalence $!(A \& B) \equiv !A \otimes !B$ Introduce a comonad $(!, \delta, \epsilon)$ taking the comonoid structure $(A, A \rightarrow A \times A, A \rightarrow 1)$ to $(!A, \text{dupl}: !A \rightarrow !A \otimes !A, \text{er}: !A \rightarrow I)$.

Categorical Semantics of ILL: problems

- Using the isomorphism $!(A \& B) \cong !A \otimes !B$ can't model multiplicatives & modalities *without* additives.
- Moreover, this semantics is not *sound*:
There are proofs π_1 and π_2 , equivalent under cut-elimination, which are not equal morphisms in the category.
see Bierman "What is a model of ILL?"

- Better idea: ask for a *monoidal* comonad

$$m_{A,B}: !(A \otimes B) \rightarrow !A \otimes !B \text{ and } m_I: I \rightarrow !I$$

satisfying conditions relating comonad to comonoid structure.
which conditions?

Categorical Semantics of ILL: first solution

A linear category comprises
— an SMCC \mathcal{C} , (with products and coproducts),
— a symmetric monoidal comonad $(!, \epsilon, \delta, m_I, m_{A,B})$

such that:

- (a) For every free co-algebra $(!A, \delta)$ there are nat. transfns.
 $er_A: !A \rightarrow I$ and $dupl_A: !A \rightarrow !A \otimes !A$, forming a commutative comonoid,
which are coalgebra maps.
- (b) Every map of free coalgebras is also a map of comonoids.

Bierman 1995, based on Benton et al'93

- long definition, lots of diagrams to check..
- Prove that $A \Rightarrow B \cong !A \multimap B$ by constructing CCC out of SMCC plus comonad $!$.
- Extended Curry-Howard works!! but complicated.

Categorical Semantics of ILL: second solution

- (Benton 1995) A model of LNL (Linear/NonLinear Logic) consists of a symm monoidal closed category \mathbf{L} , a cartesian closed category \mathbf{C} , and a *monoidal adjunction* between these categories.
- Modelling two logics, intuitionistic and linear at the same time. Logics on equal footing.
- First solution builds the cartesian closed category out of the monoidal comonad; this definition asks for the adjunction directly
- Easier to remember, same diagrams to check.
- Is it the *same* logic?

Categorical Semantics of ILL: third solution

- (Barber 1996) A model of DILL (Dual Intuitionistic and Linear Logic) consists of a symmetric monoidal closed category \mathbf{L} , a cartesian (not necessarily closed) category \mathbf{C} , and a *monoidal adjunction* between these categories. small simplification of semantics
- Extended Curry-Howard works, Barber's thesis.
Linear λ -calculus much nicer as PL
- DILL sound and complete for adjunctions above, but not *internal language*. Does it matter?
- Each model of DILL in bijection with models of ILL. But looking at categories of models have **DILL** not equivalent to **LIN**, only a retraction ???!!!

Kripke-style Semantics for LL

- Some work done
 - Hodas/Miller and Ambler (only fragment of ILL)
 - Allwein/Dunn
(three-valued, off-shelf: best use of Kripke models, counterexamples, lost? too many accessibility relations?)
 - B. Mitchell's
 - Ursini
- how to choose from? Use categorical semantics plus duality, I say.
- NOT DONE. want to lend a hand?

Conclusions

- discussed several alternatives for algebraic and categorical semantics of LL
- Is this plurality good or bad?
 - similar to IL, various ways to approach the essence

Some References

- Categorical Proof Theory and Linear Logic –preliminary notes for Summer School
- Lineales: algebras and categories in the semantics of Linear Logic
- G. Bierman, What's a Model of ILL? TLCA95.
- P. Scott, Category Theory for Linear Logicians
- A. Troelstra Lectures on Linear Logic
- Girard, Lafont and Regnier Advances in Linear Logic
- Relating Categorical Semantics for Intuitionistic Linear Logic (with M. Maietti, P. Maneggia and E. Ritter), available from webpage