An Institutional View on Categorical Logic and the Curry-Howard-Tait-Isomorphism

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Abstract. We introduce a generic notion of propositional categorical logic and provide a construction of an institution with proofs out of such a logic, following the Curry-Howard-Tait paradigm. We then prove logic-independent soundness and completeness theorems. The framework is instantiated with a number of examples: classical, intuitionistic, linear and modal propositional logics. Finally, we speculate how this framework may be extended beyond the propositional case.

1 Introduction

The well-known Curry-Howard-Tait isomorphism establishes a correspondence between

- propositions and types
- proofs and terms
- proof reductions and term reductions.

Moreover, in this context, a number of deep correspondences between categories and logical theories have been established (identifying 'types' with 'objects in a category'):

conjunctive logic	cartesian categories	[19]
positive logic	cartesian closed categories	[19]
intuitionistic propositional logic	bicartesian closed categories	[19]
classical propositional logic	bicartesian closed categories with $\neg\neg$-elimination	[19]
linear logic	$*$-autonomous categories	[33]
first-order logic	hyperdoctrines	[31]
Martin-Löf type theory	locally cartesian closed categories	[32]

Here, we present work aimed at casting these correspondences in a common framework based on the theory of *institutions*. The notion of institution arose within computer science as a response to the population explosion among logics in use, with the ambition of doing as much as possible at a level of abstraction independent of commitment to any particular logic [16, 18]. Its key idea is to focus on abstractly axiomatizing the *satisfaction* relation between sentences and models. A surprisingly large amount of meta-logical reasoning can be carried out in this abstract framework; e.g., institutions have been used to give general foundations for modularization of theories and programs [29, 10], and substantial portions of classical model theory can be lifted to the level of institutions [34, 7–9]. The notion of institution has been extended to cover also deduction [23] and proof theory [25].

Below, we give a general notion of propositional categorical logic, a construction of an institution with proofs out of such a logic, and logic-independent soundness and completeness theorems. We then discuss how the Curry-Howard-Tait isomorphism can be cast as a so-called *comorphism* of institutions. While all this is clearly limited to propositional logics, we conclude with some speculations on how to extend this program to other logics.

2 Institutions and Logics

We assume that the reader is familiar with basic notions from category theory; e.g., see [1, 20] for introductions to this subject. By way of notation, $|\mathbb{C}|$ denotes the class of objects of a category $\mathbb{C}$, and composition is denoted by "$\circ$". Let $\mathbb{CAT}$ be the the quasi-category of all categories (quasi-categories are categories that live in a higher set-theoretic universe [1]). The basic concept of this paper in its so-called set/cat variant is as follows.

Definition 1. An *institution* $\mathcal{I} = (\mathbb{S}ign^{\mathcal{I}}, \mathsf{Sen}^{\mathcal{I}}, \mathsf{Mod}^{\mathcal{I}}, \models^{\mathcal{I}})$ consists of

- a category $\mathbb{S}ign^{\mathcal{I}}$ of *signatures*;
- a functor $\mathsf{Sen}^{\mathcal{I}}\colon \mathbb{S}ign^{\mathcal{I}} \to \mathbb{S}et$ giving, for each signature Σ, the set of *sentences* $\mathsf{Sen}^{\mathcal{I}}(\Sigma)$, and for each signature morphism $\sigma\colon \Sigma \to \Sigma'$, the *sentence translation map* $\mathsf{Sen}^{\mathcal{I}}(\sigma)\colon \mathsf{Sen}^{\mathcal{I}}(\Sigma) \to \mathsf{Sen}^{\mathcal{I}}(\Sigma')$, where $\mathsf{Sen}^{\mathcal{I}}(\sigma)(\varphi)$ is written $\sigma\varphi$;
- a functor $\mathsf{Mod}^{\mathcal{I}}\colon (\mathbb{S}ign^{\mathcal{I}})^{op} \to \mathbb{CAT}$ giving, for each signature Σ, the category of *models* $\mathsf{Mod}^{\mathcal{I}}(\Sigma)$, and for each signature morphism $\sigma\colon \Sigma \to \Sigma'$, the *reduct functor* $\mathsf{Mod}^{\mathcal{I}}(\sigma)\colon \mathsf{Mod}^{\mathcal{I}}(\Sigma') \to \mathsf{Mod}^{\mathcal{I}}(\Sigma)$, where $\mathsf{Mod}^{\mathcal{I}}(\sigma)(M')$, the *$\sigma$-reduct* of M', is written $M'|_{\sigma}$; and
- a satisfaction relation $\models^{\mathcal{I}}_{\Sigma} \subseteq |\mathsf{Mod}^{\mathcal{I}}(\Sigma)| \times \mathsf{Sen}^{\mathcal{I}}(\Sigma)$ for each $\Sigma \in \mathbb{S}ign^{\mathcal{I}}$,

such that for each $\sigma\colon \Sigma \to \Sigma'$ in $\mathbb{S}ign^{\mathcal{I}}$, the *satisfaction condition*

$$M' \models^{\mathcal{I}}_{\Sigma'} \sigma\varphi \leftrightarrow M'|_{\sigma} \models^{\mathcal{I}}_{\Sigma} \varphi$$

holds for all $M' \in \mathsf{Mod}^{\mathcal{I}}(\Sigma')$ and all $\varphi \in \mathsf{Sen}^{\mathcal{I}}(\Sigma)$.

Proof-theoretic institutions add proof-theoretic structure to the sentences by turning the sentence sets into proof categories, with proofs as morphisms between sentences. Moreover, reductions between proof terms are modeled by a preorder on these morphisms; that is, proof categories are preorder-enriched categories, where $f \leq g$ for proofs f, g expresses the fact that g reduces to f. (Recall that a preorder-enriched category is a category where hom-sets are preorders, such that composition preserves the preorder. We let $\mathbb{O}rdCat$ denote the category of preorder-enriched small categories.) Explicitly:

Definition 2. A *preordcat/cat institution* is an institution $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Mod}, \models)$ equipped with a proof system $\mathsf{Pr}\colon \mathbb{S}ign \to \mathbb{O}rdCat$ such that $\mathsf{Sen} = U \circ \mathsf{Pr}$, where $U\colon \mathbb{O}rdCat \to \mathbb{S}et$ is the obvious forgetful functor.

In an institution, we can easily define the usual notion of *semantic consequence*: Given a Σ-sentence ψ and a set Φ of Σ-sentences, we say that ψ is a consequence of Φ, and write $\Phi \models_\Sigma \psi$, iff $M \models_\Sigma \Phi$ implies $M \models_\Sigma \psi$ for each Σ-model M (where by definition $M \models_\Sigma \Phi$ iff $M \models_\Sigma \varphi$ for each $\phi \in \Phi$). Instead of $\{\varphi\} \models \psi$, we briefly write $\varphi \models \psi$. In a preordcat/cat institution, we can moreover define an *entailment* relation $\vdash_\Sigma$ between Σ-sentences as follows: $\varphi \vdash_\Sigma \psi$ iff there exists a morphism $\varphi \to \psi$ in $\mathsf{Pr}(\Sigma)$. A preordcat/cat institution is *sound* if $\varphi \vdash_\Sigma \psi$ implies $\varphi \models_\Sigma \psi$; it is *weakly complete* if the converse implication holds. In the sequel, all institutions will be assumed to be sound.

Remark 3. Usually, also the entailment relation is defined for *sets* of sentences as premises. In the presence of conjunction, we extend the entailment relation to sets by putting $\Phi \vdash \psi$ if there exist $\varphi_1, \dots, \varphi_n \in \Phi$ such that

$$\varphi_1 \wedge \cdots \wedge \varphi_n \vdash \psi$$

This defines what has been called an institution with proofs in [25], and should be called a powerord/cat institution in the present terminology. We speak of *strong completeness* if $\Phi \models \psi$ implies $\Phi \vdash \psi$.

Example 4 (Classical propositional logic). The institution **CPL** of classical propositional logic is defined as follows. Signatures are just sets Σ of propositional variables, i.e. $\mathbb{S}ign = \mathbb{S}et$. Models of Σ are truth valuations, represented as subsets of Σ; the model category $\mathsf{Mod}(\Sigma)$ is formed by turning the subset ordering into a category. (This is consistent with predicates holding minimally in free models; cf. the Herbrand models of Prolog.) Model reduction is just preimage formation. Sentences in $\mathsf{Sen}(\Sigma)$ are propositional formulae over Σ, featuring the usual propositional connectives $\wedge$, $\vee$, $\neg$, $\Rightarrow$, $\Leftrightarrow$, with the evident notion of sentence translation. Finally, $M \models_\Sigma \varphi$ iff φ evaluates to true under the truth valuation M.

CPL can be turned into a preordcat/cat institution in various ways. For example, we can use a Gentzen-style proof system. A proof morphism $p\colon \varphi \to \psi$ is a Gentzen-style proof of ψ from φ. Given two proof morphisms $p, q\colon \varphi \to \psi$, $p \leq q$ if p can be obtained from q by cut elimination.

Example 5 (Partial Horn logic). The institution $PHorn^{=}$ of partial algebras with conditional existence equations is defined as follows. A signature $\Sigma = (S, F, P)$ consists of a set S of sorts, set F of partial function symbols (each with a string of argument sorts and a result sort), and a set P of predicate symbols (each with a string of argument sorts). Signature morphisms map these items in a compatible way. Models are many-sorted partial first-order structures. Sentences are conditional formulas (implicitly: in context) $\varphi_1 \wedge \ldots \wedge \varphi_n \rightarrow \varphi_0$, where the φ_i are either existence equations $t \stackrel{e}{=} t'$ or predicate applications $p(t_1, \ldots, t_n)$. Definedness of terms, written *def* t, just abbreviates $t \stackrel{e}{=} t$. Satisfaction is defined using total valuations of variables, while the valuation of terms is partial due to presence of partial operators. An existential equation holds if both sides are defined and equal. A predicate application holds if all involved terms are defined, and the resulting tuple is in the predicate. This is extended to conditional formulas as usual. (A restricted version of this institution has been introduced under the names *essentially algebraic theories* [14] and *HEP-varieties* [28], and has been used in the meta-theory of categories.)

Again, there are several proof systems for this institution; e.g. the system in [5] can easily be formalized in such a way that $PHorn^=$ becomes a preord-cat/cat institution.

Moreover, intuitionistic and modal logics can be formalized as institutions. So can substructural logics like linear logic, by taking judgements of the form $\varphi_1 \ldots \varphi_n \vdash \psi$ as sentences. Also, higher-order [4], polymorphic [30], temporal [13], process [13], behavioural [2], coalgebraic [6], and object-oriented [17] logics have been shown to be institutions.

3 Partial Conditional Rewriting Logic

Conditional rewriting logic has been introduced in [24, 22] as a model of concurrency that distinguishes *equations* from *rewriting rules* and hence naturally leads to notions of rewriting up to associativity, commutativity etc. Here, we will use conditional rewriting logic as a meta-theory for the theory of categories. Because composition is a partial operation, we need a variant of rewriting logic with partial functions (and predicates, which come at no extra cost in the presence of partial functions), i.e. a combination of rewriting logic with $PHorn^=$ as introduced above.

A (partial conditional) *rewrite theory* $\mathcal{R} = (S, F, P, E, R)$ consists of a $PHorn^=$-signature $\Sigma = (S, F, P)$, a set E of conditional formulas over Σ, and a set R of conditional *rewrite rules*, i.e. conditional formulae

$$t_1 \rightsquigarrow u_1 \wedge \cdots \wedge t_n \rightsquigarrow u_n \rightarrow t \rightsquigarrow u$$

over $\Sigma^{\rightsquigarrow}$, where $\Sigma^{\rightsquigarrow}$ is Σ enriched with a binary relation $\rightsquigarrow$ for each sort.

A *model* of a rewrite theory consists of a partial first-order structure for (S, F, P) satisfying the conditional equations in E, equipped with a preorder on the carriers such that operations are monotone and the interpretations of the rewrite rules are order-decreasing.

Extensions of rewrite theories are defined in the obvious (component wise) way. Given an extension $i\colon \mathcal{R}_1 \rightarrow \mathcal{R}_2$, model restriction gives a reduct functor

$\_|_i : \mathsf{Mod}(\mathcal{R}_2) \to \mathsf{Mod}(\mathcal{R}_1)$. From results of [28, 24] and the fact that the preorder implicitly present in rewrite theories can be axiomatized explicitly, we have

Proposition 6. *Extensions of rewrite theories admit free extensions, i.e.* $\_|_i$ *has a left adjoint for each extension* $i : \mathcal{R}_1 \to \mathcal{R}_2$.

Rewriting logic can also be formalized as an institution; we do not make this explicit here, because we only use rewriting logic as a meta-logic. Note that our use of rewriting logic differs from that in [22], where rewrite steps are used to model proof steps, whereas we use them to model proof *reduction* steps.

4 Categorical Logic

Lambek and Scott [19] introduce *deductive systems* as directed graphs with a composition structure, and only later impose the usual axioms for (cartesian, cartesian closed, bi-cartesian closed) categories on these. Note that objects in categories serve as 'types' for morphisms, hence the 'propositions as types' paradigm becomes 'propositions as objects'. In order to formalize this notion of deductive systems, consider the two-sorted specification of small categories as a partial equational theory in $PHorn^{=}$, with sorts Ob and Mor. Call the resulting rewrite theory with no rewrites Cat. In the notation of the language CASL [26], it looks as follows:

spec CAT =
  sorts Ob, Mor
  ops $dom, cod : Mor \to Ob$;
    $id : Ob \to Mor$;
  $\_\_\circ\_\_ : Mor \times Mor \to ? \ Mor$
preds $\_\_::\_\_\to\_\_ : Mor \times Ob \times Ob$;
$\forall\, f, g, h: Mor;\ a, b: Ob$
• $f :: a \to b \Leftrightarrow dom(f) \overset{e}{=} a \wedge cod(f) \overset{e}{=} b$
• $id(a) :: a \to a$ %(id)%
• $def\ f \circ g \Leftrightarrow cod(g) \overset{e}{=} dom(f)$ %(compatibility of comp)%
• $def\ f \circ g \Rightarrow (f \circ g) :: dom(g) \to cod(f)$ %(comp)%
• $def\ f \circ (g \circ h) \Rightarrow f \circ (g \circ h) \overset{e}{=} (f \circ g) \circ h$ %(associativity of comp)%
• $f \circ id(dom(f)) \overset{e}{=} f$ %(unit-right)%
• $id(cod(f)) \circ f \overset{e}{=} f$ %(unit-left)%
end

Via the Curry-Howard interpretation, elements of sort Ob are interpreted as logical formulae, while elements of sort Mor are interpreted as proofs between formulae.

Definition 7. A *propositional categorical logic* $\mathcal{L}$ is an extension of Cat that does not add sorts (i.e. it may add new operations and (oriented) conditional equations) and whose additional axioms mention only equations and rewrites between morphisms (not between objects). The category of categorical logics, denoted by $\mathbb{C}atLog$, has such theories $\mathcal{L}$ as objects and theory extensions as morphisms.

(Forbidding the mention of object equality other than in the basic axiomatization of composition is both technically useful and in tune with the underlying philosophy of category theory, which maintains that equality of objects, as opposed to their isomorphism, is a spurious concept.)

Example 8 (Conjunctive logic). Take $\mathcal{L} = \mathit{CartesianCat}$ to be the theory of cartesian categories, i.e. enrich *Cat* with a constant $\top$ for a terminal object and a binary product operation. On the logic side, this corresponds to adding "true" and conjunction. An extension of *CartesianCat* will be called a cartesian categorical logic.

Note that we have some freedom in specifying the universal properties. For example, concerning the pairing of arrows (as part of the specification of the products), we can either specify the equation $\pi_1 \circ \langle f, g\rangle \stackrel{e}{=} f$, which identifies both proof terms and hence regards their difference as merely bureaucratic and caused by the needs of syntactic representation, or we can specify the rewrite rule $\pi_1 \circ \langle f, g\rangle \rightsquigarrow f$, which keeps $\pi_1 \circ \langle f, g\rangle$ and f distinct. In the sequel, we always assume that universal properties are specified using rewrite rules.

Example 9 (Intuitionistic logic). Let $\mathcal{L} = \mathit{IProp}$ be the theory of bicartesian closed categories [19]. Products are interpreted as conjunctions and coproducts as disjunctions, exponentials are implications, and the initial object corresponds to false. Hence, we arrive at propositional intuitionistic logic.

Example 10 (Classical logic). Extend *IProp* from Example 9 to *Prop* by adding an operation $\mathit{classical}\colon \mathit{Ob} \to \mathit{Mor}$ that is axiomatized to deliver, for each object a, a morphism

$$\mathit{classical}(a)\colon (a \Longrightarrow \bot) \Longrightarrow \bot \to a,$$

where $\Longrightarrow$ is exponential and $\bot$ is the initial object. Of course, this amounts to the classical axiom of excluded middle.

Example 11 (Modal logic). Extend $IProp$ from Example 9 by adding operations $\Box, \Diamond\colon \mathit{Ob} \to \mathit{Ob}$ modeling the necessity and possibility operators from constructive modal S4, as defined by [3]. The necessity modality corresponds to a monoidal comonad, while the possibility modality corresponds to a monad, which is *strong relative to* the necessity comonad.

Example 12 (Linear logic). For a different kind of categorical logic, which does not build up from $IProp$, but simply from Cat, consider intuitionistic linear logic, and a posteriori, classical linear logic. Take ILL to be the extension of Cat with operations $\otimes$, $\multimap$, ! modeling, respectively, multiplicative conjunction, linear implication and the exponential operator "of course!". This corresponds to an autonomous category with a 'linear' comonad, [21].

5 The Institutional Curry-Howard-Tait Construction

We now recast the Curry-Howard-Tait interpretation of categorical logic in an institutional setting. Given a categorical logic $\mathcal{L}$, let $C(\mathcal{L})$ be the category of $\mathcal{L}$-algebras. Note that since $\mathcal{L}$ is an extension of the theory *Cat* of small preorder-enriched categories, $C(\mathcal{L})$ is a category of certain small preorder-enriched categories. Moreover, let $C'(\mathcal{L})$ denote the category of $\mathcal{L}$-algebras with discrete preorder. We define a functor $C(\mathcal{L}) \to C'(\mathcal{L})$ by quotienting out the preorder; i.e. given a preorder-enriched category A, $\tilde{A}$ is the quotient of A by the equivalence generated by the preorder on hom-sets (which by monotonicity of the operations in $\mathcal{L}$ is already a congruence). Given a preordcat/cat institution I, let $\tilde{I}$ be the cat/cat institution obtained by replacing each $\mathsf{Pr}(\Sigma)$ with $\widetilde{\mathsf{Pr}(\Sigma)}$.

Given a categorical logic $\mathcal{L}$ and a set X, let $T_{\mathcal{L}}(X)$ be the free $\mathcal{L}$-algebra over X, which exists by Prop. 6. $T_{\mathcal{L}}(X)$ consists of equivalence classes of terms formed from the operations in $\mathcal{L}$ and the elements of X, regarded as variables of sort *Ob*. The elements of sort *Ob* and *Mor* in $T_{\mathcal{L}}(X)$ will be our sentences and proofs, respectively.

Definition 13. Given a categorical logic $\mathcal{L}$, define the preordcat/cat institution $I(\mathcal{L})$ as follows:

- $\mathbb{S}ign$ is the category of sets. A signature hence is just a set (of propositional symbols).
- Let $\mathsf{Pr}(\Sigma) = T_{\mathcal{L}}(\Sigma)$, where Σ is used as a set of variables of sort *Ob*.
- Given a signature morphism (that is, a function) $\sigma\colon \Sigma_1 \to \Sigma_2$ and a Σ_1-sentence φ, let $\mathsf{Pr}(\sigma)$ be the extension $T_{\mathcal{L}}(\Sigma_1) \to T_{\mathcal{L}}(\Sigma_2)$ of $\sigma\colon \Sigma_1 \to \Sigma_2$ to terms.
- A model of Σ consists of a category $A \in C'(\mathcal{L})$ and a valuation $m\colon \Sigma \to |A|$ of propositional variables in Σ to objects in A. A model morphism $(F,\mu)\colon (A,m) \to (A',m')$ consists of an $\mathcal{L}$-functor $F\colon A \to A'$ (i.e. a morphism in $\mathsf{Mod}(\mathcal{L})$) and a family of morphisms $\mu_a\colon F(m(a)) \to m'(a)$, indexed over $a \in \Sigma$.
- For a signature morphism $\sigma\colon \Sigma_1 \to \Sigma_2$, model reducts are given by composition: $\mathsf{Mod}(\sigma)(m\colon \Sigma_2 \to |A|) = m \circ \sigma$, analogously for model morphisms.
- Satisfaction is defined by $(A,m) \models_\Sigma \varphi$ iff $m(\varphi)$ is a weakly terminal object in A. An object W of a category A is called weakly terminal if there is a (not necessarily unique) morphism into W from every object of A. In a category with a terminal object $\top$, W is weakly terminal iff there is a morphism ("global element") $\top \to W$ (intuitively: iff W is inhabited).

The Curry-Howard-Tait correspondence then takes the following shape.

Propositions as types (categorically: objects) Sentences can be defined as a derived notion: $\mathsf{Sen}(\Sigma) = |\mathsf{Pr}(\Sigma)|$, i.e. sentences are $\mathcal{L}$-terms of sort *Ob* with variables in Σ. Similarly $\mathsf{Sen}(\sigma)$ is given by the action of $\mathsf{Pr}(\sigma)$ on objects.

Proofs as terms Proofs are the morphisms of $\mathsf{Pr}(\Sigma)$. That is, a Σ-proof between sentences φ and ψ is simply an equivalence class of terms $p \in T_{\mathcal{L}}(\Sigma)$ of sort *Mor* such that $\mathcal{L} \vdash p :: \varphi \to \psi$. Note that the $\vdash$ sign in the latter statement refers to derivability in rewriting logic, which serves as meta-logic here. Assuming that the only equations in $\mathcal{L}$ are those of *Cat*, this means that Σ-proofs are strings of composition-free terms of sort *Mor*.

Proof reduction as morphism ordering The reducibility of proofs is given by the rewriting relation in $T_{\mathcal{L}}(\Sigma)$ which is a preorder on morphisms.

Categorical models A Σ-model is determined by a model A of $\mathcal{L}$ and a homomorphism/functor from $\mathsf{Pr}(\Sigma)$ into A. If A is given such a functor is defined by the homomorphic extension of an assignment of objects of A to the elements in Σ.

Proposition 14 (Soundness). *For any categorical logic $\mathcal{L}$, $I(\mathcal{L})$ is a sound preordcat/cat institution.*

Proof. The satisfaction condition is seen as follows. Let $\sigma\colon \Sigma \to \Sigma'$ be a morphism, let φ be a Σ-sentence, and let (A', m') be a Σ'-model. Then $m'|_\sigma \models_\Sigma \varphi$ iff $m' \circ \sigma \models \varphi$ iff $(m' \circ \sigma)(\varphi) = m(\sigma(\phi))$ is weakly terminal in A' iff $m' \models_\Sigma \sigma(\varphi)$.

To prove soundness, assume $\varphi \vdash_\Sigma \psi$, i.e. there is a morphism $p\colon \varphi \to \psi$ in $\mathsf{Pr}(\Sigma)$. Then there is a morphism $m(p)\colon m(\varphi) \to m(\psi)$ in every Σ-model (A, m). Thus if $m(\varphi)$ is weakly terminal in A then so is $m(\psi)$. Therefore $\varphi \models \phi$. □

Given a cartesian categorical logic $\mathcal{L}$, we can extend it with an extra assumption of a proof x of a sentence φ: let $\mathcal{L}(x : \top \to \phi)$ denote the theory $\mathcal{L}$ extended with a new constant x and the axiom $x :: \top \to \varphi$.

Definition 15 (Deduction Theorem). If $\mathcal{L}$ is cartesian, then we say that the *deduction theorem* holds in $\mathcal{L}$ if for any $\mathcal{L}(x : \top \to \varphi)$-proof term $p(x) : \psi \to \chi$, there is an $\mathcal{L}$-proof term $\kappa x \,.\, p(x) : \varphi \wedge \psi \to \chi$.

Proposition 16. *The deduction theorem holds in CartesianCat, CCC, Prop, IProp and any extension of these, provided that newly introduced operations that return morphisms do not take morphisms as arguments.*

Proof. See Proposition I.2.1 of [19]. The addition of operations adhering to the above restriction does not destroy the induction proof given there. □

(The above proposition allows e.g. the introduction of falsity and disjunction, which require only operations that construct morphisms from objects [19].)

Theorem 17 (Completeness). If the deduction theorem holds in a cartesian categorical logic $\mathcal{L}$, then $I(\mathcal{L})$ is a complete institution.

Proof. Let $\varphi \models_\Sigma \psi$ in $I(\mathcal{L})$, and let $\eta\colon \Sigma \to F$ be the free $\mathcal{L}(x : \top \to \phi)$-algebra over Σ, regarded as an $\mathcal{L}$-algebra (here, η is the insertion of generators). Then $\eta \models_\Sigma \varphi$ and hence $\eta \models_\Sigma \psi$, i.e. there is $p(x) : \top \to \eta^{\#}(\psi)$. Since in a free algebra, a ground atomic sentence holds iff it is provable [28], $\mathcal{L}(x\colon \top \to \varphi) \vdash p(x)\colon \top \to \psi$. By the deduction theorem, $\mathcal{L} \vdash \kappa x \,.\, p(x)\colon \varphi \wedge \top \to \psi$, therefore $\mathcal{L} \vdash \kappa x \,.\, p(x) \circ \langle id, ! \rangle\colon \varphi \to \psi$. Hence $\varphi \vdash \psi$ in $I(\mathcal{L})$. □

In cases where one does not wish to distinguish between different proofs of the same formula, one can collapse the proof category into a preorder. Explicitly:

Definition 18. Let $\mathcal{L}$ be a categorical logic. Then the institution $I_{thin}(\mathcal{L})$ is defined like $I(\mathcal{L})$, except for restricting the notion of model to valuations in *thin* categories $A \in C'(\mathcal{L})$.

In the context of the Curry-Howard correspondence, the cut rule corresponds to the composition of morphisms. Then cut elimination means that the composition operation can be eliminated from proof terms.

We say a categorical logic $\mathcal{L}$ has the property of cut admissibility if for all sets Σ and all objects a, b in $T_{\mathcal{L}}(\Sigma)$ there is a proof term $p : a \to b$ iff there is a proof term $p' : a \to b$ such that p' does not contain the operation $\circ$. We say that $\mathcal{L}$ has the property of cut elimination if in addition $p \rightsquigarrow p'$ holds in $T_{\mathcal{L}}(\Sigma)$.

Both cut admissibility and cut elimination need to be established independently for every categorical logic: Minor changes in the specification can require to redo large portions of the proof. While some of the above examples have cut admissibility, additional rewrites are necessary to establish cut elimination. These additional rewrites correspond to the various cases and subcases of the induction step of a constructive cut elimination proof.

For the simple case of a cartesian logic a cut elimination proof can be found in [12]. An example for a rewrite that needs to be added to yield cut elimination is $\langle f, g\rangle \circ h \rightsquigarrow \langle f \circ h, g \circ h\rangle$; this rewrite replaces one top-level cut with two lower-level cuts.

Example 19. Consider the theory *IProp* of bicartesian closed categories (Example 9). The institution $I(\mathit{IProp})$ is described as follows. Signatures are sets of propositional variables. Sentences are the usual propositional formulae. A model consists of a bicartesian closed category together with an interpretation of the propositional variables as objects in this category. Evaluation of sentences is just term evaluation in the bicartesian closed category.

If $p\colon C \to A$ and $q\colon C \to B$ are proofs, they can be combined to $\langle p, q\rangle\colon C \to A \wedge B$, and $\pi_1 \circ \langle p, q\rangle$ is another proof of $C \to A$. The rewriting structure gives us $\pi_1 \circ \langle p, q\rangle \rightsquigarrow p$.

Usually, intuitionistic logic is interpreted in either Kripke structures or Heyting algebras, rather than in bicartesian closed categories. This traditional view is captured in the institution $I_{thin}(\mathit{IProp})$, where models are valuations into Heyting algebras. The connection between $I(\mathit{IProp})$ and $I_{thin}(\mathit{IProp})$ will be made more explicit in the next section.

Exactness of institutions is a property important for modular specifications and proofs [11]:

Definition 20. An institution is said to be *exact*, if Mod takes colimits in $\mathbb{S}ign$ to limits in $\mathbb{C}at$.

Since reduct is composition, $I(\mathcal{L})$ is "almost" exact:

Proposition 21. *For any categorical logic $\mathcal{L}$, the model functor of $I(\mathcal{L})$ takes colimits of non-empty diagrams to limits. The model category of the initial signature is weakly terminal only.*

Definition 22. An institution is said to be *(weakly) liberal*, if the reduct functor of each *theory morphism* $\sigma\colon (\Sigma_1, \Psi_1) \to (\Sigma_2, \Psi_2)$ has a (weak) left adjoint. Here, a theory morphism is a signature morphism mapping elements of Ψ_1 into semantic consequences of Ψ_2. A functor F is a weak left adjoint to U via unit $\eta\colon Id \to UF$, if any morphism $f\colon X \to U\,A$ factors (not necessarily uniquely) as $U\,g \circ \eta_X$.

Example 23. $I(IProp)$ and $I(Prop)$ are not liberal: the theory $(\{A, B\}, \{A \vee B\})$ (viewed as extension of the empty theory) has no free model.

However:

Proposition 24. *For any categorical logic $\mathcal{L}$, $I(\mathcal{L})$ is weakly liberal.*

Proof. Given a Σ_1-model $m\colon \Sigma_1 \to |A|$, the (Σ_2, Ψ_2)-*diagram* of m is obtained as follows: add all objects of A as propositional variables to Σ_2, arriving at the signature $\Sigma_2(A)$; m is easily extended to model of that signature. Add all $\Sigma_2(A)$-sentences holding in m to Ψ_2, obtaining a theory $\Psi_2(A)$. The free $\mathcal{L}(\Sigma_2(A) \cup \{x_\varphi\colon \top \to \varphi \mid \varphi \in \Psi_2(A)\})$-algebra canonically is a Σ_2-model, which is a weakly free extension of m. □

6 Some Institution Comorphisms Related to Curry-Howard-Tait

Relationships between institutions are captured mathematically by 'institution morphisms', of which there are several variants. Here, we use so-called institution comorphisms [18], which capture the coding of one logic in another.

Definition 25. Given institutions $\mathcal{I}$ and $\mathcal{J}$, an *institution comorphism* $\rho = (\Phi, \alpha, \beta)\colon \mathcal{I} \to \mathcal{J}$ consists of

- a functor $\Phi\colon \mathbb{S}ign^{\mathcal{I}} \to \mathbb{S}ign^{\mathcal{J}}$,
- a natural transformation $\alpha\colon \mathsf{Sen}^{\mathcal{I}} \to \mathsf{Sen}^{\mathcal{J}} \circ \Phi$,
- a natural transformation $\beta\colon \mathsf{Mod}^{\mathcal{J}} \circ \Phi^{op} \to \mathsf{Mod}^{\mathcal{I}}$

such that the following *satisfaction condition* holds for all $\Sigma \in |\mathbb{S}ign^{\mathcal{I}}|$, $M' \in |\mathsf{Mod}^{\mathcal{J}}(\Phi(\Sigma))|$ and $\varphi \in \mathsf{Sen}^{\mathcal{I}}(\Sigma)$:

$$M' \models^{\mathcal{J}}_{\Phi(\Sigma)} \alpha_\Sigma(\varphi) \text{ iff } \beta_\Sigma(M') \models^{\mathcal{I}}_{\Sigma} \varphi.$$

Together with the natural composition and identities, this defines a quasicategory $\mathbb{C}oIns$ of institutions and institution comorphisms.

A *preordcat/cat institution comorphism* between preordcat/cat institutions $\mathcal{I}$ and $\mathcal{J}$ consists of an institution comorphism $(\Phi, \alpha, \beta)\colon \mathcal{I} \to \mathcal{J}$ and a natural transformation $\gamma\colon \mathsf{Pr}^{\mathcal{I}} \to \mathsf{Pr}^{\mathcal{J}} \circ \Phi$ such that $U(\gamma_\Sigma) = \alpha_\Sigma$.

Definition 26. By $thin(C)$, we denote the partial order obtained from a category C by quotienting all its hom-sets to singletons.

Thanks to the restriction on axioms admissible in a categorical logic, we have

Proposition 27. *Let $\mathcal{L}$ be a categorical logic. If C is a model of $\mathcal{L}$, then $thin(C)$ is also a model of $\mathcal{L}$, and the canonical functor $C \to thin(C)$ is an $\mathcal{L}$-model morphism.*

Proposition 28. *There is an institution comorphism from $I(\mathcal{L})$ to $I_{thin}(\mathcal{L})$) and vice versa. In particular, semantic consequence in $I(\mathcal{L})$ and $I_{thin}(\mathcal{L})$ coincide.*

Proof. The comorphisms are identities on signatures, sentences and proofs. Models translation from $I_{thin}(\mathcal{L})$ to $I(\mathcal{L})$ is just inclusion. In the other direction, a model $m : \Sigma \to |A|$ is translated into $m : \Sigma \to |thin(A)|$, which is an $I_{thin}(\mathcal{L})$-model by Proposition 27. Since satisfaction is defined by the existence of certain morphisms, the satisfaction condition for these comorphisms is straightforward. □

We are now ready to formalize the Curry-Howard-Tait isomorphism as a comorphism between certain institutions. We cannot expect this to be constructed in an institution independent manner. Rather, for some given propositional categorical logic $\mathcal{L}$, we can try to relate $I(\mathcal{L})$ to some well-known institution. For example, consider the case of classical propositional logic:

There is an institution comorphism (disregarding the proof structure) from $I(Prop)$ to **CPL**:

- on signatures it is the identity;
- on sentences, it is an obvious bijection (note that although sentences in $I(Prop)$ are technically equivalence classes, the definition of categorical logic ensures that no equations are imposed on sentences);
- model translation (from **CPL**-models into $I(Prop)$-models) is just inclusion, regarding a valuation of propositional variables as a map into the boolean algebra (thus, *Prop*-category) $Bool = \{\bot, \top\}$.

In a Boolean algebra, regarded as a category, an object has a global element iff it is terminal (i.e. equal to the 1 of the boolean algebra). There is no mapping from valuations into boolean algebras to valuations into *Bool* that preserves and reflects truth (i.e. terminalhood) of formulas: let $\nu\colon \{p, q\} \to Bool^2$ map p to $(\top, \bot)$ and q to $(\bot, \top)$. Assume that $\nu'\colon \{p, q\} \to Bool$ makes true the same formulae as ν. Then $\nu'(p)$ and $\nu'(q)$ must both be $\bot$, because neither $\nu(p)$ nor $\nu(q)$ is terminal. However, $\nu(p \vee q)$ *is* terminal, while $\nu'(p \vee q)$ is not, a contradiction. Hence, the model translation of the above comorphism cannot be reversed.

A related observation is that **CPL** and $I(Prop)$ differ in their behaviour of disjunction: while in **CPL**, a model M satisfies a disjunction iff it satisfies either of the disjuncts (this is called "model-theoretic disjunction" in [25]), this is not the case for $I(Prop)$: consider the weakly free model over the theory $(\{A, B\}, \{A \vee B\})$, existing by Prop. 24.

However, from the point of view of semantic consequence, we can restrict ourselves to the boolean algebra *Bool*: semantic consequence in **CPL** and $I(Prop)$ coincide. We turn this observation into a general notion:

Definition 29. Let $\mathcal{L}$ be a categorical logic, and let $A \in C'(\mathcal{L})$ (cf. Section 5). We write $\Phi \models^A \phi$ if $(A, m) \models \Phi$ implies $(A, m) \models \psi$ for every model (A, m), and abbreviate $\{\varphi\} \models^A \psi$ by $\varphi \models^A \psi$. The category A is called *weakly complete* for $\mathcal{L}$ if $\varphi \models^A \psi$ implies $\varphi \models \psi$ for all sentences ϕ, ψ, and *strongly complete* if $\Phi \models^A \psi$ implies $\Phi \models \psi$ for each sentence ψ and each set Φ of sentences.

Example 30. The two-element boolean algebra *Bool* (qua classical bicartesian closed category) is strongly complete for *Prop*. For any dense-in-itself metric space X, the Heyting algebra $\mathfrak{O}(X)$ of open sets in X (qua bicartesian closed category) is weakly complete for *IProp* [27]. We do not, at present, know whether $\mathfrak{O}(X)$ is also strongly complete for *IProp*.

Proposition 31. *Let $\mathcal{L}$ be a categorical logic extending the theory of cartesian closed categories (i.e. minimal intuitionistic logic with $\top$, $\wedge$, $\Rightarrow$), and let $A \in C'(\mathcal{L})$ be thin and skeletal, with induced ordering $\leq$. Then A is strongly complete for $\mathcal{L}$ iff the following condition holds.*

($*$) *Let $B \in C'(\mathcal{L})$, and let $a, b \in |B|$. If $f(a) \leq f(b)$ for each $\mathcal{L}$-morphism $f : B \to A$, then $\hom_B(a, b) \neq \emptyset$.*

(If B is a thin category, i.e. a preorder, then condition ($*$) states that the source of all morphisms from B into A is jointly order-reflecting.)

Proof. Assume that ($*$) holds, let $\Phi \models^A \psi$, and let $(B, m) \models \Phi$. Then for every $f : B \to A$, $(A, f \circ m) \models \Phi$ and hence $(A, f \circ m) \models \psi$, i.e. $\top \leq f(m(\psi))$. By ($*$), it follows that $\hom_B(\top, m(\psi)) \neq \emptyset$, i.e. $(B, m) \models \psi$.

Conversely, let A be strongly complete, let $B \in C'(\mathcal{L})$, and let $a, b \in |B|$ such that $\hom_A(f(a), f(b)) \neq \emptyset$ for all $f : B \to A$. Let $\Sigma = |B|$, and let Φ be the theory (i.e. the set of valid formulae) of the Σ-model $(id_{|B|}, B)$. Since morphisms $f : B \to A$ are then just the Σ-valuations in A that validate Φ (because by the assumption on $\mathcal{L}$, formulae χ_1 and χ_2 denote isomorphic objects of B iff $(id, |B|) \models \chi_1 \Leftrightarrow \chi_2$), the premise says that $\Phi \models^A a \Rightarrow b$ (note that the assumption on $\mathcal{L}$ implies that an implication holds iff there exists a morphism between the corresponding objects). Thus, $\Phi \models a \Rightarrow b$, so that $a \Rightarrow b$ holds in (id, B); i.e. $\hom_B(a, b) \neq \emptyset$ as claimed. □

Finally, we consider relations between different categorical logics:

Proposition 32. *The construction I is functorial, i.e. it can be extended to a functor $I \colon \mathbb{C}at\mathbb{L}og \to \mathbb{C}o\mathbb{I}ns$.*

Proof. Given two categorical logics $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\mathcal{L}_2$ is an extension of $\mathcal{L}_1$, we can construct an institution comorphism from $I(\mathcal{L}_1)$ to $I(\mathcal{L}_2)$ as follows:

– The action on signatures is just the identity
– Since $\mathcal{L}_2$ is an extension of $\mathcal{L}_1$, each $\mathcal{L}_1$-term also is an $\mathcal{L}_2$-term. This gives us the sentence translation.
– Take an $\mathcal{L}_2$-model $m\colon \Sigma \to |A|$, where A is an $\mathcal{L}_2$-category. Again since $\mathcal{L}_2$ is an extension of $\mathcal{L}_1$, A also is a model of $\mathcal{L}_1$, hence $m\colon \Sigma \to |A|$ is a model in $\mathcal{L}_1$ as well.
– Since both sentence and model translations do not modify anything, the satisfaction condition follows trivially.
– Since $\mathcal{L}_1$-terms are $\mathcal{L}_2$-terms, proof morphisms can be translated by leaving them as they are.

□

7 Conclusions

We have presented a canonical way of obtaining proof-theoretic institutions for propositional categorical logics, following the spirit of the Curry-Howard-Tait isomorphism. The institutional structure sheds light on the usual collapsing problem (classical bicartesian closed categories are boolean algebras); the collapsing can be avoided by using preorder-enriched categories, as in [15]. We have proved generic deduction, soundness, and completeness theorems. Moreover, while trying to recover the Curry-Howard-Tait isomorphism as an explicit isomorphism between institution, only a limited correspondence could be set up. The main obstacle is here the difference between, e.g., Boolean algebra-valued and Bool-valued models. We have provided some general result about the relations between such models.

An interesting question is whether the institutions $I(\mathcal{L})$ have elementary diagrams in the sense of [8]. Our current notion of model is obviously too weak to ensure this; for ensuring elementary diagrams one would need "intensional models" over signatures containing proof variables, to be valuated with proofs — such models would not only determine which propositions are true, but also why. With such models, it is also easy to show liberality of $I(\mathcal{L})$. The study of other properties, such as Craig interpolation and Beth definability, is future work.

An extension to first-order logic is left as future work as well; this will require a different treatment of signatures. The natural categorical analogue of FOL are hyperdoctrines [31]. Since hyperdoctrines are certain indexed categories, it might be possible to extract the Signature category from the index category of a hyperdoctrine. This would be in accordance with the treatment of quantifiers as adjoints to sentence translation functions $\mathsf{Sen}(\sigma)$ in [25]; indeed, the cited work comes close to imposing a hyperdoctrine structure on the proof functor Pr.

Note that the correspondence between intuitionistic higher-order logic and toposes [19] is *not* based on a correspondence between propositions and objects in a category. However, toposes are in particular also hyperdoctrines; it is an open question where this view might lead to.

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