Intuitionistic Hybrid Logic

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Abstract

Hybrid logics are a principled generalization of both modal logics and description logics, a standard formalism for knowledge representation. In this paper we give the first constructive version of hybrid logic, thereby showing that it is possible to hybridize constructive modal logics. Alternative systems are discussed, but we fix on a reasonable and well-motivated version of intuitionistic hybrid logic and prove essential proof-theoretical results for a natural deduction formulation of it. Our natural deduction system is also extended with additional inference rules corresponding to conditions on the accessibility relations expressed by so-called geometric theories. Thus, we give natural deduction systems in a uniform way for a wide class of constructive hybrid logics. This shows that constructive hybrid logics are a viable enterprise and opens up the way for future applications.

Key words: Hybrid logic; modal logic; intuitionistic logic; natural deduction

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1 Introduction

Classical hybrid logic is obtained by adding to ordinary classical modal logic further expressive power in the form of a second sort of propositional symbols called nominals, and moreover, by adding so-called satisfaction operators. A nominal is assumed to be true at exactly one world, so a nominal can be considered the name of a world. Thus, in hybrid logic a name is a particular sort of propositional symbol whereas in first-order logic it is an argument to a predicate. If a is a nominal and A is an arbitrary formula, then a new formula $a : A$ called a satisfaction statement can be formed. The part $a :$ of $a : A$ is called a satisfaction operator. The satisfaction statement $a : A$ expresses that the formula A is true at one particular world, namely the world at which the nominal a is true.

The present paper concerns *constructive* hybrid logics, that is, hybrid logics where the classical logic basis has been replaced by a constructive logic basis. A question we can ask is of course *why* should one worry about constructive hybrid logics.

A first, philosophical answer might be that since we "believe" in constructive logics as well as in hybrid logics, we would like to combine them in one logical system.

A second, more mathematical answer may be simply that we should be able to define "constructive hybrid logics" since we presume that the main concerns of hybrid logic are orthogonal to as whether the logic basis is constructive or not. We note that this supposition of orthogonality is justified by a distinction between the way of reasoning and what the reasoning is about. If we do define basic constructive hybrid logics and prove for them the kinds of results that we usually prove for constructive logics (normalization, subformula property, cut-elimination), we learn more about extant hybrid logics and we provide more evidence that hybrid logics are important in their own right [3].

A third, pragmatic answer, or perhaps one geared to applications, is that if one needs to construct a logic of contexts with certain characteristics, perhaps a constructive basic hybrid logic might be the right foundation for this kind of application. The basic intuition here is that the satisfaction operators of hybrid logic might be just the syntactic tool required to construct logics that pay attention to contexts with certain desirable features. This hypothesis can only be checked, once we have defined and investigated one or more basic constructive hybrid logics, at least to some extent. Moreover, we reckon that

[3] This would also provide evidence that these proof-theoretical properties are worth investigating for any logical system. But justifying proof-theory is not the issue here.

a modal type theory based on hybrid logic could prove itself useful [4], as other constructive modal type theories and as other classical hybrid logics have already proven themselves.

Of course, even if the supposition above, of orthogonality between hybridness and constructivity is valid, we still do not know exactly *how* to define constructive hybrid logics. We do not have a recipe for defining constructive hybrid logics: Several open possibilities are discussed, but we fix on a reasonable and well-motivated natural deduction system and prove some essential proof-theoretical results for this system. Moreover, we show how to extend the system with additional inference rules corresponding to first-order conditions on the accessibility relations. The conditions we consider are expressed by so-called geometric theories. Different geometric theories give rise to different constructive hybrid logics, so natural deduction systems for new constructive hybrid logics can be obtained in a uniform way simply by adding inference rules as appropriate.

One of the hallmarks of contemporary modal logic is the view that modal logics are languages for talking about relational structures, that is, models in the sense of model-theory. Thus, modal logics are alternatives to (classical) first-order logics. This view is well justified, but we believe that proof-theory adds another important perspective to modal logics, emphasizing the notion of proof and the fundamental differences between various formal systems for representing proofs, see [14]. The perspective of this paper is proof-theoretic, so we shall focus on formal proof systems for constructive hybrid logic.

This paper is structured as follows. In the second section of the paper we make some preliminary considerations, in the third section we give a Kripke semantics, and in the fourth section we introduce a natural deduction system, and moreover, we prove a normalization theorem. The natural deduction system is an intuitionistic version of a natural deduction system for classical hybrid logic originally given in the paper [5]. In the fifth section we prove soundness and completeness with respect to Kripke semantics. In the final section we draw some conclusions and discuss future work. This paper is an extended version of [7]. While the previous version only discussed pure hybrid systems, the extended version here discusses additional inference rules corresponding to first-order conditions on the accessibility relations.

[4] Actually a constructive modal type theory, based on a constructive and hybrid logical version of S5, has independently of the present work been proposed by Jia and Walker[11] in 2004.

2 Preliminaries

Having decided to investigate *constructive* hybrid logics, we must describe, what their main components are. We start with a short description of constructive modal logic. This is followed by another short description of what constitutes classical hybrid logic. Finally we describe ways of putting these components together.

2.1 Constructive modal logic

'Constructive' is an umbrella term for logics that worry about deciding which disjunct is true and/or about providing witnesses for existential statements. Here we mean much more restrictively that we take our basis to be intuitionistic logic, henceforth IL for the propositional fragment and IFOL for intuitionistic first-order logic.

While the syntax (in axiomatic form) and the semantics (in usual Kripke style) of traditional, classical modal logics are relatively well-known, it is fair to say that even axioms for *constructive* modal logics, but especially natural deduction formalizations for these are subject to much debate.

We start by describing which basic axioms we require for a minimal K constructive modal logic. First we define our formulas. Given a basic set of propositional symbols ranged over by the metavariables p, q, r, ..., the well-formed formulas of our constructive modal logic are built by conjunction, disjunction, and implication, together with falsum and the necessity and possibility modalities. Thus, the formulas are defined by the grammar

$$S \ ::= \ p \mid S \wedge S \mid S \vee S \mid S \rightarrow S \mid \bot \mid \Box S \mid \Diamond S$$

where p is a propositional symbol. In what follows, the metavariables A, B, C, ... range over formulas. Negation, as usual in constructive logic, is defined by the convention that $\neg A$ is an abbreviation for $A \rightarrow \bot$. Also, the nullary conjunction $\top$ is an abbreviation for $\neg\bot$ and the bi-implication $A \leftrightarrow B$ is an abbreviation for $(A \rightarrow B) \wedge (B \rightarrow A)$.

Axioms for this system consist of any basic axiomatization of intuitionistic propositional logic (IL) together with Simpson's axioms for the modalitites. These are:

(**taut**) $\vdash A$ for all *intuitionistic* tautologies A

(K) $\vdash \Box(A \to B) \to (\Box A \to \Box B)$

($\Diamond_1$) $\vdash \Box(A \to B) \to (\Diamond A \to \Diamond B)$

($\Diamond_2$) $\vdash (\Diamond A \to \Box B) \to \Box(A \to B)$

(**dist$_0$**) $\vdash \neg\Diamond\bot$

(**dist$_1$**) $\vdash \Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$

together with the proof rules of Necessitation (*Nec*) and Modus Ponens (*MP*)

$$(Nec)\ \frac{\vdash A}{\vdash \Box A} \qquad\qquad (MP)\ \frac{\vdash A \to B \qquad \vdash A}{\vdash B}$$

We start by defining the possible-worlds semantics for the intuitionistic *modal* logic that we shall be using. We remark that this notion of a model for intuitionistic modal logic was originally introduced in a tense-logical version by Ewald in [8] and it has also been used by Simpson in [17], p. 88.

Definition 1 *A* model *for basic intuitionistic modal logic is a tuple*

$$\mathfrak{M} = (W, \leq, \{D_w\}_{w\in W}, \{R_w\}_{w\in W}, \{V_w\}_{w\in W})$$

where

(1) W is a non-empty set partially ordered by $\leq$;
(2) for each w, D_w is a non-empty set such that $w \leq v$ implies $D_w \subseteq D_v$;
(3) for each w, R_w is a binary relation on D_w such that $w \leq v$ implies $R_w \subseteq R_v$; and
(4) for each w, V_w is a function that to each propositional symbol p assigns a subset of D_w such that $w \leq v$ implies $V_w(p) \subseteq V_v(p)$.

Given a model $\mathfrak{M}$ as defined above, the relation $\mathfrak{M}, w, d \models A$ is defined by induction, where w is an element of W, d is an element of D_w, and A is a formula.

$$\mathfrak{M}, w, d \models p \text{ iff } d \in V_w(p)$$

$$\mathfrak{M}, w, d \models A \wedge B \text{ iff } \mathfrak{M}, w, d \models A \text{ and } \mathfrak{M}, w, d \models B$$

$$\mathfrak{M}, w, d \models A \vee B \text{ iff } \mathfrak{M}, w, d \models A \text{ or } \mathfrak{M}, w, d \models B$$

$$\mathfrak{M}, w, d \models A \rightarrow B \text{ iff for all } v \geq w, \mathfrak{M}, v, d \models A \text{ implies } \mathfrak{M}, v, d \models B$$

$$\mathfrak{M}, w, d \models \bot \text{ iff falsum}$$

$$\mathfrak{M}, w, d \models \Box A \text{ iff for all } v \geq w, \text{ for all } e \in D_v, dR_v e \text{ implies } \mathfrak{M}, v, e \models A$$

$$\mathfrak{M}, w, d \models \Diamond A \text{ iff for some } e \in D_w, dR_w e \text{ and } \mathfrak{M}, w, e \models A$$

This notion of model is very much a notion of a model of intuitionistic *quantificational* logic used simply for propositional modal logic. The introduction of D's might look an excessive complication.

Simpson's natural deduction system consists of labelled versions of the usual rules for the propositional connectives plus the following rules for the modalities:

$$\frac{y : A \quad xRy}{x : \Diamond A}\ (\Diamond I) \qquad \frac{x : \Diamond A \qquad \begin{matrix} [y : A][xRy] \\ \vdots \\ z : B \end{matrix}}{z : B}\ (\Diamond E)$$

$$\frac{\begin{matrix} [xRy] \\ \vdots \\ y : A \end{matrix}}{x : \Box A}\ (\Box I) \qquad \frac{x : \Box A \quad xRy}{y : A}\ (\Box E)$$

The rules $(\Diamond E)$ and $(\Box I)$ are equipped with appropriate side-conditions in connection with the label y, that is, y does not occur in $x : \Diamond A$, in $z : B$, or in any undischarged assumptions other than the specified occurrences of $y : A$ and xRy. Moreover, y does not occur in $x : \Box A$ or in any undischarged assumptions other than the specified occurrences of xRy. Simpson's natural deduction system is different from, say natural deduction in Prawitz's book, in two aspects: First it has all of its formulas labelled; they have the form $x : A$ where x is a label, that is, a variable, and A is a formula. Intuitively, the variable x denotes a world in a modal model and the metalinguistic expression $x : A$ is to be read as 'formula A holds at world x.' Secondly, Simpson's system has relational premises on the form xRy, and similarly, relational assumptions. The metalinguistic expression xRy is to be read as 'world x sees world y.'

There are several versions of natural deduction systems for (mostly classical) modal logics of this form in the literature [1,12,16,9]. Their basic idea is always to give introduction and elimination rules for necessitiy ($\Box$) and possibility ($\Diamond$) which capture their possible worlds interpretation [5]. Recall that Simpson's particular version of natural deduction presentation satisfies normalization, as well as being "extendable" to a large collection of other logics given by so-called geometric theories. Hence, this system is proof-theoretically well-behaved, at least as far as the criteria of normalization and subformula property are concerned. A philosophical objection to this kind of system is that it builds-in the (desired) semantics into the given syntax, but the trade-off is that this way one obtains a uniform framework for a large class of modal logics.

2.2 Hybrid logic

Hybrid logics are extensions of modal logics where to the usual stock of syntactic and semantic structures, we add a new kind of propositional symbols, the so-called *nominals*, which reference individual possible worlds, that is, a nominal is true at exactly one world. It is assumed that a set of ordinary propositional symbols and a countably infinite set of nominals are given. The sets are assumed to be disjoint. The metavariables p, q, r, ... range over ordinary propositional symbols and a, b, c, ... range over nominals. We also add a new kind of operators called the *satisfaction operators*. The formulas of hybrid modal logic are defined by the grammar

$$S \ ::= \ p \mid a \mid S \wedge S \mid S \vee S \mid S \rightarrow S \mid \bot \mid \Box S \mid \Diamond S \mid a : S$$

where p is an ordinary propositional symbol and a is a nominal. Formulas of the form $a : C$ are called *satisfaction statements*, cf. a similar notion in [3]. In this paper we will define a system of intuitionistic hybrid logic, using the same formulas of classical hybrid logic, which will be denoted by IHL.

There are many kinds of hybrid logics in the literature, one of the parameters being which extra operators one considers. Here we restrict our attention to only satisfaction operators, a system that in the classical case has been called HL(@) where @ is one notation for satisfaction operators.

[5] But the systems themselves and their models vary widely.

2.3 *Which Constructive Hybrid Logic?*

Now there are at least two different approaches that one can take when trying to put together constructivity, modality and hybridness of logic systems.

First, one might think that since hybrid logic is obtained from classical modal logic by adding nominals and satisfaction operators, one should obtain constructive hybrid logic by adding to constructive modal logic nominals and satisfaction operators. This is sensible, and will be our approach to defining IHL, but of course it depends on the constructive modal logic chosen. Our goal is to reason constructively within constructive modal logic and we want to be able to "jump" to other modal worlds using some version of constructive satisfaction operators and modalities.

Second, one might think of "hybridizing" intuitionistic logic to enhance the expressive power of intuitionistic logic, considered as a language for talking about intuitionistic Kripke structures. Choosing this option puts an excessive emphasis on Kripke semantics as a guiding principle.

It is not clear whether these two lines of research will lead to different logical systems or not. But one suspects that the results will be different. In particular, considering only constructive modal logics, one of the reasons for the multiplicity of systems is whether one believes that only the propositional basis of the logic should be constructive or whether the modalities themselves should have a constructive component of their own. This can be ascertained by asking whether adding, say the excluded middle axiom, gives you back the classical modal system you first thought of, or not. Choosing Simpson's axiom system we're choosing that the addition of the excluded middle gives you back the classical modal logic system you started from, in our case modal K. Similarly, if the natural deduction system $\mathbf{N}_{\mathsf{IHL}}$ for intuitionistic hybrid logic given later in this paper is extended with the natural deduction rule corresponding to the excluded middle (technically, we just modify the rule for $\perp$ as appropriate), then will we get back the classical hybrid logic of [5], that is, the modal operator $\Diamond$ becomes definable in terms of $\Box$ ($\Diamond$ becomes equivalent to $\neg\Box\neg$) and we also have that $\wedge$ and $\vee$ become definable in terms of $\rightarrow$ and $\perp$.

3 Kripke Semantics

In this section we give the possible-worlds semantics for intuitionistic hybrid logic, IHL. This semantics is an extension of a possible-worlds semantics for intuitionistic modal logic which was originally introduced in a tense-logical version in the Ewald paper [8].

The main intuition is that since we want to consider a constructive reading of hybrid logic where a distinction is made between the way of reasoning and what the reasoning is about, we need to separate the intuitionistic partial order from the interpretation of the nominals as well as the binary relation corresponding to the two modal operators.

Definition 2 *A* model *for intuitionistic hybrid logic is a tuple*

$$(W, \leq, \{D_w\}_{w \in W}, \{\sim_w\}_{w \in W}, \{R_w\}_{w \in W}, \{V_w\}_{w \in W})$$

where

(1) W is a non-empty set partially ordered by $\leq$;
(2) for each w, D_w is a non-empty set such that $w \leq v$ implies $D_w \subseteq D_v$;
(3) for each w, $\sim_w$ is an equivalence relation on D_w such that $w \leq v$ implies $\sim_w \subseteq \sim_v$;
(4) for each w, R_w is a binary relation on D_w such that $w \leq v$ implies $R_w \subseteq R_v$; and
(5) for each w, V_w is a function that to each ordinary propositional symbol p assigns a subset of D_w such that $w \leq v$ implies $V_w(p) \subseteq V_v(p)$.

It is assumed that if $d \sim_w d'$, $e \sim_w e'$, and dR_we, then $d'R_we'$, and similarly, if $d \sim_w d'$ and $d \in V_w(p)$, then $d' \in V_w(p)$.

Intuitively, the elements of the set W are "states of knowledge" and for any such state w, the set D_w is the set of possible worlds known in the state of knowledge w, the relation $\sim_w$ corresponds to the known identities between possible worlds, and the relation R_w is the known relationship between possible worlds. Note that the definition requires that the partial order $\leq$ on the states of knowledge, which we call the "epistemic" partial order, preserves knowledge, that is, if an advance to a greater state of knowledge is made, then what is known is preserved.

Given a model $\mathfrak{M}$ and an element w of W, a *w-assignment* is a function that to each nominal assigns an element of D_w. The relation $\mathfrak{M}, g, w, d \models A$ is defined by induction, where w is an element of W, g is a w-assignment, d is an element of D_w, and A is a formula.

$$\mathfrak{M}, g, w, d \models p \text{ iff } d \in V_w(p)$$

$$\mathfrak{M}, g, w, d \models a \text{ iff } d \sim_w g(a)$$

$$\mathfrak{M}, g, w, d \models A \wedge B \text{ iff } \mathfrak{M}, g, w, d \models A \text{ and } \mathfrak{M}, g, w, d \models B$$

$$\mathfrak{M}, g, w, d \models A \vee B \text{ iff } \mathfrak{M}, g, w, d \models A \text{ or } \mathfrak{M}, g, w, d \models B$$

$$\mathfrak{M}, g, w, d \models A \rightarrow B \text{ iff for all } v \geq w,\ \mathfrak{M}, g, v, d \models A \text{ implies } \mathfrak{M}, g, v, d \models B$$

$$\mathfrak{M}, g, w, d \models \bot \text{ iff falsum}$$

$$\mathfrak{M}, g, w, d \models \Box A \text{ iff for all } v \geq w, \text{ for all } e \in D_v,\ dR_v e \text{ implies } \mathfrak{M}, g, v, e \models A$$

$$\mathfrak{M}, g, w, d \models \Diamond A \text{ iff for some } e \in D_w,\ dR_w e \text{ and } \mathfrak{M}, g, w, e \models A$$

$$\mathfrak{M}, g, w, d \models a : A \text{ iff } \mathfrak{M}, g, w, g(a) \models A$$

By convention $\mathfrak{M}, g, w \models A$ means $\mathfrak{M}, g, w, d \models A$ for every element d of D_w and $\mathfrak{M} \models A$ means $\mathfrak{M}, g, w \models A$ for every element w of W and every w-assignment g. A formula A is *valid* if and only if $\mathfrak{M} \models A$ for every model $\mathfrak{M}$. Note the difference in the interpretations of the two modal operators: The interpretation of the $\Box$ operator involves quantification over states of knowledge whereas the interpretation of $\Diamond$ does not. This is because the modal operators correspond to quantifiers in intuitionistic first-order logic where the interpretation of the $\forall$ uses the accessibility relation whereas the interpretation of $\exists$ does not.

An example of a formula valid in classical hybrid logic but not valid in the constructive semantics given here is $a : b \vee a : \neg b$, where $a : b$ is interpreted as the possible worlds $g(a)$ and $g(b)$ being related by $\sim_w$ but $a : \neg b$ is interpreted as $g(a)$ and $g(b)$ not being related by $\sim_v$ for any $v \geq w$. This formula corresponds to the formula $a = b \vee \neg a = b$ in the first-order correspondence language we introduce below. Since in intuitionistic first-order logic we do not have a general excluded middle, a constructivist thinks that this formula should only be valid, if the equality predicate for nominals is a *decidable* predicate. (Incidentally, this formula *would* be valid if for any w, the relation $\sim_w$ is taken to be the identity on the set D_w. Thus, the relation $\sim_w$ is needed.)

Another example is the formula $a : A \leftrightarrow \neg a : \neg A$. This formula should not be valid constructively, but it should be valid classically and it is actually taken as an axiom for classical hybrid logic by some authors.

The semantics satisfies the following important proposition.

Proposition 3 *(Monotonicity) If $\mathfrak{M}, g, w, d \models A$ and $w \leq v$, then $\mathfrak{M}, g, v, d \models A$.*

PROOF. Induction in the structure of A. $\square$

A model for intuitionistic hybrid logic can be considered a model for intuitionistic first-order logic with equality and conversely, a model for intuitionistic first-order logic with equality can be considered a model for intuitonistic hybrid logic (see [17] for the simpler correspondence between modal-logical models and intuitionistic first-order models without equality and see also [18] which gives a definition of intuitionistic first-order models with equality). The first-order language under consideration here has a 1-place predicate symbol corresponding to each ordinary propositional symbol of modal logic, a 2-place predicate symbol corresponding to the modalities, and a 2-place predicate symbol corresponding to equality. The language does not have constant or function symbols. It is assumed that a countably infinite set of first-order variables is given. The metavariables a, b, c, ...range over first-order variables. So the formulas of the first-order language we consider are defined by the grammar

$$S \ ::= \ p(a) \mid R(a,b) \mid a = b \mid S \wedge S \mid S \vee S \mid S \rightarrow S \mid \bot \mid \forall a S \mid \exists a S$$

where p is an ordinary propositional symbol of hybrid logic, and a and b are first-order variables. The connectives $\neg$, $\top$, and $\leftrightarrow$ are defined as in intuitionistic hybrid logic. Moreover, if nominals of hybrid logic are identified with first-order variables, then a w-assignment in the sense of intuitionistic hybrid logic can be considered as a w-assignment in the sense of intuitionistic first-order logic and vice versa.

Given a model $\mathfrak{M} = (W, \leq, \{D_w\}_{w \in W}, \{\sim_w\}_{w \in W}, \{R_w\}_{w \in W}, \{V_w\}_{w \in W})$ for intuitionistic hybrid logic, considered as a model for intuitionistic first-order logic, the relation $\mathfrak{M}, w \models A[g]$ is defined by induction, where w is an element of W, g is a w-assignment, and A is a first-order formula.

$$\begin{aligned}
\mathfrak{M}, w \models p(a)[g] &\text{ iff } g(a) \in V_w(p) \\
\mathfrak{M}, w \models a = b[g] &\text{ iff } g(a) \sim_w g(b) \\
\mathfrak{M}, w \models R(a,b)[g] &\text{ iff } g(a) R_w g(b) \\
\mathfrak{M}, w \models A \wedge B[g] &\text{ iff } \mathfrak{M}, w \models A[g] \text{ and } \mathfrak{M}, w \models B[g] \\
\mathfrak{M}, w \models A \vee B[g] &\text{ iff } \mathfrak{M}, w \models A[g] \text{ or } \mathfrak{M}, w \models B[g] \\
\mathfrak{M}, w \models A \rightarrow B[g] &\text{ iff for all } v \geq w,\ \mathfrak{M}, v \models A[g] \text{ implies } \mathfrak{M}, v \models B[g] \\
\mathfrak{M}, w \models \bot[g] &\text{ iff falsum} \\
\mathfrak{M}, w \models \forall a A[g] &\text{ iff for all } v \geq w, \text{ for all } g' \overset{a}{\sim} g,\ g'(a) \in D_v \text{ implies } \mathfrak{M}, v \models A[g'] \\
\mathfrak{M}, w \models \exists a A[g] &\text{ iff for some } g' \overset{a}{\sim} g,\ g'(a) \in D_w \text{ and } \mathfrak{M}, w \models A[g']
\end{aligned}$$

By convention $\mathfrak{M} \models A$ means $\mathfrak{M}, w \models A[g]$ for every element w of W and every w-assignment g. In Section 5 we shall make use of the first-order semantics above in connection with the interpretation of geometric theories.

4 The Natural Deduction System

In this section a natural deduction system for intuitionistic hybrid logic is given and it is shown how to extend the system with additional rules corresponding to conditions on the accessibility relations. Moreover, a normalization theorem is proved. The natural deduction system is an intuitionistic version of a natural deduction system for classical hybrid logic originally given in [5]. An axiomatic formulation for intuitionistic hybrid logic can be found in the paper [4]. See the books [13] and [19] for the basics of natural deduction. See also the book [10] for an introduction to natural deduction with a slant towards intuitionistic logic.

We make use of the following conventions. The metavariables π, τ, ... range over derivations. The metavariables Γ, Δ, ... range over sets of formulas. A derivation π is a *derivation of* A if the end-formula of π is an occurrence of A and π is a *derivation from* Γ if each undischarged assumption in π is an occurrence of a formula in Γ (note that numbers annotating undischarged assumptions are ignored). If there exists a derivation of A from $\emptyset$, then we simply say that A is *derivable*. Moreover, $B[c/a]$ is the formula B where the nominal c has been substituted for all occurrences of the nominal a and $\pi[c/a]$ is the derivation π where each formula occurrence B has been replaced by $B[c/a]$. The *degree* of a formula is the number of occurrences of non-nullary connectives in it.

Natural deduction inference rules for intuitionistic hybrid logic are given in Figure 1 and Figure 2. All formulas in the rules are satisfaction statements. Note that the modal rules for our system are like Simpson's modal rules, except that we replaced first-order relational formulas like $R(a, c)$ by the hybrid formula $a : \Diamond c$, thereby internalizing the accessibility relation. Note that this internalization process can be accomplished for hybrid logic, but for modal logic, these relational formulas belong the metalevel. The nominal natural deduction rules are similar to the usual equality rules for first order logic. The system thus obtained will be denoted $\mathbf{N}_{\mathsf{IHL}}$.

$$\frac{a : A \qquad a : B}{a : (A \wedge B)}\,(\wedge I) \qquad \frac{a : (A \wedge B)}{a : A}\,(\wedge E1) \quad \frac{a : (A \wedge B)}{a : B}\,(\wedge E2)$$

$$\frac{a : A}{a : (A \vee B)}\,(\vee I1) \quad \frac{a : B}{a : (A \vee B)}\,(\vee I2) \qquad \frac{a : (A \vee B) \qquad \begin{array}{c}[a : A]\\ \vdots \\ C\end{array} \qquad \begin{array}{c}[a : B]\\ \vdots \\ C\end{array}}{C}\,(\vee E)$$

$$\frac{\begin{array}{c}[a : A]\\ \vdots \\ a : B\end{array}}{a : (A \to B)}\,(\to I) \qquad \frac{a : (A \to B) \qquad a : A}{a : B}\,(\to E)$$

$$\frac{a : \bot}{C}\,(\bot E)$$

$$\frac{a : A}{c : a : A}\,(: I) \qquad \frac{c : a : A}{a : A}\,(: E)$$

$$\frac{e : A \qquad a : \Diamond e}{a : \Diamond A}\,(\Diamond I) \qquad \frac{a : \Diamond A \qquad \begin{array}{c}[c : A] \;\; [a : \Diamond c]\\ \vdots \\ C\end{array}}{C}\,(\Diamond E)^{*}$$

$$\frac{\begin{array}{c}[a : \Diamond c]\\ \vdots \\ c : A\end{array}}{a : \Box A}\,(\Box I)^{\star} \qquad \frac{a : \Box A \qquad a : \Diamond e}{e : A}\,(\Box E)$$

$*$ c does not occur in $a : \Diamond A$, in C, or in any undischarged assumptions other than the specified occurrences of $c : A$ and $a : \Diamond c$.
$\star$ c does not occur in $a : \Box A$ or in any undischarged assumptions other than the specified occurrences of $a : \Diamond c$.

Fig. 1. Natural deduction rules for connectives

$$\frac{}{a : a}\ (\mathit{Ref}) \qquad \frac{a : c \quad a : A}{c : A}\ (\mathit{Nom}1)^* \qquad \frac{a : c \quad a : \Diamond b}{c : \Diamond b}\ (\mathit{Nom}2)$$

$*$ A is a propositional symbol (ordinary or a nominal).

Fig. 2. Natural deduction rules for nominals

$$\dfrac{s_1 \;\ldots\; s_n \qquad \begin{matrix}[s_{11}] \ldots [s_{1n_1}] \\ \vdots \\ C\end{matrix} \qquad \ldots \qquad \begin{matrix}[s_{m1}] \ldots [s_{mn_m}] \\ \vdots \\ C\end{matrix}}{C}\ (R_G)^*$$

$*$ None of the nominals in $\overline{c}$ occur in C or in any of the undischarged assumptions other than the specified occurrences of s_{jk}. (Recall that nominals are identified with first-order variables and that $\overline{c}$ are the first-order variables existentially quantified over in the formula G.)

Fig. 3. Natural deduction rules for geometric theories

4.1 Extensions to Geometric Theories

In what follows we shall consider natural deduction systems obtained by extending $\mathbf{N}_{\mathsf{IHL}}$ with additional inference rules corresponding to first-order conditions on the accessibility relations. The conditions we consider are expressed by so-called geometric theories. A first-order formula is *geometric* if it is built out of atomic formulas of the form $R(a, c)$ and $a = c$ using only the connectives $\bot$, $\wedge$, $\vee$, and $\exists$. In what follows, the metavariables S_k and S_{jk} range over atomic formulas of the mentioned forms. Atomic formulas of the mentioned forms can be translated into first-order hybrid logic in a truth preserving way as follows.

$$HT(R(a, c)) = a : \Diamond c$$
$$HT(a = c) = a : c$$

See [20] for an introduction to geometric logic.

Now, a *geometric theory* is a finite set of closed first-order formulas each having the form $\forall \overline{a}(A \to B)$ where the formulas A and B are geometric, $\overline{a}$ is a list $a_1, \ldots, a_l$ of first-order variables, and $\forall \overline{a}$ is an abbreviation for $\forall a_1 \ldots \forall a_l$. It can be proved, cf. [17], that any geometric theory is intuitionistically equivalent to a *basic geometric theory* which is a geometric theory in which each formula has the form

$$(*) \qquad \forall \overline{a}((S_1 \wedge \ldots \wedge S_n) \rightarrow \exists \overline{c} \bigvee_{j=1}^{m} (S_{j1} \wedge \ldots \wedge S_{jn_j}))$$

where $n, m \geq 0$ and $n_1, \ldots, n_m \geq 1$. For simplicity, we assume that the variables in the list $\overline{a}$ are pairwise distinct, that the variables in $\overline{c}$ are pairwise distinct, and that no variable occurs in both $\overline{a}$ and $\overline{c}$. Note that a formula of the form displayed above is a Horn clause if $\overline{c}$ is empty, $m = 1$, and $n_m = 1$.

We now give hybrid natural deduction rules corresponding to a basic geometric theory. The metavariables s_k and s_{jk} range over hybrid-logical formulas of the forms $a : \Diamond c$ and $a : c$. With a first-order formula G of the form $(*)$ displayed above, we associate the natural deduction inference rule (R_G) given in Figure 3 where s_k is of the form $HT(S_k)$ and s_{jk} is of the form $HT(S_{jk})$. For example, if G is the formula

$$\forall a \forall c((R(a, c) \wedge R(c, a)) \rightarrow a = c)$$

then (R_G) is the natural deduction rule

$$\dfrac{a : \Diamond c \qquad c : \Diamond a \qquad \begin{array}{c}[a : c] \\ \vdots \\ C\end{array}}{C} \ (R_G)$$

The formula, and hence the inference rule, corresponds to the accessibility relation R being antisymmetric. Now, let $\mathbf{T}$ be any basic geometric theory. The natural deduction system obtained by extending $\mathbf{N}_{\mathsf{IHL}}$ with the set of rules $\{(R_G) \mid G \in \mathbf{T}\}$ will be denoted $\mathbf{N}_{\mathsf{IHL}} + \mathbf{T}$. We shall assume that we are working with a fixed basic geometric theory $\mathbf{T}$ unless otherwise specified.

It is straightforward to check that if a formula in a basic geometric theory is a Horn clause, then the rule (R_G) given in Figure 3 can be replaced by the following simpler rule (which we have also called (R_G)).

$$\dfrac{s_1 \quad \ldots \quad s_n}{s_{11}} \ (R_G)$$

For example, if G is the formula corresponding to the accessibility relation being antisymmetric, cf. above, then the following rule will do.

$$\dfrac{a : \Diamond c \qquad c : \Diamond a}{a : c} \ (R_G)$$

Natural deduction rules corresponding to Horn clauses were discussed already in [14].

4.2 An eliminable rule

Below we state a small proposition regarding an eliminable rule.

Proposition 4 *The rule*

$$\frac{a : c \quad a : A}{c : A}\,(Nom)$$

is eliminable.

PROOF. A straightforward extension of a proof in [5]. □

Note in the proposition above that A can be any formula; not just a propositional symbol. Thus, the rule (Nom) generalises $(Nom1)$ (and the rule $(Nom2)$ as well). The side-condition on the rule $(Nom1)$ enables us to prove a normalization theorem such that normal derivations satisfy a version of the subformula property called the quasi-subformula property. We shall return to this issue later.

4.3 Normalization

In what follows we give reduction rules for the natural deduction system and we prove a normalization theorem. First some conventions. If a premise of a rule has the form $a : c$ or $a : \Diamond c$, then it is called a *relational premise*, and similarly, if the conclusion of a rule has the form $a : c$ or $a : \Diamond c$, then it is called a *relational conclusion*. Moreover, if an assumption discharged by a rule has the form $a : \Diamond c$, then it is called a *relationally discharged assumption*. The premise of the form $a : A$ in the rule $(\rightarrow E)$ is called *minor* and the premises of the form C in the rules $(\vee E)$, $(\Diamond E)$, and (R_G) are called *parametric premises*. A premise of an elimination rule that is neither minor, relational, or parametric is called *major*.

A *maximum formula* in a derivation is a formula occurrence that is both the conclusion of an introduction rule and the major premise of an elimination rule. Maximum formulas can be removed by applying *proper reductions*. The rules for proper reductions are as follows. We have omitted the reduction rules involving the connectives $\wedge$, $\rightarrow$, $:$, and $\Box$ which can be found in [5].

$(\vee I1)$ followed by $(\vee E)$ (analogously in the case of $(\vee I2)$)

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ a : A\end{array}}{a : (A \vee B)} \qquad \begin{array}{c}[a : A]\\ \vdots\,\pi_2\\ C\end{array} \quad \begin{array}{c}[a : B]\\ \vdots\,\pi_3\\ C\end{array}}{C} \quad \rightsquigarrow \quad \begin{array}{c}\vdots\,\pi_1\\ a : A\\ \vdots\,\pi_2\\ C\end{array}$$

$(\Diamond I)$ followed by $(\Diamond E)$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ e : A\end{array} \qquad \begin{array}{c}\vdots\,\pi_2\\ a : \Diamond e\end{array}}{a : \Diamond A} \qquad \begin{array}{c}[c : A]\ [a : \Diamond c]\\ \vdots\,\pi_3\\ C\end{array}}{C} \quad \rightsquigarrow \quad \begin{array}{c}\begin{array}{cc}\vdots\,\pi_1 & \vdots\,\pi_2\\ e : A & a : \Diamond e\end{array}\\ \vdots\,\pi_3[e/c]\\ C\end{array}$$

It turns out that we need further reduction rules in connection with the inference rules $(\bot E)$, $(\vee E)$, $(\Diamond E)$, and (R_G). A *permutable formula* in a derivation is a formula occurrence that is both the conclusion of $(\bot E)$, $(\vee E)$, $(\Diamond E)$, or (R_G) and the major premise of an elimination rule. Permutable formulas in a derivation can be removed by applying *permutative reductions.* The rules for permutative reductions are as follows in the case where the elimination rule has two premises. We have omitted the reduction rule where (R_G) is followed by an elimination which can be found in [5].

$(\bot E)$ followed by a two-premise elimination

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ a : \bot\end{array}}{C} \qquad \begin{array}{c}\vdots\,\pi\\ E\end{array}}{D} \quad \rightsquigarrow \quad \dfrac{\begin{array}{c}\vdots\,\pi_1\\ a : \bot\end{array}}{D}$$

$(\vee E)$ followed by a two-premise elimination

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ a : (A \vee B)\end{array} \quad \begin{array}{c}[a : A]\\ \vdots\,\pi_2\\ C\end{array} \quad \begin{array}{c}[a : B]\\ \vdots\,\pi_3\\ C\end{array}}{C} \qquad \begin{array}{c}\vdots\,\pi\\ E\end{array}}{D} \quad \rightsquigarrow \quad \dfrac{\begin{array}{c}\vdots\,\pi_1\\ a : (A \vee B)\end{array} \quad \dfrac{\begin{array}{c}[a : A]\\ \vdots\,\pi_2\\ C\end{array} \quad \begin{array}{c}\vdots\,\pi\\ E\end{array}}{D} \quad \dfrac{\begin{array}{c}[a : B]\\ \vdots\,\pi_3\\ C\end{array} \quad \begin{array}{c}\vdots\,\pi\\ E\end{array}}{D}}{D}$$

($\Diamond E$) followed by a two-premise elimination

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \pi_1\\ a:\Diamond A\end{array}\qquad \begin{array}{c}[c:A]\ [a:\Diamond c]\\ \vdots\ \pi_2\\ C\end{array}}{C}\qquad \begin{array}{c}\vdots\ \pi\\ E\end{array}}{D} \quad\rightsquigarrow\quad \dfrac{\begin{array}{c}\vdots\ \pi_1\\ a:\Diamond A\end{array}\qquad \dfrac{\begin{array}{c}[b:A]\ [a:\Diamond b]\\ \vdots\ \pi_2[b/c]\\ C\end{array}\quad \begin{array}{c}\vdots\ \pi\\ E\end{array}}{D}}{D}$$

The cases where the elimination rule has one or three premises are obtained by deleting or adding derivations as appropriate.

A derivation is *normal* if it contains no maximum or permutable formula. In what follows we shall prove a normalization theorem which says that any derivation can be rewritten to a normal derivation by repeated applications of reductions. To this end we need a number of definitions and lemmas.

Definition 5 *The* $\Diamond$-graph *of a derivation* π *is the binary relation on the set of formula occurrences in* π *of the form* $a : \Diamond c$ *which is defined as follows. A pair of formula occurrences* (A, B) *is an element of the* $\Diamond$*-graph of* π *if and only if it satisfies one of the following conditions.*

(1) *A is the relational premise of an instance of* ($\Diamond I$) *which has B as the conclusion.*
(2) *A is the major premise of an instance of* ($\Diamond E$) *at which B is relationally discharged.*
(3) *A is a parametric premise of an instance of* ($\vee E$), ($\Diamond E$), *or* (R_G) *which has B as the conclusion.*

Note that the $\Diamond$-graph of π is a relation on the set of formula occurrences of π; not the set of formulas occurring in π. Also, note that every formula occurrence in a $\Diamond$-graph is of the form $a : \Diamond c$.

Lemma 6 *The* $\Diamond$*-graph of a derivation* π *does not contain cycles.*

PROOF. Induction on the structure of π. □

Definition 7 *The* potential *of a chain in the* $\Diamond$*-graph of* π *is the number of formula occurrences in the chain which are major premises of instances of* ($\Diamond E$)*. A* stubborn formula *in a derivation* π *is a maximum or permutable formula of the form* $a : \Diamond c$ *and the* stubbornness *of a stubborn formula in* π *is the maximal potential of a chain in the* $\Diamond$*-graph of* π *that contains the stubborn formula.*

Note that the notion of potential in the definition above is well-defined since Lemma 6 implies that the number of formula occurrences of a chain in a $\Diamond$-graph is bounded.

Lemma 8 *Let π be a derivation where all stubborn maximum formulas have stubbornness less than or equal to d and all stubborn permutable formulas have stubbornness less than d. Assume that A is a stubborn maximum formula with stubbornness d such that no formula occurrence above A is a stubborn maximum formula with stubbornness d. Let π' be the derivation obtained by applying the reduction such that A is removed. Then all stubborn maximum formulas in π' have stubbornness less than or equal to d and all stubborn permutable formulas in π' have stubbornness less than d, and moreover, the number of stubborn maximum formulas with stubbornness d in π' is less than the number of stubborn maximum formulas with stubbornness d in π.*

PROOF. The derivations π and π' have the forms below.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ e:d\end{array}\qquad\begin{array}{c}\vdots\,\pi_2\\ a:\Diamond e\end{array}}{a:\Diamond d}\qquad\begin{array}{c}[c:d]\ \ [a:\Diamond c]\\ \vdots\,\pi_3\\ C\end{array}}{\begin{array}{c}C\\ \vdots\,\tau\\ B\end{array}}
\qquad\qquad
\begin{array}{c}\begin{array}{cc}\vdots\,\pi_1 & \vdots\,\pi_2\\ e:d & a:\Diamond e\end{array}\\ \vdots\,\pi_3[e/c]\\ C\\ \vdots\,\tau\\ B\end{array}$$

Note that any formula occurrence in π' except the indicated occurrences of $e : d$, $a : \Diamond e$, and C in an obvious way can be mapped to a formula occurrence in π. Let f be the map thus defined (note that f need not be injective as the instance of $(\Diamond E)$ in π might discharge more than one occurrence of $a : \Diamond c$). Using the map f, a map from the $\Diamond$-graph of π' to the $\Diamond$-graph of π is defined as follows. There are a number of cases to consider. Case 1: An element (F, G) of the $\Diamond$-graph of π' where the formula occurrences F and G both are in the domain of f is mapped to $(f(F), f(G))$ which straightforwardly can be shown to be an element of the $\Diamond$-graph of π (observe that no assumption in π_1 or π_2 is discharged at a rule-instance in $\pi_3[e/c]$). Case 2: An element (F, G) where G is one of the indicated occurrences of $a : \Diamond e$ (and F therefore is in the domain of f) is mapped to $(f(F), G')$ where G' is the relational premise of the instance of $(\Diamond I)$. Case 3: An element (F, G) where F is the indicated occurrence of C (and G therefore is in the domain of f) is mapped to $(F', f(G))$ where F' is the conclusion of the instance of $(\Diamond E)$. Case 4: An element (F, G) where F is one of the indicated occurrences of $a : \Diamond e$, F is different from the indicated occurrence of C, and G is in the domain of f is mapped to $(F', f(G))$ where F' is the assumption in π_3 discharged by the instance of $(\Diamond E)$ corresponding to the occurrence of $a : \Diamond e$ in question. Case 5: An element (F, G) where G is the indicated occurrence of C, G is different from each of the indicated occurrences of $a : \Diamond e$, and F is in the domain of f is mapped to $(f(F), G')$ where G' is the parametric premise of the instance of $(\Diamond I)$. Case 6: An element (F, G) where F is one of the indicated occurrences of $a : \Diamond e$ and G is the indicated occurrence of C is mapped to (F', G') where F' is the assumption in π_3 discharged by the instance of $(\Diamond E)$ corresponding to the occurrence of $a : \Diamond e$ in question and G' is the parametric premise of the instance of $(\Diamond E)$. By using the map from the $\Diamond$-graph of π' to the $\Diamond$-graph of π, any chain in the $\Diamond$-graph of π' that does not contain any of the indicated occurrences of $a : \Diamond e$ can in an obvious way be mapped to a chain in the $\Diamond$-graph of π with the same potential which does not contain the indicated occurrences of $a : \Diamond e$, $a : \Diamond d$, and $a : \Diamond c$, and similarly, any chain in the $\Diamond$-graph of π' that contains one of the indicated occurrences of $a : \Diamond e$ can in an obvious way be mapped to a chain in the $\Diamond$-graph of π with greater potential which contain the mentioned formula occurrences. The conclusions of the lemma follow straightforwardly. □

Definition 9 *A* segment *in a derivation π is a non-empty list $A_1, \ldots, A_n$ of formula occurrences in π with the following properties.*

(1) A_1 is not the conclusion of an instance of $(\vee E)$, an instance of $(\Diamond E)$, or an instance of (R_G) with more than zero parametric premises.
(2) For each $i < n$, A_i is a parametric premise of an instance of $(\vee E)$, $(\Diamond E)$, or (R_G) which has A_{i+1} as the conclusion.
(3) A_n is not a parametric premise of an instance of $(\vee E)$, $(\Diamond E)$, or (R_G).

The length *of a segment is the number of formula occurrences in the segment. A segment σ_1* stands above *a segment σ_2 if and only if the last formula occurrence in σ_1 stands above the first formula occurrence in σ_2. A* maximum segment *(*permutable segment*) is a segment in which the last formula occurrence is a maximum formula (permutable formula). A* stubborn segment *is a maximum or permutable segment where the formula that occurs in the segment is of the form $a : \Diamond c$. The* degree *of a segment is the degree of the formula that occurs in the segment.*

The following lemma is along the lines of a similar result for ordinary intuitionistic first-order logic given in [13].

Lemma 10 *Any derivation π can be rewritten to a derivation π' that does not contain permutable formulas or non-stubborn maximum formulas, by repeated applications of permutative reductions applied to permutable formulas and proper reductions applied to non-stubborn maximum formulas.*

PROOF. To any derivation π we assign the pair (d, k) of non-negative integers where d is the maximal degree of a permutable or non-stubborn maximum segment in π or 0 if there is no such segment and k is the sum of the lengths of permutable and non-stubborn maximum segments in π of degree d (note that a list of formula occurrences with only one element is a segment if the one and only formula occurrence in the list is a maximum formula). The proof is by induction on such pairs equipped with the lexicographic order. Let π be a derivation to which a pair (d, k) is assigned such that $d > 0$. It is straightforward to check that there exists a permutable or non-stubborn maximum segment σ of degree d in π such that there is i) no permutable or non-stubborn maximum segment with degree d that stands above σ and ii) no permutable or non-stubborn maximum segment with degree d that stands above or contains a minor, relational, or parametric premise of the rule instance of which the last formula occurrence in σ is the major premise. Let π' be the derivation obtained by applying the appropriate reduction rule such that the last formula occurrence in σ is removed. Then it is straightforward to check that the pair (d', k') assigned to π' is less than (d, k) in the lexicographic order $\square$

We are now ready to prove the normalization theorem.

Theorem 11 *(normalization) Any derivation can be rewritten to a normal derivation by repeated applications of proper and permutative reductions.*

PROOF. By Lemma 10 we just need to consider derivations that do not contain permutable formulas or non-stubborn maximum formulas. To any such derivation π we assign the non-negative integer d where d is the maximal stubbornness of a stubborn maximum formula in π or 0 if there is no stubborn maximum formula. Let π be a derivation to which an integer d is assigned such that $d > 0$. It is straightforward that there exists a stubborn maximum formula A with stubbornness d such that no formula occurrence above A is a stubborn maximum formula with stubbornness d. Let π' be the derivation obtained by applying the reduction such that A is removed. Then by inspecting the involved reduction rule it is trivial to check that all maximum or permutable formulas in π' are stubborn, and moreover, by Lemma 8 all stubborn maximum formulas in π' have stubbornness less than or equal to d and all stubborn permutable formulas in π' have stubbornness less than d, and furthermore, the number of stubborn maximum formulas with stubbornness d in π' is less than the number of stubborn maximum formulas with stubbornness d in π. By repeated applications of this procedure a derivation is obtained in which all maximum or permutable formulas are stubborn with stubbornness less than d. By application of Lemma 10 a derivation π'' is obtained that does not contain permutable formulas or non-stubborn maximum formulas. If all maximum or permutable formulas in a derivation τ are stubborn with stubbornness less than d, then it is trivial to check by inspecting the involved reduction rules that all maximum or permutable formulas in the derivation τ' obtained by applying a permutative reduction are stubborn, and moreover, it can be proved in a way similar to the way in which Lemma 8 is proved, that all stubborn formulas in τ' have stubbornness less than d. Thus, all maximum formulas in π'' are stubborn with stubbornness less than d. We are therefore done by induction. □

4.4 The form of normal derivations

Below we adapt an important definition from [13] to intuitionistic hybrid logic.

Definition 12 *A* path *in a derivation π is a non-empty list $A_1, \ldots, A_n$ of formula occurrences in π with the following properties.*

(1) A_1 is a relational conclusion, or the conclusion of a (R_G) rule with zero parametric premises, or an assumption that is not non-relationally discharged by an instance of $(\vee E)$ or $(\Diamond E)$.
(2) For each $i < n$, A_i is not a minor or relational premise and either
  (a) A_i is not the major premise of an instance of $(\vee E)$ or $(\Diamond E)$ and A_i stands immediately above A_{i+1}, or
  (b) A_i is the major premise of an instance r of $(\vee E)$ or $(\Diamond E)$ and A_{i+1} is an assumption non-relationally discharged by r.
(3) A_n is either the end-formula of π, or a minor or relational premise, or the major premise of an instance of $(\vee E)$ or $(\Diamond E)$ that does not non-relationally discharge any assumptions.

Note that A_1 in the definition above might be a discharged assumption.

Lemma 13 *Any formula occurrence in a derivation π belongs to some path in π.*

PROOF. Induction on the structure of π. □

The definition of a path leads us to the lemma below. The lemma says that a path in a normal derivation can be split up into three parts: An analytical part in which formulas are broken down in their components by successive applications of the elimination rules, a minimum part in which an instance of the rule $(\bot E)$ may occur, and a synthetical part in which formulas are put together by by successive applications of the introduction rules. See [14].

Lemma 14 *Let $\beta = A_1, \ldots, A_n$ be a path in a normal derivation. Then there exists a formula occurrence A_i in β, called the* minimum formula *in β, such* *that*

(1) *for each $j < i$, A_j is a major or parametric premise or the non-relational premise of an instance of* $(Nom1)$*;*
(2) *if $i \neq n$, then A_i is a non-relational premise of an introduction rule or the premise of an instance of* $(\bot E)$*; and*
(3) *for each j, where $i < j < n$, A_j is a non-relational premise of an introduction rule, a parametric premise, or the non-relational premise of an instance of* $(Nom1)$*.*

PROOF. Let A_i be the first formula occurrence in β which is not the non-relational premise of an instance of $(Nom1)$, and is not a parametric premise, and is not the major premise of an elimination rule save possibly the major premise of an instance of $(\vee E)$ or $(\Diamond E)$ that does not non-relationally discharge any assumptions (such a formula occurrence exists in β as A_n satisfies the mentioned criteria). We are done if $i = n$. Otherwise A_i is a non-relational premise of an introduction rule or the premise of an instance of $(\bot E)$ (by inspection of the rules and the definition of a path). If A_i is the premise of an instance of $(\bot E)$, then each A_j, where $i < j < n$, is a non-relational premise of an introduction rule, or the non-relational premise of an instance of $(Nom1)$, or a parametric premise (by inspection of the rules, the definition of a branch, and normality of π). Similarly, if A_i is a non-relational premise of an introduction rule, then each A_j, where $i < j < n$, is a non-relational premise of an introduction rule or a parametric premise. □

In what follows we shall consider the form of normal derivations. To this end we give the following definition.

Definition 15 *The notion of a* subformula *is defined by the conventions that*

- *A is a subformula of A;*
- *if $B \wedge C$, $B \vee C$, or $B \rightarrow C$ is a subformula of A, then so are B and C; and*
- *if $a : B$, $\Diamond B$, or $\Box B$ is a subformula of A, then so is B.*

A formula $a : A$ is a quasi-subformula *of a formula $c : B$ if and only if A is a subformula of B.*

Now we state the theorem which says that normal derivations satisfy a version of the subformula property.

Theorem 16 *(Quasi-subformula property) Let π be a normal derivation of A from a set of satisfaction statements Γ. Any formula occurrence C in π is a quasi-subformula of A, or of some satisfaction statement in Γ, or of some* *relational premise, or of some relational conclusion, or of some relationally discharged assumption.*

PROOF. First a convention: The *order* of a path in π is the number of formula occurrences in π which stand below the last formula occurrence of the path. Now consider a path $\beta = A_1, \ldots, A_n$ in π of order p. By induction we can assume that the theorem holds for all formula occurrences in paths of order less than p. Note that it follows from Lemma 14 that any formula occurrence A_j such that $j \leq i$, where A_i minimum formula in β, is a quasi-subformula of A_1, and similarly, any A_j such that $j > i$ is a quasi-subformula of A_n.

We first consider A_n. We are done if A_n is the end-formula A or a relational premise. If A_n is the minor premise of an instance of $(\rightarrow E)$, then we are done by induction as the major premise belongs to a path of order less than p. If A_n is the major premise of an instance of $(\vee E)$ or $(\Diamond E)$ that does not non-relationally discharge any assumptions, then A_n is the minimum formula and hence a quasi-subformula of A_1. Now, we are done if A_1 is a relational conclusion, or an undischarged assumption, or a relationally discharged assumption. Otherwise A_1 is discharged by an instance of $(\rightarrow I)$ with a conclusion that belongs to some branch of order less than p (note that due to normality of π, A_1 is not the conclusion of a (R_G) rule with zero parametric premises).

We now consider A_1. We are done if A_1 is a relational conclusion, or an undischarged assumption, or a relationally discharged assumption. If A_1 is the conclusion of a (R_G) rule with zero parametric premises, then A_1 has the same form as the minimum formula which is a quasi-subformula of A_n. Otherwise A_1 is discharged by an instance of $(\rightarrow I)$ with a conclusion that belongs to β or to some path of order less than p. $\Box$

Note that it is a consequence of the theorem that C is a quasi-subformula of A, or of some formula in Γ, or of a formula of the form $a : c$ or $a : \Diamond c$ (since relational premises, relational conclusions, and relationally discharged assumptions are of the form $a : c$ or $a : \Diamond c$).

5 Soundness and completeness

Having given the Kripke semantics and the natural deduction system, we are now ready to prove soundness and completeness. Recall that we are working with a fixed basic geometric theory $\mathbf{T}$. A model $\mathfrak{M}$ is called a $\mathbf{T}$-*model* if and only if $\mathfrak{M} \models C$ for every formula C in $\mathbf{T}$ (note that the model $\mathfrak{M}$ in this definition is considered a first-order model as C is a first-order formula).

Theorem 17 *(Soundness) Let B be a satisfaction statement and let Γ be a set of satisfaction statements. The first claim below implies the second claim.*

(1) B is derivable from Γ in $\mathbf{N}_{\mathsf{IHL}} + \mathbf{T}$.
(2) For any $\mathbf{T}$-model $\mathfrak{M}$, any element w of W, and any w-assignment g, if, for any formula $C \in \Gamma$, $\mathfrak{M}, g, w \models C$, then $\mathfrak{M}, g, w \models B$.

PROOF. Induction on the structure of the derivation of B where we make use of Proposition 3. □

In what follows, we shall give a Henkin-type proof of completeness. In the interest of simplicity, we shall often omit reference to the basic geometric theory $\mathbf{T}$ and to the natural deduction system $\mathbf{N}_{\mathsf{IHL}} + \mathbf{T}$.

Definition 18 *A set of satisfaction statements Γ is* inconsistent *if and only if $a : \bot$ is derivable from Γ for some nominal a and Γ is* consistent *if and only if Γ is not inconsistent.*

Let $\mathbf{C}$ be any countably infinite set of nominals and let $\mathtt{L}(\mathbf{C})$ denote the set of hybrid-logical formulas built using the nominals in $\mathbf{C}$. Moreover, let $\mathbf{C}_0$ denote the set of nominals in the language defined earlier in this paper, thus, $\mathtt{L}(\mathbf{C}_0)$ denotes the language we have considered hitherto.

Definition 19 *Let $\mathbf{C}$ and $\mathbf{E}$ be disjoint countably infinite sets of nominals. A set of satisfaction statements Γ in the language $\mathtt{L}(\mathbf{C} \cup \mathbf{E})$ is $\mathbf{E}$-saturated if and only if*

(1) Γ is consistent;
(2) if A is derivable from Γ, then $A \in \Gamma$;
(3) if $a : (A \vee B) \in \Gamma$, then $a : A \in \Gamma$ or $a : B \in \Gamma$;
(4) if $a : \Diamond A \in \Gamma$, then for some nominal e in $\mathbf{E}$, $e : A \in \Gamma$ and $a : \Diamond e \in \Gamma$; and
(5) if $e : (s_1 \wedge \ldots \wedge s_n)[\overline{d}/\overline{a}] \in \Gamma$ for some formula $\forall \overline{a}((S_1 \wedge \ldots \wedge S_n) \rightarrow \exists \overline{c} \vee_{j=1}^{m} (S_{j1} \wedge \ldots \wedge S_{jn_j}))$ in $\mathbf{T}$ where $m \geq 1$, then for some list $\overline{b}$ of nominals in $\mathbf{E}$, $e : \vee_{j=1}^{m}(s_{j1} \wedge \ldots \wedge s_{jn_j})[\overline{d}, \overline{b}/\overline{a}, \overline{c}] \in \Gamma$.

We are now ready for a saturation lemma.

Lemma 20 *(Saturation lemma) Let $\mathbf{C}$ and $\mathbf{E}$ be disjoint countably infinite sets of nominals and let A_1, A_2, A_3, ... be an enumeration of all satisfaction statements in $\mathtt{L}(\mathbf{C} \cup \mathbf{E})$. Let Γ be a set of satisfaction statements in $\mathtt{L}(\mathbf{C})$ and let B be a satisfaction statement in $\mathtt{L}(\mathbf{C})$ such that B is not derivable from Γ. An $\mathbf{E}$-saturated set of satisfaction statements $\Gamma^* \supseteq \Gamma$ from which B is not derivable is defined as follows. Firstly, Γ^0 is defined to be Γ. Secondly, Γ^{n+1} is* *defined by induction. If B is derivable from $\Gamma^n \cup \{A_{n+1}\}$, then Γ^{n+1} is defined to be Γ^n. Otherwise Γ^{n+1} is defined to be*

(1) $\Gamma^n \cup \{A_{n+1}, a : C\}$ *if A_{n+1} is of the form $a : (C \vee E)$ and B is not derivable from $\Gamma^n \cup \{A_{n+1}, a : C\}$;*
(2) $\Gamma^n \cup \{A_{n+1}, a : E\}$ *if A_{n+1} is of the form $a : (C \vee E)$ and the first clause does not apply;*
(3) $\Gamma^n \cup \{A_{n+1}, e : C, a : \Diamond e\}$ *if A_{n+1} is of the form $a : \Diamond C$;*
(4) $\Gamma^n \cup \{A_{n+1}, e : \vee_{j=1}^{m}(s_{j1} \wedge \ldots \wedge s_{jn_j})[\overline{d}, \overline{b}/\overline{a}, \overline{c}]\}$ *if there exists a formula in* $\mathbf{T}$ *of the form $\forall\overline{a}((S_1 \wedge \ldots \wedge S_n) \to \exists\overline{c} \vee_{j=1}^{m} (S_{j1} \wedge \ldots \wedge S_{jn_j}))$ such that $m \geq 1$ and $A_{n+1} = e : (s_1 \wedge \ldots \wedge s_n)[\overline{d}/\overline{a}]$ for some nominal e and some list $\overline{d}$ of nominals; and*
(5) $\Gamma^n \cup \{A_{n+1}\}$ *if none of the first four clauses apply.*

In clause 3, e is a nominal in $\mathbf{E}$ that does not occur in Γ^n or A_{n+1}, and similarly, in clause 4, $\overline{b}$ is a list of nominals in $\mathbf{E}$ such that none of the nominals in b occur in Γ^n or A_{n+1}. Finally, Γ^* is defined to be $\cup_{n\geq 0}\Gamma^n$.

PROOF. Firstly, B is not derivable from Γ^0 by definition. Secondly, to check that the non-derivability of B from Γ^n implies the non-derivability of B from Γ^{n+1}, we need to check each of the clauses in the definition of Γ^{n+1}. The first and fifth clauses are trivial. For the second clause, the derivability of B from $\Gamma^n \cup \{A_{n+1}, a : E\}$ implies the derivability of B from $\Gamma^n \cup \{A_{n+1}, a : C\}$ since the first clause does not apply, therefore B is derivable from $\Gamma^n \cup \{A_{n+1}\}$ by the rule $(\vee E)$. For the third clause, the derivability of B from $\Gamma^n \cup \{A_{n+1}, e : C, a : \Diamond e\}$ implies the derivability of B from $\Gamma^n \cup \{A_{n+1}\}$ by the rule $(\Diamond E)$. The fourth clause is analogous to the third clause. We conclude that B is not derivable from Γ^*. It is straightforward to check that Γ^* is $\mathbf{E}$-saturated. $\Box$

The canonical model given below is similar to a canonical model for first-order intuitionistic logic given in [18].

Definition 21 *(Canonical model) Let $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$, ... be pairwise disjoint countably infinite sets of nominals disjoint from $\mathbf{C}_0$ and let $\mathbf{C}_n^* = \cup_{1\leq i\leq n}\mathbf{C}_i$ where $n \geq 1$. Let Γ be a consistent set of satisfaction statements in the language $\mathtt{L}(\mathbf{C}_0)$. A model*

$$\mathfrak{M}^\Gamma = (W^\Gamma, \subseteq, \{D_w^\Gamma\}_{w\in W^\Gamma}, \{\sim_w^\Gamma\}_{w\in W^\Gamma}, \{R_w^\Gamma\}_{w\in W^\Gamma}, \{V_w^\Gamma\}_{w\in W^\Gamma})$$

and for each $w \in W^\Gamma$, a w-assignment g_w^Γ, are defined as follows.

- $W^\Gamma = \{\Delta \supseteq \Gamma \mid$ *for some n, $\Delta \subseteq \mathtt{L}(\mathbf{C}_0 \cup \mathbf{C}_n^*)$ and Δ is $\mathbf{C}_n^*$-saturated*$\}$.
- $D_\Delta^\Gamma = \mathbf{C}_0 \cup \mathbf{C}_n^*$ *where Δ is $\mathbf{C}_n^*$-saturated.*

- $a \sim^{\Gamma}_{\Delta} c$ *if and only if* $a : c \in \Delta$.
- $aR^{\Gamma}_{\Delta}c$ *if and only if* $a : \Diamond c \in \Delta$.
- $V^{\Gamma}_{\Delta}(p) = \{a \mid a : p \in \Delta\}$.
- $g^{\Gamma}_{\Delta}(a) = a$ *where* $a \in D^{\Gamma}_{\Delta}$.

Note that it follows from Lemma 20 that W^{Γ} is non-empty. It is straightforward to check the other requirements $\mathfrak{M}^{\Gamma}$ has to satisfy to be a model for intuitionistic hybrid logic. Given the saturation lemma and the definition of a canonical model, we are ready to prove a truth lemma.

Lemma 22 *(Truth lemma) For any* $\Delta \in W^{\Gamma}$ *and any satisfaction statement* $a : A$ *in* $\mathtt{L}(D^{\Gamma}_{\Delta})$*,* $a : A \in \Delta$ *if and only if* $\mathfrak{M}^{\Gamma}, g^{\Gamma}_{\Delta}, \Delta, a \models A$.

PROOF. Induction on the degree of A. We only consider the case where A is of the form $\Box C$; the other cases are simpler.

Assume that $a : \Box C \in \Delta$. Let $\Lambda \supseteq \Delta$ and $aR^{\Gamma}_{\Lambda}e$, that is, $a : \Diamond e \in \Lambda$. Then $e : C \in \Lambda$ by the rule ($\Box E$) which by the induction hypothesis implies that $\mathfrak{M}^{\Gamma}, g^{\Gamma}_{\Lambda}, \Lambda, e \models C$. It follows that $\mathfrak{M}^{\Gamma}, g^{\Gamma}_{\Delta}, \Delta, a \models \Box C$.

Assume that $a : \Box C \notin \Delta$. Assume that Δ is $\mathbf{C}^*_n$-saturated and let $e \in \mathbf{C}^*_{n+1}$. Then $e : C$ is not derivable from $\Delta \cup \{a : \Diamond e\}$ for otherwise we could derive $a : \Box C$ from Δ by the rule ($\Box I$). According to Lemma 20, there exists a $\mathbf{C}^*_{n+2}$-saturated extension Λ of $\Delta \cup \{a : \Diamond e\}$ such that $e : C$ is not derivable from Λ. It follows by the induction hypothesis that $\mathfrak{M}^{\Gamma}, g^{\Gamma}_{\Lambda}, \Lambda, e \models C$ is not the case. This contradicts $\mathfrak{M}^{\Gamma}, g^{\Gamma}_{\Delta}, \Delta, a \models \Box C$ since $a : \Diamond e \in \Lambda$ implies that $aR^{\Gamma}_{\Lambda}e$. □

We only need one more lemma before we can prove completeness.

Lemma 23 *Let* Γ *be a consistent set of satisfaction statements. Then the canonical model* $\mathfrak{M}^{\Gamma}$ *is a* $\mathbf{T}$*-model.*

PROOF. Let $G \in \mathbf{T}$. Then G has the form $\forall \overline{a}((S_1 \wedge \ldots \wedge S_n) \rightarrow \exists \overline{c} \vee^{m}_{j=1} (S_{j1} \wedge \ldots \wedge S_{jn_j}))$ where $\overline{a} = a_1, \ldots, a_l$. Assume that Δ is an element of W^{Γ} and g is a Δ-assignment for $\mathfrak{M}^{\Gamma}$ such that $\mathfrak{M}^{\Gamma}, \Delta \models S_1[g], \ldots, \mathfrak{M}^{\Gamma}, \Delta \models S_n[g]$. (Note that $\mathfrak{M}^{\Gamma}$ is considered a model for intuitionistic first-order logic.) So $g(a_1) = d_1, \ldots, g(a_l) = d_l$ for some nominals $d_1, \ldots, d_l$ in D^{Γ}_{Δ}. Then $s_1[\overline{d}/\overline{a}], \ldots, s_n[\overline{d}/\overline{a}] \in \Delta$ by the definition of a canonical model. If $m \geq 1$, then it follows from Δ being saturated that there exists a list of nominals $\overline{b}$ such that $e : \vee^{m}_{j=1}(s_{j1} \wedge \ldots \wedge s_{jn_j})[\overline{d}, \overline{b}/\overline{a}, \overline{c}] \in \Delta$ where e is an arbitrary nominal. Therefore $e : (s_{j1} \wedge \ldots \wedge s_{jn_j})[\overline{d}, \overline{b}/\overline{a}, \overline{c}] \in \Delta$ and hence $s_{j1}[\overline{d}, \overline{b}/\overline{a}, \overline{c}], \ldots, s_{jn_j}[\overline{d}, \overline{b}/\overline{a}, \overline{c}] \in \Delta$ for some j where $1 \leq j \leq m$. But then it follows from the definition of a canonical model that $\mathfrak{M}^{\Gamma}, \Delta \models \exists \overline{c} \vee^{m}_{j=1} (S_{j1} \wedge \ldots \wedge S_{jn_j})[g]$. On the other hand, if $m = 0$, then $e : \bot \in \Delta$ by the rule (R_G) which contradicts the consistency of Δ. $\Box$

Now the completeness theorem.

Theorem 24 *(Completeness) Let B be a satisfaction statement and let Γ be a set of satisfaction statements. The second claim below implies the first claim.*

(1) B is derivable from Γ in $\mathbf{N}_{\mathsf{IHL}} + \mathbf{T}$.
(2) For any $\mathbf{T}$-model $\mathfrak{M}$, any element w of W, and any w-assignment g, if, for any formula $C \in \Gamma$, $\mathfrak{M}, g, w \models C$, then $\mathfrak{M}, g, w \models B$.

PROOF. Assume that B is not derivable from Γ. Consider the canonical model $\mathfrak{M}^\Gamma$ and let Λ be a $\mathbf{C}_1^*$ saturated extension of Γ from which B is not derivable, cf. Lemma 20. It follows from Lemma 22 that $\mathfrak{M}^\Gamma, g_\Lambda^\Gamma, \Lambda \models B$ is not the case but it also follows from Lemma 22 that for any $C \in \Gamma$, $\mathfrak{M}^\Gamma, g_\Lambda^\Gamma, \Lambda \models C$ is the case. But this contradicts the second statement in the theorem since $\mathfrak{M}^\Gamma$ is a $\mathbf{T}$-model by Lemma 23 $\Box$

6 Conclusions and Further Work

We have shown that constructivity and hybridness of logics are orthogonal concerns, at least to the extent that there is a logic, which we called IHL that is both hybrid and constructive. We also provided a natural deduction formulation $\mathbf{N}_{\mathsf{IHL}}$ for this logic. The system $\mathbf{N}_{\mathsf{IHL}}$ is based on Simpson's natural deduction formulation of constructive modal logic and we have shown that our hybrid version shares with it several good properties: The system is normalizing and satisfies a version of the subformula property. Hybridizing Simpson's system is not trivial, see below. Moreover, our system can be extended with inference rules corresponding to geometric first-order conditions on the accessibility relation, as can Simpson's. The possible-worlds semantics (based on Ewald's work) is sound and complete for $\mathbf{N}_{\mathsf{IHL}}$.

In the case of ordinary first-order logic, applying a reduction to a maximum formula only generates new maximum formulas having a lower degree than the original one. In fact, the technique used in the standard normalization proof for first-order logic is based on this property. The natural deduction system given in the present paper does not have this property since the reduction rules for $\Box$ and $\Diamond$ might generate new maximum formulas of the form $a : \Diamond e$, that is, maximum formulas that do not necessarily have a lower degree than the original one (here we ignore permutable formulas). Thus, the standard technique used in the normalization proof does not work directly for our case. In this paper the problem is solved by using the so-called $\Diamond$-graph of a derivation which keeps track of such maximum formulas. The notion of a $\Diamond$-graph is similar to a notion introduced in [5] to solve the same kind of problem for classical hybrid logic. We have not seen such a notion elsewhere.

We reiterate that, although the proofs are reasonably straightforward, the results are important as they confirm the presupposition in hybrid logic that it is the process of hybridization that is central. Our work confirms that the process is at least to some extent independent of the basic logic considered. Thus our system provides some evidence of the naturalness of the hybridization process.

Concerning further work we would like to see whether we can adapt the "non-situated" version of natural deduction for hybrid logics, based on Seligman's work [15,6], to a constructive formulation. We would also like to consider an alternative version of basic modal logic [2] as a basis for a constructive hybrid logic. Finally we need to investigate the suitability of (all?) the logics just mentioned to the problem of modelling contexts in knowledge representation formalisms devised to deal with natural language semantics.

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