Constructive Description Logics: what, why and how

Valeria de Paiva [1]

Abstract. This note discusses possible conceptions of *constructive* description logics. We discuss why different communities should be interested in constructive description logics, why so little work has been done on this before and why we think it is important. Our goal is a system of constructive *contexted* description logic, whose contexts are inspired by Guha-McCarthy's ideas along the lines of the system described in [BCC+05]. In this note we only lay down the fundamentals of constructive description logics. We describe three systems, produce semantics for them and prove some basic results.

1 Motivation

Why should we worry about describing systems of *constructive* description logics? What would these be? Presumably if we can describe the systems, then we can prove theorems, like soundness and completeness, for these constructive description logics. Is this true? Who should bother? Why?

In principle, description logicians (and there are many the world over) ought to be interested in the idea of *constructive* description logics, as for most other logics one can make the distinction between its classical and its constructive versions. That is, we can consider classical first-order logic (FOL) or intuitionistic first-order logic (IFOL), classical modal logic (ML) or intuitionistic modal logic (IML), classical linear logic (LL) or intuitionitistic linear logic (ILL), etc..

Thus being able to construct and prove results about the constructive or intuitionistic based version of your logic, is a kind of *casting out nines* test of that logic. If you cannot do it, there is probably something wrong with your system. If you can do it and it works, well, it does not guarantee that your logic is good or correct or appropriate, but at least, it passed one test.

Now proof theorists, especially the ones keen on the (extended) Curry-Howard isomorphism also have good reasons to try to come up with constructive descriptions logics. By all accounts description logics are a cottage industry in logic. And proof theorists would like to extend the influence of proof-theoretical methods, especially, the Curry-Howard isomorphism, to all important/sensible/useful (or merely very used) logical systems. But the Curry-Howard isomorphism is usually easier to obtain for *constructive* systems, hence proof-theorists would also be interested in constructive description logics.

Also the arguments used for explaining why constructive *hybrid* logics elsewhere [BdP03] can be reused here. A philosophical answer might be that if we "believe" in constructive logics as well as in description logics, we might want to combine them in one system. Also many logicians have the (justified, in my opinion) conviction that differences between constructive and non-constructive logic should only matter when infinities are at stake. Description logics are very much about 'decidable' predicates, whether elephants are mammals or not and such like. These kinds of concerns should lead to calculi that are basically constructive. A mathematical answer may be simply that we should be able to define "constructive description logics" since we presume that the main concerns of description logic are orthogonal to as whether the basic logic is constructive or not. Reasoning a posteriori, if we do define constructive description logics and prove for them the kinds of results that we usually prove for constructive logics (cut-elimination, normalization, subformula property, etc), we learn more about extant description logics and we provide more evidence that these logics are important on their own. Finally, a pragmatic answer is that if one needs to construct a logic for representing knowledge obtained from natural language texts, with certain characteristics, perhaps a *contextual* version of a constructive description logic may be the right foundation for this application.

Here the argument is not so direct. The basic intuition is that description logics cut down first order logic to some kind of tractable fragment, which is good. But for a logic representing meanings of natural language sentences, one essential ingredient, missing from description logics, is some kind of partiality. Now a constructive logic has some intrinsic flavour of partiality. Hence a constructive description logic, hopefully will have the same, coupled with the restricted expressivity associated with description logics. Thus this logic might be the ideal candidate system to be made *context-sensitive*. How to make a logical system context-sensitive is, of course, open to debate, but our hypothesis can only be checked, once we have defined and investigated a basic constructive description logic, at least a little. Hence this note.

Finally and most importantly, we reckon that a type theory based on a constructive description logic could prove itself useful, as other constructive modal type theories[BdP00] and as other (classical) hybrid and description logics have already proven themselves.

2 Description Logics

Description logics are the most important knowledge representation formalism in use nowadays, unifying and giving a logical basis to more traditional frame-based systems. This is how the Description Logics webpage describes description logics. But if description logics are much more logic-based than previous formalisms for knowledge representation, they are still somewhat "unconventional" as logics. Logics usually have a syntax and a semantics (description logics are no exception) and they usually can be placed in the hierarchy of other more traditional logics without any problems. From this perspective description logics are quite different. Before discussing the

[1] PARC, 3333 Coyote Hill Road, Palo Alto CA 94304, USA

problem of placing description logics into traditional hierarchies of logics, we recapitulate the basic syntax of description logics.

Different description logics have different description languages. One of the basic description languages in the handbook of description logics is $\mathcal{ALC}$ [BCM+03]. Concepts descriptions in $\mathcal{ALC}$ are formed according to the following syntax rule:

$$C, D \to A \mid \top \mid \bot \mid \neg C \mid C \sqcap D \mid C \sqcup D \mid \forall R.C \mid \exists R.C$$

where C, D range over concepts, R ranges over names of roles and A stands for the atomic concepts.

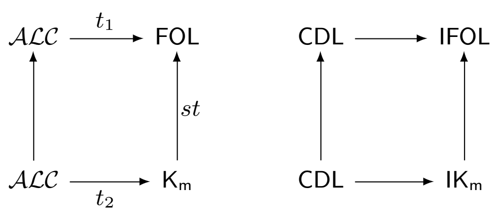
Recall that the semantics of classical $\mathcal{ALC}$ is given by interpretations $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ where $\Delta^{\mathcal{I}}$ is a non-empty set, every concept A is mapped by the interpretation function $\cdot^{\mathcal{I}}$ to a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ and every role name R is mapped to a binary relation $R^{\mathcal{I}}$ on $\Delta^{\mathcal{I}}$. This function is extended to arbitrary concepts as expected. Thus:

$$\begin{array}{rcl}
A & \rightsquigarrow & A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \\
\top & \rightsquigarrow & \Delta^{\mathcal{I}} \\
\bot & \rightsquigarrow & \emptyset \\
\neg C & \rightsquigarrow & \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \\
C \sqcap D & \rightsquigarrow & C^{\mathcal{I}} \wedge D^{\mathcal{I}} \\
C \sqcup D & \rightsquigarrow & C^{\mathcal{I}} \vee D^{\mathcal{I}} \\
\forall R.C & \rightsquigarrow & \{x \in \Delta^{\mathcal{I}} : \forall y.(R^{\mathcal{I}}(x,y) \to y \in C^{\mathcal{I}})\} \\
\exists R.C & \rightsquigarrow & \{x \in \Delta^{\mathcal{I}} : \exists y.(R^{\mathcal{I}}(x,y) \wedge y \in C^{\mathcal{I}})
\end{array}$$

In the one hand, description logics can be considered as a fragment of traditional, classical, first-order logic FOL. This goes via a translation of the basic notions of the description logic in question, (respectively *concepts* and *roles*) into FOL unary predicates (for concepts) and into FOL binary predicates (for roles) as well as translation of the chosen required connectives into themselves. Note that this translation (let us call it $t_1 \colon \mathcal{ALC} \to \mathsf{FOL}$) takes a description logic concept C into a predicate $C(x)$, with a free variable x, where we think of $C(x)$ as the collection of the x's that happen to be instances of the concept C. This translation is discussed in more detail below. This translation is the basis of several of the computational complexity results that underlay most of the theoretical work on description logics.

On the other hand, very early on, it was realized ([S91]) that description logics can also be considered as a *notational variant* of the multimodal K logic, if one thinks of roles in the description logic as acessibility relations in multimodal K_m. In this case the translation (let us call it $t_2 \colon \mathcal{ALC} \to \mathsf{K_m}$) from the description logic to the modal logic $\mathsf{K_m}$ works by transforming concepts C into atomic modal propositions C, intersections of concepts into propositional conjunctions, unions of concepts into propositional disjunctions and most importantly, taking the existential and universal role restrictions into modal operators as follows:
$t_2(\exists R_i.C) = \Diamond_i(t_2(C))$ and $t_2(\forall R_i.C) = \Box_i(t_2(C))$.

Thus when we try to place description logics into traditional hierarchies of logics, we actually have to consider *translations* and we have, in principle, two very different translations to work with. Note for instance that one translation uses predicate logic, while the other is purely propositional. We can draw these translations t_1, t_2 as the left-square below



Now if you think that reasoning constructively is a useful paradigm and you want to describe *constructive* or intuitionistic description logics, say CDL, you have two, in principle, very different routes you can take. First you can use a translation like t_1 and you can think of the system of constructive description logic you're describing as a subsystem of the well-known intuitionistic first-order logic system IFOL. Or you can use the modal translation t_2 and you can consider your target system of intuitionistic description logic as embedded into a constructive version of the modal system $\mathsf{K_m}$. There are several proposals for "the" constructive version of K in the literature and hence several proposals for the constructive version of the multimodal modal system $\mathsf{IK_m}$.

It is somewhat remarkable that these two routes can lead to the same place. Getting to the same place in the classical case is no coincidence: As van Benthem has shown and used often[vanB98], the fact that classical modal logic can be seen as a fragment of first-order logic, as well as an extension of it, is a consequence of the standard translation, plus his "Modal Invariance Theorem", which says that a first-order formula is definable by a modal formula if and only if it is invariant under bisimulation. Of course we would like an analogous result to be true about the constructive system we are trying to devise. But since we do not have (yet?) a constructive version of the modal invariance theorem, these two routes are the sources for the different kinds of constructive description logic that we introduce in this paper, in the next two sections.

In the next two sections we adapt classical $\mathcal{ALC}$ to a constructive system using the two routes outlined above. The syntax of our constructive system will be the same in both cases. Concepts descriptions in our constructive description logic CDL language obey the following syntax rule

$$C, D \to A \mid \top \mid \bot \mid C \sqcap D \mid C \sqcup D \mid C \to D \mid \forall R.C \mid \exists R.C$$

where C, D range over concepts, A is an atomic concept and R ranges over names of roles, as before. As usual in constructive logics, since $\neg C$ is simply an abbreviation for $C \to \bot$ we do not need to consider it. In compensation we must add in the constructive implication of concepts, which in classical description logic is a derived concept. Also it is just a convenience to have the true concept $\top$, as it could be defined as $\neg\bot$.

3 Constructive Description Logic via First-Order

The aim of this section is to describe the system of constructive description logic IALC obtained from $\mathcal{ALC}$ by considering its translation into (a subset of) traditional intuitionistic first order logic IFOL.

The disadvantage of IALC, when compared to classical $\mathcal{ALC}$ or even Wansing and Odintsov's inconsistency-tolerant system $\mathcal{CALC}$ [OW03] is that we cannot use the easy semantics of classical first-order logic to give "meaning" to IALC constructors. Instead we must use the translation t_1 of IALC into IFOL, which is quite standard. This translation is actually parametrized on two new intuitionistic variables, say x and y. Thus so $t_1^x \colon \mathsf{IALC} \to \mathsf{IFOL}$ is given by:

$$\begin{array}{rcl}
A & \rightsquigarrow & A(x) \\
\top & \rightsquigarrow & \top \\
\bot & \rightsquigarrow & \bot \\
C \sqcap D & \rightsquigarrow & t_1^x(C) \wedge t_1^x(D) \\
C \sqcup D & \rightsquigarrow & t_1^x(C) \vee t_1^x(D) \\
C \to D & \rightsquigarrow & t_1^x(C) \to t_1^x(D) \\
\forall R.C & \rightsquigarrow & \forall y.(R(x,y) \to t_1^y(C)) \\
\exists R.C & \rightsquigarrow & \exists y.(R(x,y) \wedge t_1^y(C))
\end{array}$$

Dually we have the t_1^y translation. These translations are very well-known for modal logic and description logics, the only difference is that here we use as our target language the intuitionistic predicate calculus, instead of the classical one. These translations follow van Benthem's slogan, "translation as a way of life" in [vanB98].

Now it is reasonable to expect a result such as "truth-preservation of the t_1 translations". To prove such a result we must produce a model for IALC. To do that, we first recall the traditional semantics of IFOL.

Definition 1 *A* model *for intuitionistic first-order logic* IFOL *is a 4-tuple*

$$\mathcal{M} = (W, \leq, \{D_w\}_{w\in W}, \{V_w\}_{w\in W})$$

where

1. *W is a non-empty set (of 'worlds') partially ordered by $\leq$;*
2. *for each w, D_w is a non-empty set such that $w \leq v$ implies $D_w \subseteq D_v$;*
3. *for each w, V_w is a function that to each n-ary predicate symbol P assigns a n-ary relation P_w contained in D_w^n such that $w \leq v$ implies $P_w \subseteq P_v$.*

*Given an element w of W, a w-*environment *is a function g from variables to D_w. Given a model $\mathcal{M}$ as defined above, the relation $\mathcal{M}, g, w \models A$ is defined by induction, where w is an element of W, g is a w-environment, and A is a formula.*

$\mathcal{M}, g, w \models P$	*iff*	$P_w(g(x_1), \ldots, g(x_n))$
$\mathcal{M}, g, w \models \bot$	*iff*	*falsum*
$\mathcal{M}, g, w \models A \wedge B$	*iff*	$\mathcal{M}, g, w \models A$ *and* $\mathcal{M}, g, w \models B$
$\mathcal{M}, g, w \models A \vee B$	*iff*	$\mathcal{M}, g, w \models A$ *or* $\mathcal{M}, g, w \models B$
$\mathcal{M}, g, w \models A \to B$	*iff*	*for all* $v \geq w$, $\mathcal{M}, g, v \models A$ *implies* $\mathcal{M}, g, v \models B$
$\mathcal{M}, g, w \models \forall x.A$	*iff*	*for all* $v \geq w$, *for all* $d \in D_v$, $\mathcal{M}, g[x := d], v \models A$
$\mathcal{M}, g, w \models \exists x.A$	*iff*	*there is* $d \in D_w$ *such that* $\mathcal{M}, g[x := d], w \models A$

Intuitively, we think of the elements of the partially ordered set W, the worlds, as "states-of-knowledge" ordered by their information content. It's also traditional to recall that this semantics is monotonic (or persistent), sound and complete.

Now we want to put together the classical semantics of $\mathcal{ALC}$ (previous section) and the constructive semantics of IFOL (above) to obtain a semantics for IALC, induced by the translations t_1. In this semantics the interpretations of concepts and of roles will be relativized to a state. Following Odintsov and Wansing's lead we say:

Definition 2 *A model for* IALC *consists of a structure $\langle W, \leq, D, (\cdot,\cdot)^{\mathcal{I}}\rangle$ where $(W, \leq)$ is a partial order, $D\colon W \to 2^{N_{\mathcal{I}}}$ is a domain function such that for each w, D_w is a non-empty set such that $w \leq v$ implies $D_w \subseteq D_v$. For each t in W, we interpret the concept A as $(A,t)^{\mathcal{I}}$ a subset of D_t.*

The interpretations of the composite concept descriptions are given by:

$A \rightsquigarrow (A,t)^{\mathcal{I}} \subseteq D_t$
$\top \rightsquigarrow \Delta^{\mathcal{I}}$
$\bot \rightsquigarrow \emptyset$
$C \sqcap D \rightsquigarrow (C,t)^{\mathcal{I}} \cap (D,t)^{\mathcal{I}}$
$C \sqcup D \rightsquigarrow (C,t)^{\mathcal{I}} \cup (D,t)^{\mathcal{I}}$
$C \to D \rightsquigarrow \{x \in D_t : (\forall s \in I)(t \leq s \to (x \in (C,s)^{\mathcal{I}} \to x \in (D,s)^{\mathcal{I}})\}$
$\forall R.C \rightsquigarrow \{x \in D_t : (\forall s \geq t)(\forall y \in D_s)((x,y) \in (r,s)^{\mathcal{I}} \to y \in (C,s)^{\mathcal{I}})\}$
$\exists R.C \rightsquigarrow \{x \in D_t : (\exists y \in D_t)((x,y) \in (r,t)^{\mathcal{I}} \wedge y \in (C,t)^{\mathcal{I}})\}$

A concept C of IALC is said to be valid (written as $\models_{\mathsf{IALC}} C$) iff for every interpretation $\mathcal{I}$ and for every t in W the interpretation of C, $(C,t)^{\mathcal{I}}$ is the whole of D_t. Given the disjunction property of IFOL we can prove:

Proposition 1 *The system* IALC *has the disjunction property:*

$$\models_{\mathsf{IALC}} (C \sqcup D) \textit{ iff } \models_{\mathsf{IALC}} C \textit{ or } \models_{\mathsf{IALC}} D$$

Proposition 2 *Truth-preservation of the translation $t1$ is straightforward.*

The proof can be easily read off [B06?].

4 Constructive Description Logic via Modal Logic

Now we want to investigate a system of constructive description logic obtained via the translation of description logic into (intuitionistic) multimodal K. It is well known that there are many, non-equivalent systems competing for the title of the intuitionistic version of the classical modal system K.

We will consider two possibilities: the system FS for G. Fischer-Servi, which turns out to be the same as Simpson's system IK and the system called CK, (for constructive K) a variation of Wijesekera's work, that has been promoted by Bellin, de Paiva and Ritter in [BdPR01].

The constructive description logic obtained from the system IK will be denoted iALC, as it's closer to IALC than cALC, obtained from CK.

This time we use the translation t_2 given by:

$$\begin{array}{rcl} A & \rightsquigarrow & A \\ \top & \rightsquigarrow & \top \\ \bot & \rightsquigarrow & \bot \\ C \sqcap D & \rightsquigarrow & t_2(C) \wedge t_2(D) \\ C \sqcup D & \rightsquigarrow & t_2(C) \vee t_2(D) \\ C \to D & \rightsquigarrow & t_2(C) \to t_2(D) \\ \forall R.C & \rightsquigarrow & \Box_R t_2(C) \\ \exists R.C & \rightsquigarrow & \Diamond_R t_2(C) \end{array}$$

Note that this translation, unlike t_1, is only propositional. Notice that R above stands for a generic role and hence we end up in multimodal K instead of a single box, diamond K.

But more importantly, note that this translation can be read as having as target either IK or CK or any other version of constructive or classical multimodal logic desired. The translation is different in each case and depends on *how* we interpret the modalities.

We provide descriptions of both iALC and cALC because iALC relates to our work on constructive hybrid logics [BdP03] while cALC can be given a computational interpretation by means of the extended Curry-Howard isomorphism, which it inherits from CK.

4.1 The system iALC

Fisher-Servi's system of constructive modal logic has been promoted by Simpson, who also provided a "labelled" Natural Deduction for it. The multimodal version of this same system, called $\mathsf{IK_m}$, is also the basis for our Natural Deduction version of hybrid logics[BdP03]. It's natural to consider it as a target for our constructive description logic.

Recall the possible-worlds semantics for $\mathsf{IK_m}$.

Definition 3 *A* model *for intuitionistic modal logic* $\mathsf{IK_m}$ *is a tuple*

$$\mathcal{M} = (W, \leq, R, V)$$

where

1. *W is a non-empty set (of 'worlds') partially ordered by $\leq$;*
2. *R is a binary relation on W;*
3. *for each w, V_w is a function that to each ordinary propositional symbol p assigns a subset of D_w such that $w \leq v$ implies $V_w(p) \subseteq V_v(p)$ and*
4. *such that the following two frame conditions below are satisfied*

Given a model $\mathcal{M}$ as defined above, the satisfaction relation between worlds w and modal formulas A $w \models_{\mathcal{M}} A$ is defined by

$w \models p$	iff $d \in V_w(p)$
$w \models \bot$	iff falsum
$w \models A \wedge B$	iff $w \models A$ and $w \models B$
$w \models A \vee B$	iff $w \models A$ or $w \models B$
$w \models A \rightarrow B$	iff for all $v \geq w$, $v \models A$ implies $v \models B$
$w \models \forall x.A$	iff for all $v \geq w$, for all $d \in D_v$, $\mathcal{M}, g[x := d], v \models A$
$\mathcal{M}, g, w \models \exists x.A$	iff there is $d \in D_w$ such that $\mathcal{M}, g[x := d], w \models A$

4.2 The system cALC

This version of constructive hybrid logic is based on Bellin, de Paiva and Ritter's description of CK[BdPR01]. This system has recently been given a Kripke semantics in [MdP05]. The reason for considering this version of constructive modal logic is that we can produce a term assignment for this system, which is provably Curry-Howard correspondent to the logic. On the negative side, this system is not as easily extensible as the previous one, despite its good proof-theoretical properties.

5 Related Work

This work is part of larger programme of constructivizing logics for AI. Others have engaged in somewhat similar programmes.

The work closest to ours is the recent paper by Hofmann [H05] on proof-theoretical methods for description logics. That paper proposes sound and complete Gentzen sequent calculus for some classes of description logics. But Hofmann concentrates efforts on description logics with fixpoints, which by definition defeats the goal of a Curry-Howard correspondence for the logic in question.

Another work close to ours is Odintsov and Wansing's "Inconsistency-tolerant Description Logics". As their title indicates they are more concerned with obtaining a paraconsistent system (or an inconsistency-tolerant system) than a constructive one. We want to follow the constructive paradigm as closely as possible.

A third line of related work is presented in the book by Gabbay, Kurucz, Wolter and Zakharyaschev [GKW+3]. They discuss modal descriptions logics and, using the semantic way of thinking of intuitionistic logic as classical S4, this suggests a new way of 'constructivizing' description logics. They also investigate the complexity issues that have been ignored in this note.

Finally, for the application to context logic that motivates this work, one might think that the work on C-OWL (Context OWL [BGvH+]) would be the blueprint. (The work on C-OWL is not constructive, but this could be easily modified.) But the work on C-OWL is not the kind of contextualization of a description logic that we need: firstly because C-OWL contextualize ontologies, while for our project we have *one* basic underlying ontology, with contexts giving us projections of that mother ontology. Secondly the kind of description logic we need (for a preliminary description see [BCC+05]) is sufficiently different from OWL to justify looking at the abstract notion of a description logic. Thirdly the mechanism for contexts themselves can be formalized either in the more distributed style of ML-systems [SG02] or in the less distributed style of modal logics, it seems to me.

6 Conclusions

We have presented three possible ways of defining constructive hybrid logics, which we find interesting and profitable, but challenging. We have not been able to prove that they end up being the same system or otherwise. Neither have we managed to produce a different set of criteria for appropriateness of the constructive versions. While one expects that expressing things constructively will bring more choices than when using classical logic,a minimum of similarity between the classical and the constructive settings is also expected. Presumably the problem is that the investigation reported here is too preliminary to yield the desired results. Meanwhile the empirical investigation into possible versions of "contexted constructive description logic" continues [BCC+05] and might yield a reasonable adequacy benchmark.

Acknowledgments

I would like to thank Torben Brauner and Natasha Alechina for many enlightening discussions. Also Carsten Lutz for initial pointers and discussion of the description logic literature.

REFERENCES

[AMPR98] N. Alechina, M. Mendler, V. de Paiva and E. Ritter. Categorical and Kripke Semantics for Constructive Modal Logics. In *Computer Science Logic (CSL'01)*, Paris, September 2001.

[BCM+03] F. Baader, D. Calvanese, D. McGuiness, D. Nardi and P. Patel-Schneider. The Description Logic Handbook: theory, implementations and applications. *Cambridge University Press*, 2003.

[BdPR01] G. Bellin, V. de Paiva and E. Ritter. Extended Curry-Howard Correspondence for a Basic Constructive Modal Logic. In Proceedings of Methods for Modalities II, November 2001.

[BdP00] G. M. Bierman and V. de Paiva. On an Intuitionistic Modal Logic. *Studia Logica*, 65: 383–416, 2000.

[BCC+05] D. Bobrow,C. Condoravdi, R. Crouch, R. Kaplan, L. Karttunen, V. de Paiva and A. Zaenen. A Basic Logic for Textual Inference. In Proceedings of the AAAI Workshop on Inference for Textual Question Answering, July 2005.

[BGvH+] P. Bouquet and F. Giunchiglia and F. van Harmelen and L. Serafini and H. Stuckenschmidt. C-OWL: Contextualizing Ontologies. In Proceedings of the Second Internatinal Semantic Web Conference,eds. Sekara and Mylopoulis, 164–179, LNCS 2870, 2003.

[BdP03] T. Brauner and V. de Paiva. Towards Constructive Hybrid Logics. In *Methods for the Modalities 3*, Nancy, France, 2003.

[B06?] T. Brauner. Axioms for Classical, Intuitionistic, and Paraconsistent Hybrid Logic. In *Journal of Logic, Language and Information*, to appear.

[GKW+3] D. Gabbay, A. Kurucz, F. Wolter and M. Zakharyaschev. *Many-Dimensional Modal Logics: Theory and Applications*. Studies in Logic and the Foundations of Mathematics. Elsevier, Amstersdam, 2003.

[G95] R. Guha. Contexts: A Formalization and Some applications. Ph.D Thesis, Stanford University, 1995.

[H05] M. Hofmann. *Proof-theoretic Approach to Description-Logic*. In *Proc. of Logic in Computer Science (LICS-05)*, 229–237, 2005.

[S91] K. Schild. *A correspondence theory for terminological logics: preliminary report*. In *Proc. of the 11th Joint Conference on Artificial Intelligence (IJCAI-91)*, 466–471, 1991.

[McC93] J. McCarthy. *Notes on Formalizing Context*. In *Proc. of the 13th Joint Conference on Artificial Intelligence (IJCAI-93)*, 555–560, 1993.

[MdP05] M. Mendler and Valeria de Paiva. *Constructive CK for Contexts*. In Proceedings of the First Workshop on Context Representation and Reasoning, CONTEXT'05, Paris, France. 2005.

[OW03] S. Odintsov and H Wansing. *Inconsistent-toleratn Description Logic: motivation and basic systems*. In *Trends in Logic: 50 Years of Studia Logic*, eds. V. Hendriks and J. Malinowski, 2003.

[SG02] L. Serafini and F. Giunchiglia. ML systems: A Proof Theory for Contexts. Journal of Logic Language and Information, 11:4, pp 471-518, 2002. Also ITC-IRST Technical Report 0006-01.

[S91] K. Schild. *A correspondence theory for terminological logigs: preliminary report*. In *Proc. of the 11th Joint Conference on Artificial Intelligence (IJCAI-91)*, 466–471, 1991.

[Sim94] A.K. Simpson. *The Proof Theory and Semantics of Intuitionistic Modal Logic*. PhD thesis, University of Edinburgh, 1994.

[vanB98] J. van Benthem. Modal Logic in Two Gestalts. In *Advances in Modal Logic, vol. II*, eds. M. de Rijke, H. Wansing and M. Zakharyashev, CSLI Publications, Stanford, 73–100, 1998.

[Wij90] D. Wijesekera. Constructive modal logic I. *Annals of Pure and Applied Logic*, 50:271–301, 1990.