

Natural Deduction and Context as (Constructive) Modality

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Abstract. This note describes three formalized logics of context and their mathematical inter-relationships. It also proposes a Natural Deduction formulation for a *constructive* logic of contexts, which is what the described logics have in common.

Key Words: Logical formalisms for context, Representing context and contextual knowledge, Context in knowledge representation.

1 Introduction

The word “context” has too many different meanings, so we should start by explaining that we are interested in logics of context designed to help automated reasoning in Artificial Intelligence (AI), more specifically, in knowledge representation. Thus we are interested in mathematically understanding and clarifying work that, starting with McCarthy’s seminal papers[McC96,McC93,McCB97], aims at giving the (informal) notion of context the role of a first-class object in a logical system.

Our goal is a mathematically well-behaved logical system that models reasoning that happens when we say, for example, that in the context of Sherlock Holmes stories it is true that Sherlock Holmes lives in Baker Street, London. For a traditional mathematical logician, this informal notion of context is modeled by considering different logical theories and the burden of deciding how these logical theories interact is shifted to the metalogic and the human reasoner. In this paper we take for granted that the reader has been convinced by McCarthy’s, Giunchiglia’s (and others’) arguments that context should be a first-class object in a logical system and that the question to be solved is *which* logical system should one use. Narrowing our focus, we concentrate not in deciding which logical system to use, but on the much smaller question of comparing, in terms of their mathematical properties, the systems¹ in the literature where context is modeled via a modality operator, usually written as $\text{ist}(\kappa, A)$. Here the basic intuition is that formulas, such as A , are true not in absolute terms, but in

¹ A referee has rightly complained that we do not discuss how well these systems match the intuitions they are trying to model. While this task seems very important, this author does not have the right intuitions to carry it out. Moreover, the project [CC+02] that our theoretical investigation underpins has moved to a new direction.

certain contexts, in particular, in the context named by the constant κ . There are many reasons why this is a good idea for AI and these, as well as examples of the applications of these ideas, are discussed in the literature. But even narrowing down the problem to choosing between systems and considering only the systems based on some kind of modality, the task is daunting. The literature on notions of context and on formalizations thereof, i.e. logics of context is really vast [AS96]. This paper discusses three propositional² systems: Buvac and Mason’s propositional logic of contexts, henceforth PLC [BBM95], Nayak’s system (here called \mathcal{N}) a logic of contexts for multiple domain theories [Nay94] and Mas-sacci’s system \mathcal{T} [Mas95], described as a tableaux version of PLC. The Trento group framework for logics of context, called LMS/MCS, for Local Model Semantics/MultiContext systems [BS00,SG00] was also originally considered, but that comparison is now in a companion paper [deP]. This is because, strictly speaking, MCS/LMS has no explicit modality. However, it is well-known that the *bridge rules* of their main system MR correspond, technically, to a K necessity operator.

In the next section we discuss why worry about Natural Deduction, why constructivity is important for us, what constitutes a Natural Deduction formulation of a logic and why obtaining a Natural Deduction formulation for logics of context is problematic and worthwhile. Then, in the following sections, we give succinct descriptions of the systems of contexts we consider. After that we compare and evaluate those systems. The upshot is that we can produce a very stringent Natural Deduction formulation for what these systems have in common. The natural deduction formulation for this core constructive language is spelled out in detail in the following section.

2 Natural Deduction: why?

McCarthy, when first discussing the idea of contexts in AI, suggested that a “strong form of Natural Deduction” should hold for an intuitively appealing logic of contexts. His suggestion of a logic of contexts is based on the notion of a modality $\text{ist}(\kappa, A)$. The intuition of using a modality operator to deal with logics of context is common to all the systems we discuss (and many others we do not). But the systems differ along three different dimensions. First they differ on which *properties* the modalities are supposed to have, then they differ on *how* they are described mathematically, e.g. whether one uses axioms or tableaux systems or Natural Deduction rules and finally they differ on which properties do they prove of the system they consider, whether they have soundness and completeness and with respect to what kind of model.

We advocate the view that a logic should be independent of its different presentations, that is, that one should be able to give different presentations (using axioms, sequents, rules) for any *decent* logic, as we can do for e.g. classical or constructive first-order logic. Moreover, since these formalizations are only different presentations of the same logic, we believe that one must be able to

² There are first-order systems in the literature, but we restrict our attention to propositional systems.

prove them all equivalent, using syntactic translations between the systems. Thus our first aim is to prove that there is a decent logic of contexts, that is, there is a formal logic of contexts which can be given in several different presentations, all proved equivalent.

McCarthy’s and Guha’s intuitions were formalized by S. Buvac, V. Buvac and I. Mason in [BBM95]. Their formalization was done in a Hilbert-style system, usually the easiest kind of formalism as far as modal logics are concerned. That paper leaves open how to formalize their propositional logic of contexts in a Natural Deduction setting. Actually, when discussing future work they say:

We also plan to define non-Hilbert style formal systems for context. Probably the most relevant is a natural deduction system, which would be in line with McCarthy’s original proposal of treating contextual reasoning as a strong version of natural deduction. In such a system, entering a context would correspond to making an assumption in natural deduction, while exiting a context corresponds to discharging an assumption.

But this future work has not, as yet, come to fruition, which is not surprising, considering the amount of controversy surrounding Natural Deduction for Modal Logics in general. For some of this controversy (and a detailed explanation) the reader is directed to [BdPR01].

A formal description of what constitutes a Natural Deduction formalism will not be attempted here, but we take as paradigmatic the work of Prawitz [Pra65], which is sometimes described as Gentzen-style Natural Deduction, by contrast to Fitch-style Natural Deduction. Gentzen-style Natural Deduction derivations are tree-shaped, usually with one introduction and one elimination rule for each logical connective. More importantly, the introduction and elimination rules give rise to a notion of normalization (elimination of the ‘detour’ in the proof, that consists of one introduction rule followed immediately by the elimination of the same connective). For intuitionistic logic this paradigm works very well, both for first-order and for higher-order calculi. For other logics, especially modal logics, the formalism does not work so well. Prawitz, for example, only deals with the systems called S4 and S5 in his treatise and even that treatment is not optimal [BdP00]. In a nutshell, the problem is that it is hard³ to provide introduction and elimination rules for a K-style necessity (\Box) operator that satisfies only the sequent calculus (Scott’s) rule:

$$\frac{\Gamma \vdash B}{\Box \Gamma \vdash \Box B}$$

For a start, this rule is clearly both an introduction and an elimination rule. But the crux of the problem is how to write, using a tree-like derivation that, after the use of the necessitation rule, all the premises become boxed. Proof-theoretic trees only grow downwards, not upwards. If instead of usual Prawitz-style trees,

³ So hard that Bull and Segerberg in [BS84] discuss whether modal logic is not *natural* enough to have a Natural Deduction formulation.

one tries to use Natural Deduction in sequent-style, as advocated by Martin-Loëf (which corresponds to writing the rule as above) the problem persists. One essential component of Natural Deduction is its ability to put proofs together. If you have proofs $\pi: A_1, \dots, A_K \vdash B$ and $\sigma: C \vdash A_1$, you must be able to compose them in ND, obtaining $\sigma; \pi$ a proof of $C, A_2, \dots, A_k \vdash B$. But if you apply the Box rule to π , obtaining $\Box A_1, \dots, \Box A_k \vdash \Box B$, then you cannot compose it with σ anymore. This is an unfortunate situation and there are several very different solutions to this problem in the literature. Most of the solutions build-in some of the semantics into the syntax of modal logic: this is the case for Gabbay's labelled deductive systems, Simpson's framework and Basin et al's framework. The solution we prefer is merely syntactic, see section 6, but there are tradeoffs, discussed later.

The (proof theoretic) received wisdom about logical formalisms is that:

- Axiomatic systems are the easiest ones to devise and also the ones where it is easier to prove theorems *about* the system;
- Sequent calculi are the systems that are easy to mechanize and
- Natural Deduction systems are the ones most similar to the way humans construct proofs.

It is also the case that given a Gentzen-style Natural Deduction system one can, automatically derive both sequent calculus and axiomatic systems from it, but the converses are not always true. Hence Natural Deduction systems are the most informative formalism. But exactly what constitutes a Natural Deduction system and, given that modal logics must depart somehow from the traditional setting, what are the most important properties to preserve is subject to personal taste and warrants discussion.

Given that sequent calculi (and tableaux systems) are, arguably, better formalisms for automatic *proof search*, whereas Natural Deduction comes into its own when dealing with proof normalization, one may wonder why we worry about a Natural Deduction version of a constructive logic of contexts. In the one hand, we are interested in *deep* understanding of the logic in question and a Natural Deduction formalization gives the ability to change formalisms as explained above. Since the different formalisms do not constitute different systems, but are simply different presentations of a given system, a Natural Deduction presentation, together with its translations, affords logical respectability. On the other hand, our emphasis on *constructivity* of the logic explains an ulterior (and eventual) goal: we would like to use the Curry-Howard correspondence to provide a functional programming language for dealing with proofs of statements in context.

But even discounting the motivation of a Curry-Howard system for contexts, it is true that the exercise of comparing logics tends to clarify our understanding. This explains the emphasis on the comparison of the systems in this paper. Both Buvac, Buvac and Mason's PLC and Nayak's \mathcal{N} are given as axiomatic systems, while Massacci's calculus is given as a tableaux system – a close cousin of a sequent calculus. Thus we start by describing PLC and \mathcal{N} and then we discuss Massacci's system. After that we introduce our own Natural Deduction system.

3 The propositional logic of contexts PLC

Buvac, Buvac and Mason’s paper “Metamathematics of Contexts” [BBM95] is the most developed formalization of McCarthy’s ideas [McC93] about a propositional logic of contexts. Their propositional logic of contexts extends *classical* propositional logic in (at least) two ways: first, it adds a new modality $\text{ist}(\kappa, \phi)$, used to express that the sentence ϕ holds or is true in context κ . Second, they postulate that each context has its own vocabulary, ie a set of propositional atoms that is meaningful in that context. They describe a basic logic of contexts, describe a semantics for this basic logic similar to the traditional semantics for first-order logic, discuss various extensions of this basic logic and give a correspondence theory, relating axioms to extensions of the basic semantics.

We start with two given, distinct, countably infinite sets \mathcal{K} , a set of labels (intuitively denoting some basic contexts) and \mathcal{P} the set of all propositional atoms. Then well-formed formulas \mathcal{F} are built from the sets \mathcal{K} and \mathcal{P} by negation and implication, together with the $\text{ist}(\kappa, \phi)$ operator.

$$\mathcal{F} := \mathcal{P} \cup (\neg\mathcal{F}) \cup (\mathcal{F} \rightarrow \mathcal{F}) \cup \text{ist}(\mathcal{K}, \mathcal{P})$$

Instead of using simply the set \mathcal{K} of basic labels PLC uses the set of finite sequences over \mathcal{K} , \mathcal{K}^* . A context, denoted $\bar{\kappa}$, consists of a finite sequence $(\kappa_1 \dots \kappa_n)$ of elements of \mathcal{K} (or in the degenerate case ϵ , the empty sequence). But when one writes $\text{ist}(\bar{\kappa}, A)$ this actually means $\text{ist}(\kappa_1, (\text{ist}(\kappa_2, \dots \text{ist}(\kappa_n, A) \dots))$. This use of sequences of basic contexts corresponds to PLC’s intuition that what holds in a context depends on how you arrived at this context, so that $\kappa_1 \kappa_2$ represents how context κ_1 is seen from context κ_2 .

We also need to explain the role of vocabularies. The intuitive idea is that a vocabulary (the set of meaningful propositional atoms) is defined for each context. Thus we have a relation Vocab between \mathcal{K}^* and \mathcal{P} . The notion of derivability ($\vdash_{\bar{\kappa}} A$) that defines PLC is also dependent on the vocabulary used, so it should be written as $\vdash_{\bar{\kappa}}^{\text{Vocab}} A$, but PLC makes the simplifying assumption that given any formula A and context $\bar{\kappa}$ we can calculate the vocabulary of the formula A in context $\bar{\kappa}$ using a function $\text{Vocab}(\bar{\kappa}, A)$. Moreover, PLC’s Definedness Condition asserts that whenever we state $\vdash_{\bar{\kappa}} A$, we implicitly assume that the $\text{Vocab}(\bar{\kappa}, A)$ is contained in (the previously given and forever fixed) Vocab .

Buvac, Buvac and Mason assume the following axioms:

(**taut**) $\vdash_{\bar{\kappa}} A$ for all classical tautologies A

(**K**) $\vdash_{\bar{\kappa}} \text{ist}(\kappa, A \rightarrow B) \rightarrow (\text{ist}(\kappa, A) \rightarrow \text{ist}(\kappa, B))$

(Δ) $\vdash_{\bar{\kappa}} \text{ist}(\kappa, \text{ist}(\kappa_1, A) \vee B) \rightarrow \text{ist}(\kappa, \text{ist}(\kappa_1, A)) \vee \text{ist}(\kappa, B)$

together with the proof rules of context switching (CS) and Modus Ponens (MP) below.

$$(CS) \frac{\vdash_{\bar{\kappa}\kappa_1} A}{\vdash_{\bar{\kappa}} \text{ist}(\kappa_1, A)} \qquad (MP) \frac{\vdash_{\bar{\kappa}} A \rightarrow B \quad \vdash_{\bar{\kappa}} A}{\vdash_{\bar{\kappa}} B}$$

The axioms⁴ and rules above constitute the Hilbert-style system for PLC. Note that derivations are always in context, i.e. the turnstile is always decorated with the context sequence where the derivation occurs. We say A is provable in context $\bar{\kappa}$ iff $\vdash_{\bar{\kappa}} A$ is an instance of an axiom schema or follows from provable formulae by one of the inference rules.

The axiom schemas (**taut**) and (**K**) are traditional, in that logics with modalities usually satisfy all tautologies of the basic (in their case classical) logic and the axiom **K** is generally considered the bare minimum to require of a modality. The Modus Ponens rule (MP) is also traditional, but adapted to hold in each and every context $\bar{\kappa}$.

The context switching rule (CS) and the axiom (Δ) deserve some discussion. It is easy to see that the context switching rule is more general than the usual modal necessitation rule. If one erases contexts from the derivability relation the context switching rule becomes the necessitation rule. But it is not immediately clear that whenever one uses the context switching rule in a PLC proof, the modal necessitation rule could have been used instead.

Let us call *localized multimodal K*, the system consisting of two axiom schemas:

(**taut**) $\vdash_{\bar{\kappa}} A$ for all classical tautologies A

(**K**) $\vdash_{\bar{\kappa}} \text{ist}(\kappa, A \rightarrow B) \rightarrow (\text{ist}(\kappa, A) \rightarrow \text{ist}(\kappa, B))$

together with rules

$$(\text{Nec}^*) \frac{\vdash_{\bar{\kappa}} A}{\vdash_{\bar{\kappa}} \text{ist}(\kappa_1, A)} \quad (\text{MP}) \frac{\vdash_{\bar{\kappa}} A \rightarrow B \quad \vdash_{\bar{\kappa}} A}{\vdash_{\bar{\kappa}} B}$$

Proposition 1 (Serafini) *Assume that all contexts have the same vocabulary. Given a proof π of A in PLC, there exists a proof π' of A in the system localized multimodal **K** plus Δ .*

Proof: Consider the first appearance of the context switching rule in π . Assume it uses $\vdash_{\bar{\kappa}\kappa_1} A$ to give $\vdash_{\bar{\kappa}} \text{ist}(\kappa_1, A)$. The proof till this use of context switching (CS) was all done in the context $\bar{\kappa}\kappa_1$. Since all axioms in the context $\bar{\kappa}\kappa_1$ are also axioms in $\bar{\kappa}$ and whatever uses of (MP) in $\bar{\kappa}\kappa_1$ are also uses in $\bar{\kappa}$ we can remove κ_1 from the whole proof and after this transformation the proof looks like

$$\frac{\vdots}{\frac{\vdash_{\bar{\kappa}} A}{\vdash_{\bar{\kappa}} \text{ist}(\kappa_1, A)}}$$

⁴ The axiom (**taut**), valid for all systems considered in this note, is disputed by a referee, who suggests that truth in a context should be constrained by relevance to a context. But relevance is a much harder problem than localization of truth, which is the simplified aim of these logics of context.

Applying this transformation to all occurrences of the context switching rule, we obtain a proof that only uses the localized necessitation rule. \square .

The reader will have noticed the assumption of all contexts having the same vocabulary. Recent work[BS00] of Bouquet and Serafini’s shows semantically that the vocabularies of PLC play no essential logical role. They say that their “Reduction to Complete Vocabularies” theorem allows them to conclude that PLC really is the normal multimodal logic K extended with the extra axiom Δ .

The axiom (Δ) is problematic from the proof-theoretic perspective. Buvac, Buvac and Mason say that axiom Δ corresponds to the validity reading of the modality $\text{ist}(\kappa, A)$. They justify their adoption of axiom (Δ) (for their most generic logic of contexts) by saying that if we disregard vocabulary restrictions then Δ can be written as

$$(\Delta') \quad \text{ist}(\kappa_1, \text{ist}(\kappa_2, A)) \vee \text{ist}(\kappa_1, \neg \text{ist}(\kappa_2, A))$$

which they read as saying that “it is true in knowledge base κ_1 that A is valid in the knowledge base κ_2 , or it is true in knowledge base κ_1 that A is not valid in the κ_2 knowledge base”. Thus each knowledge base behaves as if it can *see* into another knowledge base and decide for any formula A whether or not it is valid in the second knowledge base. But it is not clear that this kind of property is *essential* (or even sensible) for a basic logic of contexts. Actually [CP98] states: “This axiom [Δ] does not seem justified, even for the applications they consider. There is no reason why a database should have complete information about the contents of other databases.”

3.1 Evaluating PLC

Buvac and Mason say that

Modelling truth or validity in a context by a Kripke model, ie by a relation between worlds would not be intuitive, because we want contexts to be reified as first class objects in the semantics. This will allow us (in the predicate case) to state relations between contexts, define operations on contexts and specify how sentences from one context can be lifted into another contexts.

But PLC is a propositional logic and its extension to the 1st order case is far from trivial. Also in the context of PLC no relations, nor operations between contexts are specified. Thus the only reason given by Buvac and Mason for not considering a Kripke-style semantics, that “it is not intuitive to model validity in a context by a relation between worlds” seems too vague. A matter of taste, like saying that you should always use first-order logic, if you can.

It is satisfying to have a sound and complete (first-order-like) semantics for PLC, and for some of its reasonable extensions, but it is not clear how much the semantics presented forces one to accept axiom Δ ⁵. It is also not clear to

⁵ This is actually an usual problem with any axiomatic system, it is always the case that other axioms might be better, less redundant or more informative. This is another reason for considering other formalisms for a “minimal” logic of contexts.

me, why such a first-order-like semantics is or would be better than a possible-worlds semantics. Thirdly the role of vocabularies and whether one should have contexts modelled as sequences of basic contexts (or not) is still unclear.

Finally note that to consider a constructive version of PLC we need to take as basis any axiomatization of constructive logic and if we decide that the axiom (Δ) is not required, we just keep (CS), (MP) and \mathbf{K} , nothing more needs to be done.

4 The logic of contexts for multiple theories \mathcal{N}

Nayak [Nay94] takes a different view of the problem of devising a useful logic of contexts: he suggests that, for the purposes of representing and reasoning with multiple domain theories, rather than developing new syntax and new semantics for a logic, we can simply stick with a natural (multimodal) extension of a traditional modal logic. Nayak suggests to write a necessity modal operator for each context (contexts are simply labeled by natural numbers) and to allow different contexts to have different vocabularies.

Nayak presents two main reasons for treating contexts as modal operators, instead of extended terms, as in PLC. First, he says, in the propositional case the context operators and terms are effectively equivalent. Second, the advantage of contexts as terms is that it allows reasoning about contexts *within* the logic, but, he contends, most of the reasoning he wants to do about contexts and about relations between contexts can be done in a meta-theory. Hence it should be worthwhile investigating the properties of a simpler logic of contexts.

The syntax of Nayak's logic of contexts has a set of propositions \mathcal{P} , as before, as contexts \mathcal{K} it has natural numbers $\{1, 2, 3, \dots, n, \dots\}$, and instead of $\text{ist}(i, A)$ for A in \mathcal{P} , Nayak denotes that formula A is valid in a context i , by an indexed necessity operator $C_i(A)$. To facilitate the comparison we will use PLC's notation instead. Well-formed formulae are given by

$$\mathcal{F} := \mathcal{P} \cup (\neg\mathcal{F}) \cup (\mathcal{F} \rightarrow \mathcal{F}) \cup \text{ist}(i, \mathcal{F}), i \in \mathcal{K}$$

Because Nayak's logic wants to pay attention to different vocabularies for different contexts, it defines a function vocabulary, which maps contexts to the collection of propositions defined for that context, $\text{voc}: \mathcal{K} \rightarrow 2^{\mathcal{P}}$. Since some propositions are not part of the vocabulary of some contexts, we say that a well-formed formula A is *meaningful* with respect to voc if for any propositional letter p occurring in A , if p is immediately within a context $\text{ist}(i,)$ then p must be in the vocabulary of that context.

Nayak assumes the following axioms:

(A1) $\vdash A$ for all (classical) *meaningful* tautologies A

(A2) $\vdash \text{ist}(i, (A \rightarrow B)) \rightarrow (\text{ist}(i, A) \rightarrow \text{ist}(i, B))$, for $1 \leq i \leq n$

(where all formulae in axiom A2 are assumed meaningful) together with the proof rules of Necessitation and Modus Ponens below, where $\text{ist}(i, A)$ is assumed meaningful.

$$(Nec) \frac{\vdash A}{\vdash \text{ist}(i, A)} \quad (MP) \frac{\vdash A \rightarrow B \quad \vdash A}{\vdash B}$$

In a nutshell Nayak proposes using a normal multimodal system **K** as the basic logic, but goes on to say that this axiomatization does not restrict enough the properties of contexts or their inter-relationships. For the purpose of modelling these extra properties, he introduces three new axioms:

$$\begin{aligned} (\mathbf{A3}) \quad & \text{ist}(\kappa, A) \rightarrow \text{ist}(\kappa_1, \text{ist}(\kappa, A)) \\ (\mathbf{A4}) \quad & \neg \text{ist}(\kappa, A) \rightarrow \text{ist}(\kappa_1, \neg \text{ist}(\kappa, A)) \\ (\mathbf{A5}) \quad & \text{ist}(\kappa, A) \rightarrow \neg \text{ist}(\kappa, \neg A) \end{aligned}$$

The system consisting of multimodal **K** together with axioms **A3**, **A4**, **A5** (called \mathcal{F}_n in Nayak's work) is called here \mathcal{N} , for Nayak. Note that axiom **A5** is a generalization of modal axiom **D**, ie it is **D** for every context operator, discussed in the extensions of PLC. Axioms **A3** and **A4** are the generalizations of positive introspection and negative introspection that appear in converse form in other extensions of PLC. Nayak's logic \mathcal{N} is greatly simplified, it does not need to deal with sequences of contexts and these generalizations "ensure that every context knows about what every other context does and does not know, i.e. the facts true in a context are context independent".

Making Nayak's system *constructive* is a matter of making the propositional basis constructive and the basic modal operators constructive. Thus it is clear that it depends on deciding which shape of constructive modal logic one prefers.

4.1 Evaluating \mathcal{N}

There is much to recommend the use of 'off the shelf' logical systems. But it must be pointed out that the draconian simplifications brought about, especially by axioms **A4** and **A5** make Nayak's theory applicable only to situations where the contexts are almost not related at all, as in his example of Saturn's moon Titan and tropical forests.

The simplifications brought about by the extra axioms seem too strong for a minimal logic of contexts. Having said that, it would be good to have Nayak's system at one end of the spectrum of useful context logics. One problem is providing a natural deduction formulation for axioms **A4** and **A5**.

5 Massacci's Tableaux System

Massacci's papers [Mas95, Mas96] deal with a tableaux version of a logic of contexts. Massacci seems to be referring to PLC, as defined by Buvac, Buvac and Mason, but as we will discuss his logic proves more theorems than basic PLC.

To describe the system Massacci calls \mathcal{T} (for tableaux) we start with two distinct countably infinite sets \mathcal{K} and \mathcal{P} , the set of all basic contexts and the set of all propositional atoms. Then well-formed formulas \mathcal{F} are built from the sets \mathcal{K} and \mathcal{P} by negation and implication, together with the $\text{ist}(\kappa, A)$ operator. As

in PLC, contexts are actually sequence of basic contexts and contexts determine the *vocabulary* of an application or theory. The vocabulary is as before described by a function $\text{vocab}: \mathcal{K}^* \rightarrow 2^{\mathcal{P}}$ assigning to each context sequence $\bar{\kappa}$ a subset of the basic propositions that are supposed to be meaningful in that context.

But instead of axioms, Massacci introduces tableaux rules, together with a semantics in terms of “superficial valuations”. Massacci’s tableaux system uses formulae with labels and labelled deduction rules. The labels on the formulae have a double role: given a contextualized formula $\langle \bar{\kappa}[n]: A \rangle$, $\bar{\kappa}$ is a sequence of basic contexts, n is an integer and A is a well-formed formula as above. Intuitively the prefix $\bar{\kappa}[n]$ ‘names’ the n -th superficial valuation, where A holds.

The first three rules correspond to the propositional *classical* basis and are standard for tableaux systems, except that they carry annotations telling you in which context/world you are working.

$$(\&) \frac{\bar{\kappa}[n]: A \& B}{\bar{\kappa}[n]: A, \bar{\kappa}[n]: B} \quad (\neg/\vee) \frac{\bar{\kappa}[n]: \neg(A \vee B)}{\bar{\kappa}[n]: \neg A \mid \bar{\kappa}[n]: \neg B} \quad (\neg\neg) \frac{\bar{\kappa}[n]: \neg\neg A}{\bar{\kappa}[n]: A}$$

The next two rules, called “databases rules” require some explanation. The local contextual database LB is a set of formulae holding in the initial context κ_0 . The global contextual database GB contains the formulae holding in every context sequence $\bar{\kappa}_0$ extending the initial sequence κ_0 . These rules are necessary to deal with logical consequence in modal logics, but are not related to the essence of contexts.

$$(\text{Loc}) \frac{\vdots}{\kappa_0[1]: A} \text{ If } A \text{ is in } LB, \text{ where } \kappa_0 \text{ is the initial context}$$

$$(\text{Glob}) \frac{\vdots}{\bar{\kappa}[n]: A} \text{ If } A \text{ is in } GB, \bar{\kappa} \text{ is present and extends the initial context}$$

The last two rules, *positive and negative lifting* deal with the essence of contexts. They somehow reproduce the effects of the modal axiom **K** and of the necessitation rule, plus the effect of the extra axiom Δ .

$$(\text{P-lift}) \frac{\bar{\kappa}[n]: \text{ist}(\kappa, A)}{\bar{\kappa}\kappa[m]: A} \text{ If } \bar{\kappa}\kappa[m] \text{ is present in the branch}$$

$$(\text{N-lift}) \frac{\bar{\kappa}[n]: \neg \text{ist}(\kappa, A)}{\bar{\kappa}\kappa[m]: \neg A} \text{ If } \bar{\kappa}\kappa[m] \text{ is new for the branch}$$

Massacci shows that the axiom (Δ) is derivable in his tableaux system, but does not prove syntactic equivalence between the systems PLC and \mathcal{T} : ideally we should like a theorem like $\vdash_{\text{PLC}} A$ iff $\vdash_{\mathcal{T}} A$. To obtain such a theorem we need to show how to derive the rules of positive and negative lifting, using the axioms and rules of PLC.

5.1 Evaluating Massacci’s systems

Massacci claims two main advantages for his system: Firstly that the rules reflect “epistemic properties (lifting, use of assumptions, etc)”. This seems too subjective. But secondly he proves computational properties: the system allows for local and incremental computation, satisfies strong confluence and can be adapted to different search heuristics. These advantages are clear, usually tableaux calculi are better for proof search than axiomatic systems. Also his kind of tableaux were devised for efficient automated theorem proving, which is always useful. Hence it would seem a good idea to constructivize \mathcal{T} and to try to prove the conjecture above that $\vdash_{PLC} A$ iff $\vdash_{\mathcal{T}} A$. But I do not see how to mimick the positive and negative lifting rules of \mathcal{T} using PLC’s axioms and rules, and I guess the proof that this works goes via the semantics. Since Massacci’s work builds on his adaptation [Mas94] of Fitting’s work on prefixed tableaux, this whole theory needs to be used, which is very unsatisfactory.

6 Our Natural Deduction Formulation

The Natural Deduction system of contexts we have developed works only for the normal multimodal \mathbf{K} fragment of PLC, \mathcal{N} and \mathcal{T} , that is, for a system we could call \mathbf{K}_n . Of course one could always add up the axiom Δ to this system, but adding any axiom to a natural deduction system seems a bad idea. Our natural deduction system comprises the usual natural deduction rules for the propositional connectives, plus the following schema of rules, one for each modality $\mathbf{ist}(\kappa, -)$.

$$\frac{\Gamma \vdash \mathbf{ist}(\kappa, A_i) \quad A_1, A_2, \dots, A_k \vdash B}{\Gamma \vdash \mathbf{ist}(\kappa, B)} \square_{\kappa}$$

where by $\Gamma \vdash \mathbf{ist}(\kappa, A_i)$ we mean a sequence of derivations $\Gamma \vdash \mathbf{ist}(\kappa, A_1), \Gamma \vdash \mathbf{ist}(\kappa, A_2), \dots, \Gamma \vdash \mathbf{ist}(\kappa, A_k)$. This is an old formulation of normal modal \mathbf{K} , dating back at least to the mid-eighties [Bel85].

People familiar with the formulation of the necessitation rule for system \mathbf{K} in sequent calculus, need to note that the new rule \square_{κ} “builds-in” substitutions.

The monomodal system using rule \square_{κ} (for a single modality \square) over a constructive basis is discussed in detail in [BdPR01]. On that paper, several possible formulations of a natural deduction formulation for a basic notion of necessity are discussed and compared. In particular a discussion of Fitch-style Natural Deduction [F52], and its formulation as a framework for constructive modal logics versus Prawitz-style Natural Deduction and why we prefer the latter is sketched⁶. The reason is simply that it is not obvious how to provide categorical semantics for Fitch-style Natural Deduction formalisms, whereas it is so for Prawitz-style

⁶ One preliminary answer to how would the Curry-Howard isomorphism help context logics is that a type theory with context modalities could be easily implemented in an interactive theorem prover such as Isabelle or PVS and this would facilitate the creation/interconnection of large repositories of theories.

natural deduction. It is also briefly mentioned in [BdPR01] that we do not discuss approaches to constructive modal logics that use the semantics of modal logics, in terms of Kripke models and accessibility relations, as part of the syntactic information used to characterize these systems. Using the intended semantics to define your syntax may not be cheating, but feels somehow underhand, especially when proving soundness of your system. Approaches along these lines include Gabbay’s labelled deductive systems and Simpson’s framework. Clearly our system is not a framework: we can only do a few modal systems (**K** and **S4**, possibly **KT**, **KD**, **K4**) and indeed rules change according to the system that we are considering. Our only advantage at the moment, when compared to the frameworks mentioned before, is to produce *semantics of proofs* for the systems we can deal with. This was the goal from the beginning.

The multimodal extension of the system does not appear to present any problems. Localized derivation (ie. derivation in context) can also be done, by labelling the turnstyle with a given context $\bar{\kappa}$.

$$\frac{\Gamma \vdash_{\bar{\kappa}} \text{ist}(\kappa, A_i) \quad A_1, A_2, \dots, A_k \vdash_{\bar{\kappa}} B}{\Gamma \vdash_{\bar{\kappa}} \text{ist}(\kappa, B)} \text{loc}\Box_{\kappa}$$

We call the system without localization \mathcal{B} , because of Bellin’s 1984 paper and we can prove that \mathcal{B} satisfies strong normalization/cut-elimination, subformula property and also enjoys a simple categorical semantics, in terms of a cartesian closed category together with a finite collection of endofunctors, one for each modality. We expect similar properties to hold for the localized system, but have not had time to verify it.

7 Comparing Systems

Both PLC and Nayak’s system \mathcal{N} are given as axiomatic systems and comparing them first seems natural. Nayak’s system \mathcal{N} is clearly too simplified to compare with PLC, but given the system \mathcal{N} without axioms **A3**, **A4**, **A5** and PLC, without Δ , do we have the same system? The question hinges on the effect of sequences of contexts, versus individual modalities, decorating the derivability relation. As we have seen the context switching rule of PLC can be substituted by the necessitation rule of localized multimodal **K**, if differences of vocabulary are disregarded. But is this as general as usual multimodal **K**?

For instance, if we have two unrelated contexts κ_1 and κ_2 , which can only be concatenated to form $\kappa_1 * \kappa_2$ and A is a theorem, the following derivations are perfectly fine in **K_n**:

$$\frac{\frac{\vdash A}{\vdash C_{\kappa_1} A}}{\vdash C_{\kappa_2} C_{\kappa_1} A} \qquad \frac{\frac{\vdash A}{\vdash C_{\kappa_2} A}}{\vdash C_{\kappa_1} C_{\kappa_2} A}$$

But presumably in PLC, only one of them, would be valid, as if $\kappa_1 * \kappa_2$ is a valid context sequence whereas $\kappa_2 * \kappa_1$ is not, then the context switching rule can

only be applied to $\kappa_1 * \kappa_2$. At least for PLC this seems to be the case, as if the sequence of contexts doesn't matter, they describe it a *flat* model.

Only if all contexts sequences formed from a given set \mathcal{K} are valid and if only the distinct elements of any context sequence matter, ie if the derivability relation denoted by the context sequence $\vdash_{\kappa_1 * \kappa_2 \dots * \kappa_n}$ is equivalent to the derivability relation denoted by any permutation of the sequence $\kappa_1 * \dots * \kappa_n$ then Bouquet and Serafini's claim that "PLC is just the normal multimodal logic \mathbf{K} extended with the Δ axiom" is justified. If "the new theorems proved in PLC with respect to normal multimodal \mathbf{K} are only due to Δ " then PLC is indeed a sublogic of \mathcal{N} and of Massacci's \mathcal{T} and in terms of provability exactly equivalent to Bouquet and Serafini's MPLC.

Comparing Massacci's system \mathcal{T} to PLC, we see that since the axiom \mathbf{D} is not provable in PLC, but is directly derived from negative lifting rule in \mathcal{T} , \mathcal{T} seems to prove more theorems than PLC. Also Massacci has proved that \mathcal{T} proves all theorems that PLC proves. But it is not clear whether \mathcal{T} proves only these.

8 Conclusions

Comparing the four systems in this note, it seems that a designer/user of a logic of contexts has plenty of choice between systems. He may choose not to have the axiom (Δ) at all, in which case our (localized) system, the restricted version of PLC and some variation of the tableaux system \mathcal{T} should all be proved equivalent. This corresponds to a decent minimal logic of contexts. A context logic can also have the axiom Δ explicitly, as in PLC, or have its effect via multimodal \mathbf{KD} , as Massacci's system seems to indicate. If the effect of Δ is desired, the second route may be best, as one has at least the axiomatic and the tableaux versions already in place. Finally our context logic user may opt for a simplified logic of contexts along the lines of Nayak's system. In that case, I don't know how to provide a sequent calculus or a Gentzen-style Natural Deduction formulation. Proof-theoretic tricks, as taking the formulas of the system considered only up to the equivalence relation that identifies $\Box_i A$ with $\Box_j \Box_i A$ can be used, but the effect is not elegant. Lastly the comparison between our system and the MCS/LMS work deserves more discussion that could be given here [deP]. Briefly the MCS/LMS systems seem to be able to embed all traditional modal logics, constructively or not, very easily, in what is a generalization of Natural Deduction. But it is not clear to me how to decorate MCS/LMS proofs with terms in a Curry-Howard isomorphic way. This, as well as proof semantics for those systems is subjecto for further work.

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