Relating algebraic models of intuitionistic predicate logic

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This note sets out to explain the identity criteria for logics that we arrived at, when considering the problem of deciding when two logical systems should be taken as "the same" from our perspective in categorical logic.

It is well known that a logic can have different kinds of models for which it is sound and complete. If we consider intuitionistic predicate logic, for example, complete Heyting algebras and topological spaces give rise to two examples of sound and complete models. When considering complete Heyting algebras, it is also well known that the algebra of formulas of the logic does not naturally satisfy the requirements of a complete Heyting algebra. In this case we need to 'complete' the algebra of formulas to get a complete Heyting algebra. In a certain sense (to be made precise in the paper) the structure present in the class of complete Heyting algebras is richer than that offered by the calculus. The same can be said to hold for, say, topological spaces models. But if we consider categorical models for intuitionistic predicate logic as given by Lawvere in terms of hyperdoctrines in "Adjointness in Foundations", Dialectica, 1969 we can see that this mismatch disappears: indeed, universal and existential quantifiers are modelled by adjunctions and the algebra of formulas naturally have these adjunctions. More mathematically, using category theory, we can express this perfect match between syntax and semantics by saying that if we organize predicative intuitionistic logic into a category of suitable theories and translations, we can see that this category is equivalent to the category of a certain kind of hyperdoctrines, called representable hyperdoctrines, with its associated morphisms. Hence, we say that intuitionistic predicate logic provides the *internal language* of representable hyperdoctrines.

More generally we say that a logic provides the internal language of a class of categorical models if the category of theories built upon it is equivalent to its category of models. In this way the category of models of a logic is *uniquely* identified up to categorical equivalence.

Thus we notice that category theory offers a correspondence between a logic and its models much tighter than the traditional correspondence given by soundness and completeness theorems alone. Soundness and completeness do not necessarily uniquely identify a class of models, with or without apropriate morphisms.

The syntactic category of formulas of intuitionistic predicative logic fails to be a complete Heyting algebra due to the fact that models based on complete Heyting algebras do not have theories of intuitionistic predicative logic as their internal language. To prove this we show that the category of models based on complete Heyting algebras is not equivalent to the category of representable hyperdoctrines, but it is, instead, in a suitable adjunction with it. An analogous correspondence can be said to hold for classical predicative logic. Contrary to the predicative case, in the propositional case, Heyting algebras (Boolean algebras) have already theories of propositional intuitionistic (classical) logic as their internal language(s). This can be verified by the fact that the propositional algebras of formulas have already the needed algebraic structure.

In conclusion we propose to define the *semantics of a logic* $\mathcal{L}$ as the class of models uniquely identified by the internal language criterion as its most general class of models. As a corollary of this kind of definition (which involves proving a few theorems) we get that two logics $\mathcal{L}$ and $\mathcal{L}'$ are equivalent iff the category of theories of $\mathcal{L}$ is equivalent to the category of theories of $\mathcal{L}'$. We also have that equivalent logics have equivalent categories of models, as it was to be expected.

For readers not too interested in the categorical apparatus, the lesson to take home is that, given a logic $\mathcal{L}$, assuming one wants to use categories to organize semantics, assuming one has managed to produce a decent (sound and complete) notion of model for the logic in question, the work is not finished, yet. If one is interested in freely moving from syntax to semantics (and backwards) one must prove some version of the internal language result described above. Once this is done, we can talk about "the" most adequate categorical semantics for a logic, which also induces a well-behaved notion of equivalence between logics.

This criterion has been already used to classify models of intuitionistic linear logic in our paper below.

References

F. W. Lawvere. Adjointness in Foundations. *Dialectica*, 23, 281–296, 1969.

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