Modalities in Constructive Logics and Type Theories

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This volume of the Journal of Logic and Computation collects together papers on constructive modal logics, modal type theories and their applications. The idea for this volume arose out of the Second Workshop on Intuitionistic Modal Logics and Applications (IMLA'02 [GMdP02]) held in Copenhagen in June 2002. As already with the first workshop at Trento three years earlier (IMLA'99 [FMM99, FMM01]) it was felt that many of the ongoing activities in this young and exciting research area should be presented to a wider audience in the form of a special journal issue. In view of the diversity of the topic a general call for papers was issued. Papers presented at the workshop as well as new papers have been submitted and reviewed, following the stringent criteria of this journal.

The Topic Since our topic is a new, interdisciplinary subfield of logic, computer science and philosophy we believe that we should as least discuss some questions that the researchers interested in constructive modal logics and modal type theories might pose themselves.

- What do we mean by constructive modal logics and associated modal type theories?
- What are the boundaries of this emergent subfield?
- Which are the important problems for this area?
- Why should any one pursue a programme of investigation in this area?
- Can we see ways of increasing the interaction between the largely parallel communities (computer scientists and traditional logicians) involved?
- Most importantly, what are the trends for the future?

The aim of this editorial is not to answer these questions, but simply to give any interested researcher a few pointers to the literature since the questions are hard, several answers are possible and many are already addressed in the literature.

To begin with, to explain properly what is meant by “constructive modal logics and associated modal type theories”, we would have to explain what we take to be constructive modal logics (and definitions differ even among the three authors of this note), what are “modal type theories” (again, several non-equivalent definitions are possible) and then we would have to explain the operative word “associated”.

It is fairly clear what a modal logic is, but what do we mean by "constructive"? There is not a single, generally accepted definition of constructiveness. Is it the disjunction and existential properties that matter or that the logic admits a realisability interpretation? Does a classical set-theoretic realisability interpretation (Kolmogorov) suffice, does it have to be recursive functions (Heyting), or perhaps proof polynomials (Artemov)? Whatever the interpretation and separation of the adjectives non-classical, intuitionistic, and constructive, there is presumably consensus among scientists that "constructive" implies the existence of some kind of proof theory with explicit computational content. Proofs carry concrete intensional information over and above the fact that a formula is valid. From this point of view modalities come in naturally, viz. as syntactic reflections of this underlying intensional level hidden in the proof theory of a constructive logic. A constructive modal logic might arise in an attempt to handle the intensional dimension, or parameter, of modalities (time, nondeterminism, security margin, knowledge, beliefs, etc.) in a computational way at the level of the proof theory. It seems obvious that such a program has much to offer not only from a computer science perspective.

Modal type theories start from a different perspective. There, the primary focus is on the computational level, typically some lambda-calculus, which is thought of as a programming language. Types act as interface specifications for a program or a program part constraining the permissible contexts in which it can be used (static type checking). The more such a lambda-calculus is intended to model a real programming language the more the need arises to handle implementation-related, intensional (some would say "impure") features like side-effects, non-determinism, execution complexity, strictness and uniqueness of reference to function parameters, exception handling, continuations, etc. If these issues are taken seriously at the level of the calculus itself (rather than at the "implementation" level as it was the case, traditionally, at least throughout the 1980s) then we obtain some variant of typed lambda-calculus with extra intensional operators. The type theory must then be extended to reflect these intensional operators. This is the birth of modal type theory. Generally speaking, as the existing literature proves convincingly, modal types are a flexible mechanism to control and integrate different intensional features consistently and coherently within the same calculus. This has taken away much of the initial unease among lambda calculus purists to translate implementation-related phenomena into a functional setting; once this barrier has passed there is hardly a limit to the researcher's imagination.

Constructive modal logics and modal type theories, by themselves, are flourishing research themes. Yet, just like their modal-free predecessors, they may be intimately linked through the Curry-Howard Correspondence which creates many opportunities for cross-fertilisation that have hardly been fully explored yet. When we erase the terms from (perhaps some class of typing) derivations in a modal type theory, they look just like Natural Deduction proofs in an associated constructive modal logic. This "modal logic of inhabitation" can be used to single out a specific type theory among many other rivals addressing the same intensional features, but in different ways. The model theory of the associated modal logic, developed perhaps by logicians, may suggest new ways of interpreting the computational mechanism that generated the modal type theory in the first place. For the logician, on the other hand, a modal type theory (or some variation of it) may provide a new, or perhaps the first, computational proof-theory for a well-known modal logic. It is known that good proof systems for modal logics are notoriously hard to come by. Modal type theories may have something to offer here.

Some of this is wild speculation, of course, but the kind of speculation that drives research. Also pragmatic motivation drives research. While some of us feel that the relationship between logics and their associated type theories – via extensions of the Curry-Howard isomorphism – is the main target, others are more interested in the constructivization of the traditional modalities, either for specific areas of application or for general philosophical and mathematical considerations.

For some of the researchers involved, constructive modal type theories are simply a way of extending functional programming so as to do exception-handling, continuations, etc, in a principled way. For others, constructive modal logics should be developed simply because modal logics and constructive logics "belong together" as the most successful non-classical logics. While some would like to see constructive modal logics only as demanded by whatever application is at hand, be that circuit design or grid computing for example, others would use the mathematical elegance of the systems or of the models obtained as their over-riding criteria for investigation. We believe that there is room and reason for all these conceptions of constructive modal logics and modal type theories and would like to see all of them flourishing. To such an incipient field, the thrashing of ideas, as diverse as they may be, seems to us the only way to grow.

The Context To offer a list of pointers to the work in our area, we start by looking back at the recent history of the topic. Pioneering work was done by Fitch [Fit48], Curry [Cur52] and Prior [Pri57]. Or a little more recently one could look at the work of Bull [Bul65], Fischer Servi [Ser77], Ono [Ono77], Goldblatt [Gol81], Macnab [Mac81], Božić and Došen [BD84], Font [Fon86] or Ewald [Ewa86]. All of these can be considered mathematical investigations of constructive modal logics, despite the fact that they have very different motivations and outcomes.

Simpson's thesis [Sim94] contains a detailed survey of previous work on constructive modal logics, including the pioneering Plotkin and Stirling [PS86], as well as a framework for a large class of intuitionistic modal logics. An alternative framework, also based on natural deduction was presented by Viganò [Vig00], building on previous work with Basin and Matthews. Viganò's results offer a bridge to the concept of labelled deductive systems as proposed by Gabbay and co-workers. Unlike Simpson's investigations, the work of Viganò et al (and Gabbay et al) is not restricted to the constructive setting.

Roughly at the same time, Artemov [Art99] offered his brand of connection to the older proof-theoretical tradition of Gödel and Gentzen. Artemov's suggestion is to add what he calls "proof polynomials" to intuitionistic logic, as a mechanism for self-reference. The claim here is that proof polynomials extend the Curry-Howard isomorphism and unify intuitionistic and S4 modal logic in a modular way. A different perspective, firmly based on possible worlds semantics, was given by Wolter and Zakharyaschev. In a series of three papers [WZ97, WZ99b, WZ99a] they propose to study intuitionistic modal logics as fragments of classical bimodal logics. The approach is elegant, but only deals with classes of intuitionistic modal logics that satisfy distribution of possibility over disjunction, a premise disputed in some of the intuitionistic modal logics. In particular, Wijesekera's doctoral work [Wij90] under Nerode on a constructive modal logic designed for program verification, called "constructive concurrent dynamic logic", rejects distribution of diamond over disjunction.

Concurrently with Wijesekera's work new impulses for research on constructive modalities came from computer science, notably through the work of Moggi on computational monads [Mog91] and the development of the Curry-Howard isomorphism for Linear Logic by Benton, Bierman, de Paiva and Hyland [BBdPH93] together with the immediate realization by Bierman and de Paiva [BdP00] that this also provided a Curry-Howard isomorphism for constructive S4. Moggi's seminal insight was that intensional operators (modelled by categorical monads) in the simply typed lambda-calculus provide a uniform way of dealing with "impure" notions of computation (as opposed to pure values) in functional programming languages. It was then observed, independently, by Benton, Bierman and de Paiva [BBdP98], Fairtlough and Mendler [FM97] and by Kobayashi [Kob97] that the type system introduced by Moggi could be seen as a constructive modal logic, where a computation type corresponds to a constructive logical possibility, already proposed by Curry [Cur52] but abandoned because of its unusual properties. While Benton, Bierman and de Paiva's system, called CL, was motivated by functional programming, Fairtlough and Mendler's system, dubbed PLL (for propositional lax logic), arose from applications to formal hardware verification and the analysis of abstraction constraints according to the proofs-as-constraints principle (Mendler [Men91]). Both CL and PLL turned out to be equivalent to Goldblatt's "geometric modality" [Gol81] mentioned above. Kobayashi's system CS4, on the other hand, is a constructive version of S4. The exact relationship between CL/PLL and CS4 was studied by Alechina et al. in [AMdPR01]. It is interesting to note that all these logics like Wijesekera's (for good reasons) abandon distribution of possibility over disjunction, i.e., the axiom $\Diamond(A \vee B) \rightarrow (\Diamond A \vee \Diamond B)$, a property that was considered a *sine qua non* even for intuitionistic modal logics (see e.g., [FS80, PS86, Sim94]) for a long time.

An entirely different technique of generating constructive modalities from notions of computation is Pitts' Evaluation Logic [Pit91, Mog95]. There, the computational lambda-calculus does not play the role of a calculus of proofs of some modal logic (propositions-as-types analogy) but as the object language of a modal predicate logic. The modalities, in a traditional sense, permit qualification of truth to reason directly about computational effects of object-level terms. Further recent developments are by Plotkin and Power [PP03].

This new wave brought on board computer scientists interested in extending the simply typed lambda-calculus with intensional operators to get modal type theories. A seminal paper by Pfenning and Wong [PW95] described many possible applications of modal constructive type theories. Some of these have been pursued, others remain future work.

Benton [Ben96] exploits Moggi's computational monads and the associated distinction between values and computations to perform strictness analysis in a simply typed functional language. This relies on the observation that the well-known "lifting" modality permits a coherent two-level distinction between expressions in weak head normal form (values) and arbitrary unevaluated expressions (computations). Hatcliff and Danvy [HD97] interpret these two levels as "static" (values) and "dynamic" (computations) evaluation, and obtain a type system for binding-time analysis and partial evaluation. The generalisation of this idea naturally leads to the notion of "multi-stage" computations developed by Davies and Pfenning [Dav96, DP96]. Interestingly, these "staging" approaches use different modalities. Hatcliff, and Danvy (like Benton) build on computational monads, i.e., instances of the CL/PLL modality of possibility, while the intensional lambda-calculi λ°, $\lambda^{\Box}$ of Davies and Pfenning use a next-stage modality $\circ$ borrowed from linear temporal logic and, respectively, a necessity modality $\Box$ related to CS4. Roughly speaking, one can say that a possibility-style modality arises when some intensional (impure) feature is to be reflected inside an otherwise pure lambda calculus, while a necessity-style modality captures the fragment of pure terms inside an ambient impure system. The right point of view depends on taste and application. In general, one would expect combinations of modalities as suggested in the work of Benaissa, Moggi, Taha, Sheard, Fagorzi on MetaML [BMTS99, TS00, MF03] where computational monads are integrated with staging modalities.

Just like the monads of possibility, a necessity $\Box$, too, allows several different interpretations in an intensional lambda calculus. In the work of Despeyroux, Pfenning and Schürmann on higher-order abstract syntax [DPS97] the constructive $\Box A$ singles out those terms of type A that are "closed", i.e., which are completely built from constructors over which primitive recursion is well-defined.

Another important class of computer science applications for modal type theory links modalities with the quantitative analysis of computational resources. Gurr's Ph.D. thesis [Gur91] is an early example in this direction, similar in spirit to Mendler's work on timing analysis [Men00b, Men00a] which involves the $\Diamond$ modality (strong monad). A quite different and exciting research thread building on $\Box$-modalities has been opened up by the striking results of Hofmann [Hof97, Hof99] showing that linearity and modality restrictions on structural recursion can characterise polynomial-time complexity for functional programs.

Besides the intensional lambda calculi there is also the topological and algebraic point of view. The work on the Curry-Howard interpretation, because of the so-called *extended version* of the Curry-Howard isomorphism, relating categories to typed lambda-calculi and to logics, brought about results of a more categorical flavour. As examples we can cite Goubault-Larrecq and Goubault's intriguing models of intuitionistic S4 using simplicial sets and lower-dimensional topology [GLG03], Hilken and Rydeheard's models [HR99] that take as fundamental the duality between topological spaces and frames and Hermida's generalisation of frames to categorical spans [Her02].

More general investigations into the computational meaning of modal proofs, have been pursued by Borghuis [Bor94] (using Fitch-style natural deduction) and Martini and Masini [MM96]. Proof-theoretic explications of constructive modalities as reflections of external meta-theoretic constructions have been given by Benevides and Maibaum [BM92] for necessity and by Masini [Mas93] for both necessity and possibility introducing the notion of 2-sequents. Regarding specific systems, Pfenning and Davies [PD00] have given a judgmental reconstruction, and thus a philosophically respectable reading of CS4 and PLL/CL. Their modal type theory is related to Plotkin and Barber's ideas for the necessity-only fragment (i.e., a separation of the context of assumptions into modal and non-modal ones) and an ingenious use of the meta-level capabilities to deal with the possibility operator. A different pattern inspired by Curry was identified by Fairtlough and Mendler, who have shown that the modality of PLL/CL can be characterised fully in terms of implicational and disjunctional syntactic contexts [FM02].

A whole new area concerned with topological dynamic systems and the several (non-equivalent) ways of obtaining it, by putting together the topological interpretation of S4 due to McKinsey and Tarski and some tense logic operators is emerging right now as a way to deal with hybrid control systems. Work in this area, of a constructive flavour, can be found in [DCM$^+$02].

Finally on applications of constructive modal logics to Mathematics itself there is the work of Scott's group in Pittsburgh and of Sambin's group in Padova. The group around Scott proposes to develop a logical framework for type theories and computability that includes both standard mathematical spaces and the many constructions and spaces developed exclusively for domain theory and type theory. Their goal is a 'rapprochement' between topology as developed by computer scientists (e.g. domain theory) and topology as in usual Mathematics. This rapprochement should help facilitate the study of computable operations on data that is not necessarily computable, such as the real numbers. Meanwhile research in Padova has concentrated on formal topology, a rethinking of topology firmly based on Martin-Löf's constructive type theory, providing a new interpretation of modalities as a side-product.

The Papers The contributions in this special volume of the Journal of Logic and Computation exemplify developments of some of the lines of research mentioned above.

The paper by Schröder and Mossakowski begins by recalling the encapsulation of side-effects in functional programming using monads. They build on that to design an intuitionistic monad-independent dynamic logic and a representation of this logic in the internal logic of HasCasl. The reason for designing this special kind of dynamic logic is that dynamic logic can express not only correctness but also termination of programs. As an example of the applications they intend for their logic, they give a termination proof for Dijkstra's non-deterministic version of Euclid's algorithm.

The paper by Awodey and Bauer, on the other hand, exemplifies an application of modal type theory to a problem in mathematical logic itself: the question of the extent to which the propositions-as-types interpretation of Martin-Löf's type theory is conservative over intuitionistic first-order logic. Their partial solution (a "lower bound" so to speak) to this problem is that conservativity holds for the class of *left stable* formulas. To prove this result they use the bracket type constructor, which can be seen as a modality that erases computational content.

The papers by Bellin and Brunet exemplify applications of (traditionally conceived) constructive modal logics to new areas. Bellin investigates a logic of pragmatics, hence an application to philosophy, while Brunet's logic is geared towards knowledge representation with partial knowledge.

Bellin's logic of pragmatics, based on previous work of the author, Dalla Pozza and Garola, extends classical logic by modal operators that intuitively correspond to impersonal acts of asserting and conjecturing. The paper here develops a modal system for these new operators and, using the traditional modal translations of Gödel, McKinsey, Tarski and Kripke, proves that (fragments of) the logic of pragmatics are sound and complete with respect to Kripke's semantics for S4.

In contrast, Brunet's work proposes two intuitionistic modal logics, which emerge in his work, from considerations of partiality of descriptions of generic systems in knowledge representation. From his proposed use of Galois connections as a general way of modelling approximation come the definitions of *(closed and simply) representational systems* and these in turn give rise to the two logics, for which they are sound and complete models.

Finally, Crolard's paper shows the trend hinted at above of considering the (Curry-Howard) correspondence between the type theories and the logics as more important than the intensional modal operators themselves. Crolard considers the correspondence between bi-intuitionistic logic (i.e. intuitionistic logic with an extra operator of co-implication) and a type system for first class coroutines, a restricted form of continuations. This intriguing programming application of a logical notion exemplifies some of the strengths of the Curry-Howard correspondence as a paradigm for research in logic and computing.

Most of the lines of investigation above are still open, with new results and new application areas appearing all the time. Tempting as it is to speculate on the most promising areas for future investigation, we will resist temptation here and simply finish hoping for more meetings dedicated to the theme, as this seems clearly to have more to offer.

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