A bifibrational reconstruction of Lawvere's presheaf hyperdoctrine

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Adjointness in foundations

Bill Lawvere, *Dialectica* 23, 1969.

Conjunction and implication (ccc):

$$A \times - \quad \dashv \quad A \Rightarrow -$$

Quantifiers as adjoints to substitution (hyperdoctrine):

$$\Sigma_f \quad \dashv \quad \mathscr{P}_f \quad \dashv \quad \Pi_f$$

The subset hyperdoctrine

Define a functor $\mathscr{P}_A : \mathscr{B}^{op} \to \mathbf{Cat}$, where $\mathscr{B} = \mathbf{Set}$, as follows:

For any set A, let $\mathscr{P}_A$ be the category of subsets $R \subseteq A$, ordered by inclusion.

For any $f : A \to B$ and $S \subseteq B$, define $\mathscr{P}_f\, S \subseteq A$ by

$$\mathscr{P}_f\, S \stackrel{\text{def}}{=} \{\ a \in A \mid fa \in S\ \}$$

For any $R \subseteq A$ and $f : A \to B$, define $\Sigma_f\, R \subseteq B$ and $\Pi_f\, R \subseteq B$ by

$$\Sigma_f\, R \stackrel{\text{def}}{=} \{\ b \in B \mid \exists a \in A,\ \ fa = b \ \wedge\ a \in R\ \}$$
$$\Pi_f\, R \stackrel{\text{def}}{=} \{\ b \in B \mid \forall a \in A,\ \ fa = b \ \Rightarrow\ a \in R\ \}$$

Moreover, each $\mathscr{P}_A$ is cartesian closed.

Bifibrations

A **bifibration** is just a special kind of *type refinement system*[1]

$$p \quad : \quad \mathscr{E} \quad \longrightarrow \quad \mathscr{B}$$

equipped with operations

$$\frac{R \sqsubset A \quad f : A \to B}{\mathbf{push}_f R \sqsubset B} \qquad \frac{f : A \to B \quad S \sqsubset B}{\mathbf{pull}_f S \sqsubset A}$$

and a one-to-one correspondence of derivations:

$$\frac{R \underset{f;g}{\Longrightarrow} R'}{\mathbf{push}_f R \underset{g}{\Longrightarrow} R'} \qquad \frac{S' \underset{e;f}{\Longrightarrow} S}{S' \underset{e}{\Longrightarrow} \mathbf{pull}_f S}$$

[1] See MZ POPL 2015 + arXiv:1501.05115 for details.

One hyperdoctrine decomposed into two bifibrations

Any hyperdoctrine $\mathscr{P} : \mathscr{B}^{op} \to \mathbf{Cat}$ can be decomposed into a pair of bifibrations over $\mathscr{B}$ and $\mathscr{B}^{op}$.

In the case of the subset hyperdoctrine, one obtains

$$p^{\oplus} : \mathbf{SubSet}^{\oplus} \to \mathbf{Set} \qquad p^{\ominus} : \mathbf{SubSet}^{\ominus} \to \mathbf{Set}^{op}$$

where $\mathbf{SubSet}^{\oplus}$ and $\mathbf{SubSet}^{\ominus}$ have $(A, R \subseteq A)$ as objects, and morphisms $f : (A, R) \longrightarrow (B, S)$ given by functions $f : A \to B$ s.t.

$$\forall a \in A, \quad Ra \Rightarrow S(fa)$$

for $\mathbf{SubSet}^{\oplus}$, and functions $g : B \to A$ s.t.

$$\forall b \in B, \quad R(gb) \Rightarrow Sb$$

for $\mathbf{SubSet}^{\ominus}$.

From functions to relations

Consider the two (faithful but not full) embedding functors

$$\mathrm{emb}^{\oplus} : \mathbf{Set} \to \mathbf{Rel} \qquad \mathrm{emb}^{\ominus} : \mathbf{Set}^{op} \to \mathbf{Rel}$$

which send a set to itself, and a function $f : A \to B$ to the relations

$$f^{\oplus} : A \nrightarrow B \qquad f^{\ominus} : B \nrightarrow A$$

where

$$\begin{array}{rcl} f^{\oplus} & = & \{ (a, b) \in A \times B \mid fa = b \} \\ f^{\ominus} & = & \{ (b, a) \in B \times A \mid b = fa \} \end{array}$$

Notation: we write $M : A \nrightarrow B$ for a binary relation $M \subseteq A \times B$ which defines a morphism $A \to B$ in the category **Rel**.

A subset bifibration over sets and relations

We construct a bifibration

$$p \quad : \quad \mathbf{Rel}_\bullet \quad \longrightarrow \quad \mathbf{Rel}$$

where the category $\mathbf{Rel}_\bullet$ has objects the pairs (A, R) consisting of a set A together with a subset $R \subseteq A$, and morphisms

$$M \quad : \quad (A, R) \to (B, S)$$

given by binary relations $M : A \nrightarrow B$ satisfying the property

$$\forall a \in A, \forall b \in B, \quad (\, M(a, b) \ \wedge \ Ra \,) \ \Rightarrow \ Sb.$$

A subset bifibration over sets and relations

Given a binary relation

$$M \quad : \quad A \quad \nrightarrow \quad B$$

the adjoint pair of functors

$$\begin{array}{ccccccc} \exists_M & = & \mathbf{push}_M & : & \mathscr{P}_A & \longrightarrow & \mathscr{P}_B \\ \forall_M & = & \mathbf{pull}_M & : & \mathscr{P}_B & \longrightarrow & \mathscr{P}_A \end{array}$$

are defined in the following way:

$$\begin{array}{rcl} \exists_M R & = & \{ b \in B \mid \exists a \in A, \; M(a,b) \wedge Ra \} \\ \forall_M S & = & \{ a \in A \mid \forall b \in B, \; M(a,b) \Rightarrow Sb \} \end{array}$$

for all subsets $R \subseteq A$ and $S \subseteq B$.

A subset bifibration over sets and relations

The key observation is that $\Sigma_f = \exists_{f\oplus}$ and $\Pi_f = \forall_{f\ominus}$. By uniqueness of adjoints, this implies:

$$\forall_{f\oplus} = \mathscr{P}_f = \exists_{f\ominus}$$

Hence the adjoint triple

$$\Sigma_f \quad \dashv \quad \mathscr{P}_f \quad \dashv \quad \Pi_f$$

can be decomposed into a pair of adjunctions

$$\exists_{f\oplus} \quad \dashv \quad \forall_{f\oplus} \quad = \quad \exists_{f\ominus} \quad \dashv \quad \forall_{f\ominus}$$

together with an equality in the middle.

A subset bifibration over sets and relations

In that way, the subset bifibration

$$p : \mathbf{Rel}_\bullet \to \mathbf{Rel}$$

gains theoretical precedence over the hyperdoctrine

$$\mathscr{P} : \mathbf{Set}^{op} \to \mathbf{Set}$$

Another way of putting this:

$$\begin{array}{ccccc}
\mathbf{SubSet}^{\oplus} & \longrightarrow & \mathbf{Rel}_\bullet & \longleftarrow & \mathbf{SubSet}^{\ominus} \\
\downarrow{\scriptstyle p^{\oplus}} & \lrcorner & \downarrow{\scriptstyle p} & \llcorner & \downarrow{\scriptstyle p^{\ominus}} \\
\mathbf{Set} & \xrightarrow[\mathrm{emb}^{\oplus}]{} & \mathbf{Rel} & \xleftarrow[\mathrm{emb}^{\ominus}]{} & \mathbf{Set}^{op}
\end{array}$$

Monoidal closed refinement systems

A **monoidal closed refinement system** is defined as a functor

$$p \quad : \quad \mathscr{E} \quad \longrightarrow \quad \mathscr{B}$$

between mc cats that strictly preserves the mc structure. Such a refinement system comes equipped with operations

$$\frac{R \sqsubset A \quad S \sqsubset B}{R \otimes S \sqsubset A \otimes B} \qquad \frac{R \sqsubset A \quad T \sqsubset C}{R \multimap T \sqsubset A \multimap C} \qquad \frac{T \sqsubset C \quad S \sqsubset B}{T \circ\!- S \sqsubset C \circ\!- B}$$

and a one-to-one correspondence of derivations:

$$\frac{R \otimes S \underset{f}{\Longrightarrow} T}{S \underset{curry(f)}{\Longrightarrow} R \multimap T} \qquad \frac{R \otimes S \underset{f}{\Longrightarrow} T}{R \underset{rcurry(f)}{\Longrightarrow} T \circ\!- S}$$

Monoidal closed refinement systems

Rel is compact closed, where:

$$A \otimes B \overset{\text{def}}{=} A \times B$$
$$A \multimap B \overset{\text{def}}{=} A^* \otimes B = A \times B$$

Rel$_\bullet$ is symmetric monoidal closed, where:

$$(A, R) \otimes (B, S) \overset{\text{def}}{=} (A \times B, \{ (a, b) \in A \times B \mid Ra \wedge Sb \})$$
$$(A, R) \multimap (B, S) \overset{\text{def}}{=} (A \times B, \{ (a, b) \in A \times B \mid Ra \Rightarrow Sb \})$$

The functor $(A, R) \mapsto A$ strictly preserves the smc structure.

Hence, **Rel**$_\bullet$ $\to$ **Rel** is a smc refinement system.

The bifibrational Day construction

Proposition

If $\mathscr{E} \to \mathscr{B}$ is a monoidal closed bifibration, then every monoid

$$(A, m : A \otimes A \to A, e : 1 \to A) \quad \in \quad \mathscr{B}$$

in the basis determines a monoidal closed structure on the fiber $\mathscr{E}_A$, where the tensor and implication are defined for all $R, S \sqsubset A$ by

$$R \otimes_A S \stackrel{def}{=} \mathbf{push}_m(R \otimes S)$$
$$R \multimap_A S \stackrel{def}{=} \mathbf{pull}_{curry(m)}(R \multimap S)$$

and the tensor unit is defined by $1_A \stackrel{def}{=} \mathbf{push}_e 1$.

The bifibrational Day construction

Every set determines a comonoid

$$(A, \Delta_A : A \to A \times A, !_A : A \to 1) \quad \in \quad \mathbf{Set}$$

and hence a monoid

$$(A, \Delta_A^{\ominus} : A \otimes A \to A, !_A^{\ominus} : 1 \to A) \quad \in \quad \mathbf{Rel}$$

Applying the bifibrational Day construction to this comonoid in $\mathbf{Rel}_\bullet \to \mathbf{Rel}$ recovers the cartesian closed structure on $\mathscr{P}_A$.

From subsets to presheaves

Everything works just as nicely for the presheaf hyperdoctrine:

$$
\begin{array}{ccccc}
\mathbf{Psh}^{\oplus} & \longrightarrow & \mathbf{Dist}_{\bullet} & \longleftarrow & \mathbf{Psh}^{\ominus} \\
{\scriptstyle p^{\oplus}}\downarrow & \lrcorner & \downarrow{\scriptstyle p} & \llcorner & \downarrow{\scriptstyle p^{\ominus}} \\
\mathbf{Cat} & \xrightarrow[\mathrm{emb}^{\oplus}]{} & \mathbf{Dist} & \xleftarrow[\mathrm{emb}^{\ominus}]{} & \mathbf{Cat}^{op}
\end{array}
$$

Here **Dist** is Bénabou's (bi)category of (small) categories and *distributors*, where a distributor $M : A \nrightarrow B$ is defined as a functor $M : B^{op} \times A \to \mathbf{Set}$.

The problem of identity

Lawvere (1970) explained how to define “equality predicates” by

$$\mathbf{I}_A \quad \stackrel{\text{def}}{=} \quad \Sigma_{\Delta_A}(\top_A)$$

in any hyperdoctrine satisfying Frobenius and BC conditions.

Notably, the presheaf hyperdoctrine does not satisfy either of these conditions, and $\mathbf{I}_A$ does not seem to give the “right” notion of equality predicate for that hyperdoctrine (which should really be hom_A, as Lawvere himself acknowledged).

The problem of identity

That the presheaf hyperdoctrine does not satisfy Frobenius + BC « *should not be taken as indicative of a lack of vitality [...] or even of a lack of a satisfactory theory of equality* » for the presheaf hyperdoctrine, but rather « *that we have probably been too naive in defining equality in a manner too closely suggested by the classical conception* » (Lawvere 1970, p. 11).

The problem of identity

An alternative definition of equality can be formulated in any monoidal closed fibration $p : \mathscr{E} \to \mathscr{B}$, by

$$\mathbf{J}_A \quad \stackrel{\mathsf{def}}{=} \quad \langle \mathrm{id}_A \rangle$$

where the "graph" of a morphism $f : A \to B$ in $\mathscr{B}$ is defined by:

$$\langle f \rangle \sqsubset A \multimap B$$
$$\langle f \rangle \stackrel{\mathsf{def}}{=} \mathbf{push}_{curry(f)} \, (\, 1 \,)$$

In the case of $\mathbf{Rel}_\bullet \to \mathbf{Rel}$, we get $\mathbf{J}_A \equiv \mathbf{I}_A$.

In the case of $\mathbf{Dist}_\bullet \to \mathbf{Dist}$, we get $\mathbf{J}_A \equiv \mathrm{hom}_A$.

The problem of identity

Theorem

In any monoidal closed bifibration, there are strong equivalences:

$$\mathbf{push}_f R \equiv \mathbf{push}_{eval}(R \otimes \langle f \rangle) \quad (1)$$

$$\mathbf{pull}_f S \equiv \mathbf{pull}_{dni}(S \multimapinv \langle f \rangle) \quad (2)$$

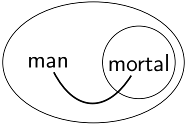
where $eval : A \otimes (A \multimap B) \longrightarrow B$ *is the left evaluation map, and where* $dni : A \longrightarrow B \multimapinv (A \multimap B)$ *is the right currying of* $eval$*.*

(1) is comparable to Lawvere's $\Sigma_f R \equiv \Sigma_{\pi_2}(\mathscr{P}_{\pi_1} R \wedge \mathbf{I}_f)$, which holds in any hyperdoctrine satisfying Frobenius + BC.

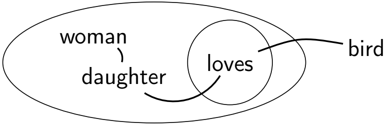
(2) is a generalization of the Yoneda lemma.

Postlude: Peirce's existential graphs

"There's something which is both black and a bird (e.g., a crow)."



"There isn't a man who ain't mortal (i.e., every man is mortal)."



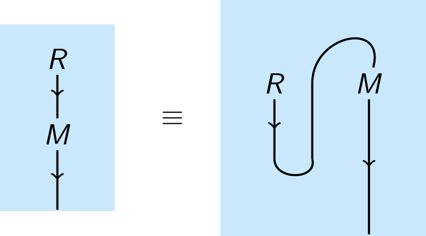
"There is a (very popular) bird that every woman's daughter loves."

We can decompose $\mathbf{Dist}_\bullet \to \mathbf{Dist}$ as a *bifibration chirality* between co- and contravariant presheaves:

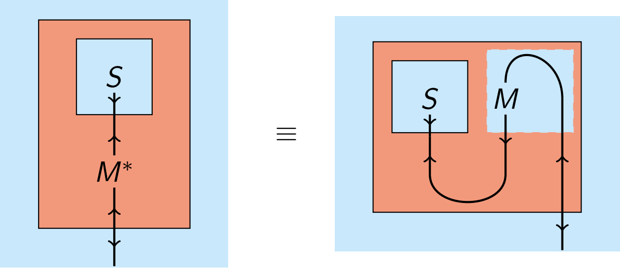
$$\begin{array}{ccc}
\mathbf{Dist}_\bullet & \underset{{}^*(-)}{\overset{(-)^*}{\rightleftarrows}} & \mathbf{Dist}_\circ^{op} \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q^{op}} \\
\mathbf{Dist} & \underset{{}^*(-)}{\overset{(-)^*}{\rightleftarrows}} & \mathbf{Dist}^{op}
\end{array}$$

This leads to a Peircean notation for presheaves, refining an earlier interpretation of existential graphs in terms of Boolean hyperdoctrines by Brady and Trimble.

$$\mathbf{push}_f R \equiv \mathbf{push}_{eval}(R \otimes \langle f \rangle)$$



$$\mathbf{pull}_f S \equiv \mathbf{pull}_{dni}(S \multimapinv \langle f \rangle)$$



Conclusions

- Reconstruct the subset (presheaf) hyperdoctrine, via the smc bifibration $\mathbf{Rel}_\bullet \to \mathbf{Rel}$ ($\mathbf{Dist}_\bullet \to \mathbf{Dist}$).
- Revise Lawvere's axiomatization of equality, yielding $\hom_A$ in the case $\mathbf{Dist}_\bullet \to \mathbf{Dist}$.
- Obtain a Peircean string diagram calculus for presheaves from bifibration chirality beween $\mathbf{Dist}_\bullet \to \mathbf{Dist}$ and $\mathbf{Dist}_\circ \to \mathbf{Dist}$.
- Derive distributivity principles familiar from linear logic.