Polarity and the Logic of Delimited Continuations

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Part I

Questions
Q: What are the meanings of proofs in classical logic?
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A [Kolmogorov, Glivenko-Kuroda, Gödel-Gentzen, ...]:
Derived by $\neg\neg$ translations into intuitionistic logic.
Q: What are the meanings of proofs in classical logic?
A [Kolmogorov, Glivenko-Kuroda, Gödel-Gentzen, …]:
   Derived by ¬¬ translations into intuitionistic logic.

Q: What are the meanings of programs with effects?
**Q:** What are the meanings of proofs in classical logic?

**A** [Kolmogorov, Glivenko-Kuroda, Gödel-Gentzen, ...]:
Derived by $\neg\neg$ translations into intuitionistic logic.

**Q:** What are the meanings of programs with effects?

**A** [Reynolds, Steele-Sussman, Plotkin, ...]:
Derived by CPS translations into lambda calculus.
Of course As beg more Qs: meaning of proofs/programs in IL/λC?
Of course As beg more Qs: meaning of proofs/programs in IL/$\lambda$C?

(Yes, answers are well-known, but we can also dodge the question! . . )
Refining these answers...

Of course As beg more Qs: meaning of proofs/programs in IL/\lambda C?

(Yes, answers are well-known, but we can also dodge the question! . . .)

Idea: directly study **canonical forms** in *image* of translations, e.g., as . . .

- strategies (game semantics)
- focusing proofs (proof theory)

**Polarity** is a guide for describing these canonical forms

Internalized as **polarized logic**
Polarity (¬¬-translation, CPS) plays a role in constructivizing classical logic. Does it have a role in constructive logic?¹

¹Cf. Intuitionistic focusing, Benton’s LNL logic, Watkins’ CLF, Levy’s CBPV, ...
Delimited continuations greatly widen the scope of continuation semantics.² What is their logical structure?

Towards positive answers

Key (simple) idea: study polarity with more than one answer type
- Introduces asymmetry between positive and negative polarity
- Yields different “¬¬”-interpretations of intuitionistic logic
- Positive answer types give rise to monadic effects

Paper works out this idea guided mainly by proof-theoretic principles
- pros: concrete, close connection between syntax and semantics
- cons: perhaps not so transparent, very partial picture
Part II

Review of Classical Polarity
The basic type distinction

\[ P \]
defined by truth
(i.e., datatypes)

vs.

\[ N \]
defined by falsehood
(e.g., records, classes, etc.)
### The basic judgments

<table>
<thead>
<tr>
<th></th>
<th>Logical</th>
<th>Operational</th>
</tr>
</thead>
<tbody>
<tr>
<td>([P])</td>
<td>“P obvious”</td>
<td>value of type (P)</td>
</tr>
<tr>
<td>(\bullet P)</td>
<td>“P false”</td>
<td>continuation accepting (P)</td>
</tr>
<tr>
<td>(N)</td>
<td>“N true”</td>
<td>value of type (N)</td>
</tr>
<tr>
<td>([\bullet N])</td>
<td>“N absurd”</td>
<td>continuation accepting (N)</td>
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<tr>
<td>(#)</td>
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<td>well-typed expression</td>
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</table>
How to explain the meanings of the judgments? Different approaches...
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- definition-by-canonical-forms [most precise, primary in paper]
How to explain the meanings of the judgments? Different approaches...

- definition-by-canonical-forms [most precise, primary in paper]
- definition-by-translation [shortly...]
How to explain the meanings of the judgments? Different approaches... 

- definition-by-canonical-forms [most precise, primary in paper]
- definition-by-translation [shortly...]
- definition-by-handwaving [now!]
“direct proof of $P$”

$[P]$
Intuition

“direct proof of $P$” \hspace{1cm} “direct proof of $P$” \hspace{1cm} $\bullet P$ \hspace{1cm} $\rightarrow$ \hspace{1cm} $\#$
Intuition

“direct proof of $P$”

$[P]$

“direct proof of $P$” $\rightarrow \#$

$\bullet P$

“direct refutation of $N$”

$[\bullet N]$
Intuition

“direct proof of $P$”

$[P]$ 

$\leadsto$ 

$\#$

“direct refutation of $N$”

$N$ 

$\Rightarrow$ 

$\#$ 

“direct refutation of $N$”

$[\bullet N]$
"direct proof of $P$"  \[\Rightarrow\]  \# $
abla P$

"direct refutation of $N$"  \[\Rightarrow\]  \# \[\Rightarrow\]  \# \[\Rightarrow\]  \# $\nabla N$

\[\Rightarrow\]  \# $\nabla N$

\[\Rightarrow\]  \# $\nabla N$
Definition by translation

Target: fragment of intuitionistic logic (or intuitionistic linear logic)

Given translations \( P^+ \) and \( N^- \) [next slide], translate judgments \( J^* \) by:
Target: fragment of intuitionistic logic (or intuitionistic linear logic)

Given translations $P^+$ and $N^-$ [next slide], translate judgments $J^*$ by:

$$[P]^* = P^+$$
Definition by translation

Target: fragment of intuitionistic logic (or intuitionistic linear logic)

Given translations $P^+$ and $N^-$ [next slide], translate judgments $J^*$ by:

$[P]^* = P^+$  \quad \bullet P^* = P^+ \supset \#$

(where $\#$ a distinguished logical atom)
Target: fragment of intuitionistic logic (or intuitionistic linear logic)

Given translations $P^+$ and $N^-$ [next slide], translate judgments $J^*$ by:

\[
\begin{align*}
[P]^* &= P^+ \\
\bullet P^* &= P^+ \supset \# \\
[\bullet N]^* &= N^-
\end{align*}
\]

(where $\#$ a distinguished logical atom)
Definition by translation

Target: fragment of intuitionistic logic (or intuitionistic linear logic)

Given translations $P^+$ and $N^-$ [next slide], translate judgments $J^*$ by:

$$[P]^* = P^+ \quad \bullet P^* = P^+ \supset #$$

$$N^* = N^- \supset # \quad [\bullet N]^* = N^-$$

(where $#$ a distinguished logical atom)
Definition by translation

Target: fragment of intuitionistic logic (or intuitionistic linear logic)

Given translations $P^+$ and $N^-$ [next slide], translate judgments $J^*$ by:

$$[P]^* = P^+ \quad \bullet P^* = P^+ \supset \#$$
$$N^* = N^- \supset \# \quad [\bullet N]^* = N^-$$
$$\#^* = \#$$

(where $\#$ a distinguished logical atom)
Definition by translation

Some connectives:

\[ 1^+ = T = \bot^- \quad 0^+ = F = \top^- \]

\[ (P_1 \otimes P_2)^+ = P_1^+ \land P_2^+ \quad (N_1 \otimes N_2)^- = N_1^- \land N_2^- \]

\[ (P_1 \oplus P_2)^+ = P_1^+ \lor P_2^+ \quad (N_1 \& N_2)^- = N_1^- \lor N_2^- \]

\[ (N^-)^+ = N^- \quad (P \rightarrow N)^- = P^+ \land N^- \]

\[ (\downarrow N)^+ = N^- \cup \# \quad (\uparrow P)^- = P^+ \cup \# \]
The classical connection

Define “polarity-collapsing” translation:

\[\begin{align*}
\otimes &= \& = \land \\
\oplus &= \circ = \lor \\
\rightarrow &= \supset \\
\neg \perp &= \neg \\
\downarrow &= \uparrow = \cdot
\end{align*}\]

Proposition

\[\vdash^c N \iff \vdash^i N^* \quad \vdash^c \neg P \iff \vdash^i (\bullet P)^*\]

Punchline: different polarizations yield different \(\neg\neg\)-translations
Definition by canonical forms

Contexts $\Delta, \Gamma ::= \cdot | \Delta_1, \Delta_2 | N | \bullet P$

\[
\begin{align*}
\Delta \vdash [P] & \quad \Gamma \vdash \Delta \\
\Gamma \vdash [P] & \\
\Delta \vdash [\bullet N] & \quad \rightarrow \quad \Gamma, \Delta \vdash \# \\
\Gamma \vdash N & \\
\Delta \vdash [\bullet N] & \quad \rightarrow \quad \Gamma, \Delta \vdash \# \\
\Gamma \vdash [\bullet N] & \\
N \in \Gamma & \quad \Gamma \vdash [\bullet N] \\
\Gamma \vdash \# & \\
\bullet P \in \Gamma & \quad \Gamma \vdash [P] \\
\Gamma \vdash \# & \\
\Gamma \vdash \cdot & \quad \Gamma \vdash \Delta_1, \Delta_2
\end{align*}
\]
Definition by canonical forms

\[
\begin{align*}
 p & \quad \sigma & \quad p[\sigma] & \quad \frac{p}{\nu} & \quad \nu \frac{K^-}{E} \\
p & \quad \nu & \quad k \quad V^+ & \quad \frac{k}{\nu} & \quad \nu \frac{V^+}{k} \\
p & \quad \nu & \quad K^- & \quad \nu \frac{K^-}{E} \\
p & \quad E & \quad \frac{1}{p} & \quad \frac{1}{E} & \quad \frac{E}{p} & \quad \frac{E}{p} \\
p & \quad \nu & \quad K^- & \quad \nu \frac{K^-}{E} \\
\end{align*}
\]
Part III

Towards Generalized Polarity
Elegant symmetry or silly redundancy?

\[
\begin{array}{c|c}
P & P \\
N & N
\end{array}
\]
Elegant symmetry or silly redundancy?

\[
\begin{array}{c|c}
[P] & \bullet P \\
\hline
N & [\bullet N]
\end{array}
\]

\[\iff\]

\[
\begin{array}{c|c}
[P] & \bullet P \\
\hline
N & [\bullet N]
\end{array}
\]

\[
\begin{array}{c|c}
[P] & \bullet P \\
\hline
N & [\bullet N]
\end{array}
\]

Before giving up on our intuitions, let's think about "contradiction"...
Elegant symmetry or silly redundancy?

\[
\begin{array}{c|c}
[P] & \bullet P \\
\hline
N & [\bullet N]
\end{array}
\]

\[\Leftrightarrow\]

\[
\begin{array}{c|c}
[P] & \bullet P \\
\hline
N & [\bullet N]
\end{array}
\]

Before giving up on our intuitions, let’s think about “contradiction” \#...
If you assume contradictory axioms, you can derive anything. It's called the principle of explosion.

Hey, you're right! I started with $P\land \neg P$ and derived your mom's phone number! That's not how that works.

Wait, this is her number! How-

Hi, I'm a friend of... why, yes, I am free tonight!

Mom!

No, box wine sounds lovely!
Key (simple) idea:

\[
\# \rightsquigarrow P \\
\bullet A \rightsquigarrow A \triangleright P
\]

Perfect symmetry between positive and negative broken!
The basic judgments

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</tr>
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- "P obvious"
- "P false"
- "N true"
- "N absurd"
- "contradiction"
## The basic judgments++

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<tr>
<td>(P)</td>
<td>expression of type (P)</td>
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</table>

- "\(P\) obvious"
- "\(P\) false"
- "\(N\) true"
- "\(N\) absurd"
- "\(P\) true"
## The basic judgments++

<table>
<thead>
<tr>
<th>Logical Interpretation</th>
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<tbody>
<tr>
<td>$[P]$ $P_1 \triangleright P_2$</td>
<td>value of type $P$</td>
</tr>
<tr>
<td>$N$</td>
<td>continuation from $P_1$ to $P_2$</td>
</tr>
<tr>
<td>$[\bullet N]$</td>
<td>value of type $N$</td>
</tr>
<tr>
<td>$P$</td>
<td>continuation accepting $N$</td>
</tr>
<tr>
<td>“$P$ obvious”</td>
<td></td>
</tr>
<tr>
<td>“$P_1$ entails $P_2$”</td>
<td></td>
</tr>
<tr>
<td>“$N$ true”</td>
<td></td>
</tr>
<tr>
<td>“$N$ absurd”</td>
<td></td>
</tr>
<tr>
<td>“$P$ true”</td>
<td></td>
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continuation from $P_1$ to $P_2$

value of type $N$

continuation accepting $N$

expression of type $P$
## The basic judgments++

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<td>[P]</td>
<td>value of type P</td>
</tr>
<tr>
<td>P₁ ⊃ P₂</td>
<td>continuation from P₁ to P₂</td>
</tr>
<tr>
<td>N</td>
<td>value of type N</td>
</tr>
<tr>
<td>[N ⊃ P]</td>
<td>continuation from N to P</td>
</tr>
<tr>
<td>P</td>
<td>expression of type P</td>
</tr>
</tbody>
</table>

- “P obvious”
- “P₁ entails P₂”
- “N true”
- “N manifests P”
- “P true”
"direct proof of $P$" \(\frac{[P]}{\bullet P}\) \(\rightarrow\) \# 

"direct refutation of $N$" \(\frac{\rightarrow\# N}{\bullet N}\) 

\[ [P] \quad \bullet P \] \# 

\[ N \quad [\bullet N] \] \#
Intuition++

```
\[
\begin{align*}
\text{"direct proof of } P\text{"} & \rightarrow [P] \\
\text{"direct refutation of } N\text{"} & \rightarrow \# [N] \\
\end{align*}
\]

```

```
\[
\begin{align*}
\text{"direct proof of } P_1\text{"} & \rightarrow P_2 \\
\]

\[
\begin{align*}
\text{"direct refutation of } N\text{"} & \rightarrow \# [\bullet N] \\
\end{align*}
\]

```

```
\[
\begin{align*}
[P] \bullet P \rightarrow \# [P] \\
N \bullet [N] \rightarrow \# [N]
\end{align*}
\]

```
"direct proof of $P$" \[ \frac{}{[P]} \]

"direct proof of $P_1$" \[ \frac{}{P_1 \Rightarrow P_2} \]

"direct refutation of $N$" \[ \frac{}{N} \]

"direct argument from $N$ to $P$" \[ \frac{}{[N \Rightarrow P]} \]

\[
\begin{align*}
[P] & \quad \bullet P \\
\# & \\
N & \quad [\bullet N] \\
\# &
\end{align*}
\]
Intuition++

\[
\begin{align*}
\text{“direct proof of } P\text{”} & \quad \frac{\text{[P]}}{[P]} \\
\text{“direct argument from } N \text{ to } \alpha\text{”} & \quad \frac{\text{[N]} \# [\alpha]}{N} \\
\text{“direct proof of } P_1\text{”} & \quad \frac{P_1 \triangleright P_2}{P_1 \triangleright P_2} \\
\text{“direct argument from } N \text{ to } P\text{”} & \quad \frac{[N \triangleright P]}{[N \triangleright P]} \\
\text{[P] } \bullet P & \quad \frac{\#}{\#} \\
N \bullet [N] & \quad \frac{\#}{\#}
\end{align*}
\]
\[
\begin{align*}
\text{“direct proof of } P\text{”} & \quad \text{“direct proof of } P_1\text{”} \quad \rightarrow \quad P_2 \\
\frac{[P]}{} & \quad \frac{P_1 \triangleright P_2}{\rightarrow} \\
\text{“direct argument from } N\text{ to } \alpha\text{”} & \quad \alpha \\
\frac{N}{\rightarrow} & \\
\frac{[P] \quad P \triangleright P'}{P'} & \quad \frac{N \quad [N \triangleright P]}{P}
\end{align*}
\]
“direct proof of $P$”

$[P]$

“direct proof of $P_1$” $\rightarrow P_2$

$P_1 \triangleright P_2$

“direct argument from $N$ to $\alpha$” $\rightarrow \alpha$

$N$

“direct argument from $N$ to $P$”

$[N \triangleright P]$

$[P] \quad P \triangleright P'$

$P'$

$N \quad [N \triangleright P]$

$P$

$[P] \quad P \triangleright P'$

$P'$
Target: fragment of intuitionistic logic (or intuitionistic linear logic)

Given type translations $P^+$ and $N^-$, translate judgments by:

\[
\begin{align*}
[P]^* &= P^+ & \bullet P^* &= P^+ \supset \# \\
N^* &= N^- \supset \# & [\bullet N]^* &= N^- \\
\#^* &= \#
\end{align*}
\]

(where $\#$ a distinguished logical atom)
Definition by translation++

Target: fragment of 2nd-order intuitionistic logic

Given type translations \( P^+ \) and \( N^{-\alpha} \), translate judgments by:

\[
[P]^{*} = P^{+} \quad (P_1 \triangleright P_2)^{*} = P_1^{+} \supset P_2^{+} \\
N^{*} = \forall \alpha. N^{-\alpha} \supset \alpha \quad [N \triangleright P]^{*} = N^{-\alpha}[P^{+} / \alpha] \\
P^{*} = P^{+}
\]
Definition by translation++

Target: fragment of 2nd-order intuitionistic logic + “monad $T$”

Given type translations $P^+$ and $N^{-\alpha}$, translate judgments by:

- $[P]^* = P^+$
- $(P_1 \triangleright P_2)^* = P_1^+ \supset TP_2^+$
- $N^* = \forall \alpha. N^{-\alpha} \supset T\alpha$
- $[N \triangleright P]^* = N^{-\alpha}[P^+/\alpha]$
- $P^* = TP^+$

(Where “monad $T$” = $[\forall \alpha. \alpha \supset T\alpha] \land [\forall \alpha \beta. (\alpha \supset T\beta) \supset (T\alpha \supset T\beta)]$)
Definition by translation++

Type translation:

\[ 1^+ = T = \bot^{-\alpha} \quad 0^+ = F = \top^{-\alpha} \]

\[ (P_1 \otimes P_2)^+ = P_1^+ \land P_2^+ \quad (N_1 \otimes N_2)^{-\alpha} = N_1^{-\alpha} \land N_2^{-\alpha} \]

\[ (P_1 \oplus P_2)^+ = P_1^+ \lor P_2^+ \quad (N_1 \& N_2)^{-\alpha} = N_1^{-\alpha} \lor N_2^{-\alpha} \]

\[ (N \bullet P)^+ = N^{-\alpha}[P^+ / \alpha] \quad (P \rightarrow N)^{-\alpha} = P^+ \land N^{-\alpha} \]

\[ (\downarrow N)^+ = \forall \alpha . N^{-\alpha} \supset T \alpha \quad (\uparrow P)^{-\alpha} = P^+ \supset T \alpha \]
The intuitionistic connection

Define “polarity-collapsing” translation:

\[
\begin{align*}
\Box &= \& = \land \\
\bigodot &= \circledast = \lor \\
\implies &= \neg\bullet = \supset \\
\downarrow &= \uparrow = \cdot
\end{align*}
\]

Proposition

\[\vdash^i |A| \iff \text{Mon}_T \vdash^{2i} A^* \text{ for } \Box, \neg\bullet \text{-free } A\]

Punchline: different “\(\neg\neg\)”-interpretations of intuitionistic logic

“Polarized IL is a restriction of a generalization of polarized CL”
Definition by canonical forms++

Contexts \( \Delta, \Gamma ::= \cdot | \Delta_1, \Delta_2 | N | P \triangleright P' \)

\[
\Delta \vdash [P] \quad \Gamma \vdash \Delta \\
\hline
\Gamma \vdash [P]
\]

\[
\Delta \vdash [P] \quad \rightarrow \quad \Gamma, \Delta \vdash P' \\
\hline
\Gamma \vdash P \triangleright P'
\]

\[
\alpha.\Delta \vdash [N] \triangleright - \quad \rightarrow \quad \Gamma, \alpha.\Delta \vdash \alpha \\
\hline
\Gamma \vdash N
\]

\[
\alpha.\Delta \vdash [N] \triangleright - \quad \Gamma \vdash \Delta[P/\alpha] \\
\hline
\Gamma \vdash [N] \triangleright P
\]

\[
N \in \Gamma \quad \Gamma \vdash [N] \triangleright P \\
\hline
\Gamma \vdash P
\]

\[
P \triangleright P' \in \Gamma \quad \Gamma \vdash [P] \\
\hline
\Gamma \vdash .P'
\]

\[
\Gamma \vdash [P] \\
\hline
\Gamma \vdash P
\]

\[
\Gamma \vdash .P \\
\hline
\Gamma \vdash P \triangleright P'
\]

\[
\Gamma \vdash \Delta_1, \Delta_2 \\
\hline
\Gamma \vdash \Delta_1 \quad \Gamma \vdash \Delta_2
\]

\[
\Gamma \vdash [P] \\
\hline
\Gamma \vdash P
\]

\[
\Gamma \vdash .P \\
\hline
\Gamma \vdash P'
\]
Definition by canonical forms++

\[
\begin{align*}
  p & \quad \sigma \quad V^+ \quad p[\sigma] \\
  d & \quad \rightarrow \quad E_d \quad d \rightarrow E_d \\
  v & \quad K^- \quad v \quad K^- \quad \frac{k \quad V^+}{.E} \quad k \quad V^+ \quad \frac{\sigma_1 \quad \sigma_2}{\overline{\sigma}} \quad (\sigma_1, \sigma_2) \\
  V^+ & \quad \frac{!V^+}{E} \quad .E \quad K^+ \quad \frac{K^+ \$.E}{E}
\end{align*}
\]
Delimited control operators are already here, really!

- Danvy & Filinski’s original type-and-effect system as derived rules
- Connections to Asai & Kameyama ’07 and Kiselyov & Shan ’07
- See paper (and Twelf code!) for a more concrete connection

Important caveat: **only the first-level of the CPS hierarchy**
Asymmetry in constructive logic is still not very well-understood.

Continuation semantics (≠ semantics of callcc) deserves to be revisited.

Filinski’s monadic reflection [POPL94/10] is an underappreciated idea.

The CPS hierarchy (= “substructural hierarchy”? ) is ripe for exploration.