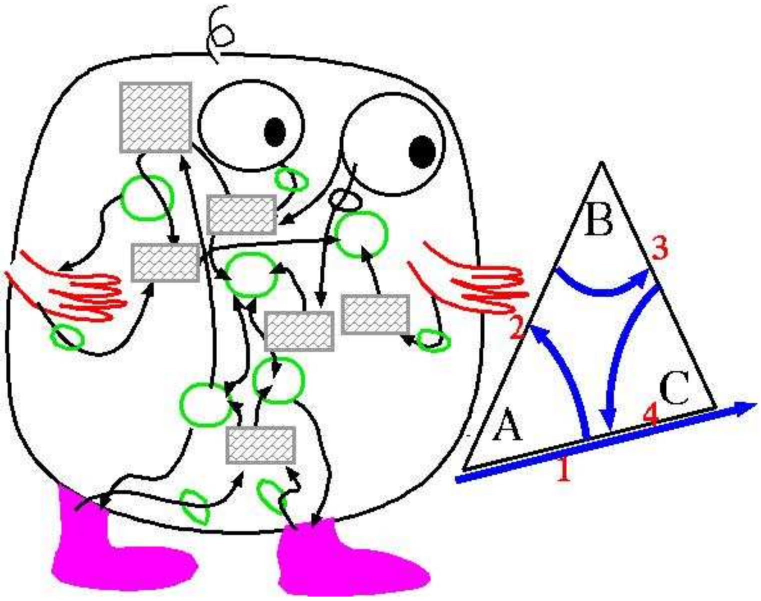
Biological/Evolutionary Foundations of Mathematics (BEFM)



How can evolution produce babies that grow up to be mathematicians?

Can we produce baby robots that grow up to be mathematicians? How?

NOTE
This paper is partly superseded by a new paper (November, 2016):
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/maths-multiple-foundations.html
Several Types of Foundation For Mathematics (e.g.):

- Neo-Kantian (epistemic) foundations,
- Mathematical foundations,
- Biological/evolutionary foundations
- Physical/chemical foundations
- Metaphysical/Ontological foundations
- Others ???

Do we need to understand all of these in order to build artificial mathematical minds comparable to ancient mathematicians, e.g. able to work out how to extend Euclidean geometry to make triangle trisection not only possible, but easy? See
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/trisect.html

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Abstract

Many philosophers and mathematicians have sought to discover or create foundations for mathematics. I argue that mathematics takes different forms with different sorts of foundations, and present examples. This is work in progress and liable to be drastically revised.
Criticisms welcome.
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"For mathematics is after all an anthropological phenomenon"
(Wittgenstein, Remarks on the Foundations of mathematics).

No! The existence of humans, like the existence of any other biological species, is a combined instance of a collection of mathematical phenomena.
See
Biology, Mathematics, Philosophy, and Evolution of Information Processing
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/bio-math-phil.html
(Evolution: the blind theorem prover.)

Installed: 19 Nov 2014
Last updated: 24 Nov 2014; 15 Dec 2014 (See the 2016 update above)
This paper is
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/maths-foundations.html
A PDF version may be added later.

A partial index of discussion notes is in
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/AREADME.html

Claim: there are different foundations for different mathematical phenomena.

The BEFM approach asks questions partly inspired by Kant's ideas about how mathematical knowledge of non-analytic necessary truths is possible and partly inspired by Turing's 1952 paper on morphogenesis, hinting at what he would have done if he had lived more than two years longer.

Above all it is inspired by a collection of biological issues: what happened when, as schoolboy I was introduced to Euclidean geometry and learnt that I could actually prove things about lines, circles, triangles, etc.? How was that possible, whereas in all my other subjects I had to take everything on trust or do experiments in a laboratory, where inaccurate measurements, or equipment problems could lead to incorrect results? What made it possible for brains to do such things? Moreover, whereas I was guided by a teacher and textbooks, there must originally have been humans, or perhaps ancestors of humans, who had no mathematics teachers, yet made the original discoveries that were later organised and presented in a (fairly) systematic way in Euclid's *Elements*, over 2,500 years ago.

Thinking about these issues, and about aspects of the design of many highly functional organisms, led me to the conclusion that evolutionary mechanisms were implicitly discovering and using mathematical theorems long before organisms evolved that could also make such discoveries, notice that they were making them, and then turn the making, discussing and teaching, of such discoveries into a socially organised activity (though sometimes the relevant society had one individual, e.g. in the case of Ramanujan, for a while).

And how is all that related to the intelligence of other animals, e.g. birds who make extraordinarily complex nests apparently requiring insight into spatial structures and constraints, even if they can't talk about that (e.g. knot-tying weaver birds), squirrels who defeat "squirrel-proof" bird-feeders, and many more, including human toddlers who, if observed carefully, seem to be capable of mathematical insights that they can use before they can talk about them -- and possibly before they can think about what they are doing? (Illustrated in a growing collection of toddler theorems here. I think Jean Piaget also noticed such phenomena.)

Yet, some of those discoveries have so far not been matched by even the most sophisticated automated theorem provers, and although robots can be trained to do many very impressive things there is so far (as far as I know) not one that can explain why it succeeded and what it would have had to change if the situation had been slightly different, and why the task would become impossible if changed in other ways: forms of reasoning about what J.J.Gibson called "affordances" (which we can generalise to proto-affordances, vicarious affordances, epistemic affordances, deliberative affordances, as explained here.)

Perhaps, if we can continue lines of thought that apparently preoccupied Alan Turing and discover what biological reasoning systems are doing that we have not learnt to replicate in machines, we'll have a new deeper understanding of foundations of mathematics than ever before. I suspect this is deeply connected with the fact, without which biological evolution as we know it would have been impossible, that the physical/chemical structure of the universe provides collections of "construction kits" with deep mathematical properties (e.g. in some cases including unbounded extendability, and types of "emergence" of qualitatively new kinds of mathematics extending old kinds). See http://www.cs.bham.ac.uk/research/projects/cogaff/misc/explaining-possibility.html

What mathematical properties of the *physical universe* make possible known features of biological evolution? E.g.

- Evolution of organisms (implicitly) using mathematical structures in their environment (e.g. homeostatic control mechanisms based on negative feedback; avian flight control mechanisms managing instability; syntax-based descriptions of environmental structures and processes, e.g. "A is between B and C"),
- Evolution of mathematical abilities to notice and reason about those structures, including inferring properties and relations, and constructing explanations ("Why do I see more of the cave as I move closer to the entrance?", "If I move three paces then five paces forward where will I be?")
- Evolution of mathematical abilities to explore possible extensions of those structures and alternatives to those structures? (E.g. adding length and other metrics to systems of comparison: from "longer" to "three paces long", or thinking of lengths of curves, areas of irregular shapes, volume of an egg...)
- Evolution of meta-cognitive meta-mathematical abilities to reflect on those reasoning abilities? (Would the answer be different in another village? On a higher mountain? When the wind is blowing? Why not? What makes P *necessarily* true? Where do constraints come from? (Kant))
- Evolution of meta-meta- abilities to reflect on possible alternatives to those reasoning and exploration abilities? (Attempting to organise results into an ordered derivational system with explicit axioms. Attempting to find syntactic criteria for valid inference. Attempting to find indisputable starting points. Engaging in foundational debates: Can arithmetic be reduced to

logic? Can geometry be reduced to arithmetic? Did Hilbert's axiomatisation of geometry change the subject(Frege 1950)? Did Frege's logicisation of arithmetic change the subject?) After reading this document David Mumford pointed out that "the big thing that Hilbert contributed to Euclid was his observation that *betweenness* was an essential component in making Euclid fully rigorous -- a topological concept that the Greeks ignored as being too obvious" -- a view of Euclid that I had previously learnt as a student. However, Guggenheimer (1977) suggests that the axioms of betweenness are not needed since the concept B is between A and C is equivalent to length(AB)+length(BC)=Length(AC).

- Products of evolution (e.g. humans) discovering how to produce working replicas (proof checkers, automated reasoners, automated mathematical discoverers...) *But why have we not yet been able to mechanise some competences, e.g. topological reasoning about continuous deformation of closed curves?* -- as illustrated here.

Some universes (e.g. one made of Newtonian point masses) could not support reasoning mechanisms [Why not? Could a useful computer be constructed out of point masses with only mass/inertia, elasticity, location, direction, velocity, acceleration, etc. of motion, and gravitational attraction?]. A key feature of our universe is *chemistry*, making possible an enormous variety of stable structures of varying size and complexity able to store usable energy, create motors, create sensors, create enclosing and supporting structures, create self-moving machines, and create *stable* but *changeable* information structures (using catalysis, thanks to quantum mechanics). (Ganti 2003). Turing machines are more restricted, unable to support mixed discrete and continuous operations.

That leads to a host of new questions: if X is possible what mathematical properties of the *processes of biological evolution* make X possible ... e.g. make possible mechanisms able to discover Euclidean geometry? Was Kant on the right track? Was it really pure logic somehow disguised? Something else? (J.S.Mill? Wittgenstein? ...?) If it was logic, what made organisms able to do logical reasoning?

This leads to further questions: e.g. what new possibilities are enabled by mathematical properties of the *products of biological evolution*? Products include information-based control mechanisms of many kinds. What mathematical properties of various information structures and information-using mechanisms make possible various forms of perception, learning, reasoning, deliberating, acting, communicating, ... and mathematical discovery?

Example: what biological mechanisms made it possible for our ancestors to discover bits of geometry that led to the production of Euclid's Elements and many discoveries in arithmetic-long before the development of formal proof methods or attempts to base mathematics on some well defined foundations. I think those early non-empirical mathematical modes of reasoning have never been fully understood. Euclid's *Elements* clearly reported and to some extent organised mathematical discoveries, not empirical discoveries (even if they started as empirical). Could the abilities of ancient mathematicians be connected with the abilities to perceive and reason about affordances in the environment - what is and is not possible, and why? (J.J.Gibson)

Those modes of perception and reasoning must in part be products of biological evolution that served needs of some intelligent animals, e.g. the need to acquire and manipulate information about what is and is not possible in an individual's environment (e.g. in a spatial configuration where some things move and others do not), and to discover consequences of realising some of the possibilities (e.g. how they would alter subsequent possibilities and impossibilities).

Example: what biological mechanisms made it possible for Descartes to notice the structural correspondences between parts of Euclidean geometry and sets of numbers, sets of pairs of numbers, sets of triples of numbers, and equations relating numbers - a mathematical discovery

linking two domains? (No current robot could do that. What needs to be added?)
What mechanisms made it possible for Newton to use Descartes' results to discover and prove things about previously unnoticed aspects of the physical world, expressible in a new mathematical formalism? Abilities to invent, see and use syntactic structures are also mathematical competences.
And later on: what features of the forms of meta-cognitive information processing in humans make it possible for some of them to notice and begin to investigate systematically possible *alternatives* to the mathematical structures discovered and thought about up to any particular time - including non-Euclidean geometries, non-standard logics, non-standard arithmetics, different transfinite structures,...
[All this presupposes a non-standard theory of the semantics of modality, discussed elsewhere.]
Some aspects of abilities that underlie adult mathematical competences seem to play a role in various kinds of animal intelligence and pre-verbal human intelligence (We need examples of toddler discoveries, as well as examples from other animals, e.g. squirrels, weaver birds, elephants...).
Example (toddler?) theorem: If a shoe-lace goes through a hole in the shoe you can extricate it by pulling one end or the other end, but not both ends simultaneously - even if the lace is stretchable. Why not? Why is it a mistake to try to put your shirt on by pushing a hand into a cuff, and pulling the sleeve up the arm? Your answers need not depend on statistical evidence from failed attempts, because you can reason mathematically about everyday things, even unwittingly. It's very hard to axiomatise such knowledge, and to justify required premises/axioms/inference rules?
Examples of spatial intelligence in young humans and other animals suggest that biological evolution (blindly) "discovered" and made effective use of mathematical structures in both the environments in which organisms evolved, and also in the space of possible biological information-processing mechanisms that we (human mathematicians, philosophers, computer scientists) have not yet understood, and which may provide new answers to old questions about the nature and scope of mathematical knowledge and how it differs from other kinds.
Those (object)-mathematical abilities can be present in some animals and young humans without the meta-cognitive abilities required to think about the discoveries and the reasoning processes, or to communicate discoveries to others, or give reasons. What has to change?
The meta-cognitive abilities are clearly not available at birth in humans, and seem to depend on later development of brain mechanisms that for important biological reasons are delayed in some intelligent species. Kant: "...faculty of knowledge ... awakened into action...", i.e. there is unreflective and (later) reflective mathematical learning.
Are foundations of mathematics and foundations of meta-mathematics necessarily different?
This may eventually provide new support for a modern variant of Kant's ideas: mathematical knowledge is synthetic and *a priori*, non-empirical, but neither innate nor infallibly derived (Lakatos 1976), and mathematical truths are necessary but not a subspecies of logical truths.
There are many unanswered questions about mathematical discoveries (e.g. discoveries about equivalence classes of curves on a planar, spherical or toroidal surface) that are close to human common-sense reasoning about spatial structures and processes but beyond the scope of current automated theorem provers. Can that be fixed without radical innovation? Brain science has shed no light on any of this. New advances in AI are needed to model these processes. What sort? Then perhaps neuroscientists will know what to look for.
It may also turn out that there are forms of computation (e.g. perhaps, as Turing hinted in 1950, chemistry-based computation, with its mixture of discrete and continuous processes) that provide a different space of mathematical mechanisms from Turing-equivalent mechanisms. Turing was thinking of chemical processes before he died (1952). He had a deep interest in how living things

worked.
I think there's a rich space of mathematical structures waiting to be explored concerned with such evolutionary and developmental processes, which ultimately depend on the mathematical structures and processes supported by physics and chemistry - work in progress.
There are connections with Jean Piaget's developmental psychology and the work of neurodevelopmental psychologist Annette Karmiloff-Smith (e.g. her ideas about representational redescription), and probably others I've not yet identified.
We need to attract clever, young, researchers from several disciplines to contribute to this type of research on foundations of mathematics. We may need new forms of education.

Incomplete (illustrative) Bibliography

(More to be added later)

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