Introduction

I conjecture that Alan Turing unwittingly held a view of mathematical discovery that was in some respects close to Immanuel Kant’s view, as explained below, and that is at least part of the reason why, around the time he wrote about the Imitation Game, he was also thinking about a possible role for chemical brain mechanisms combining continuous and discrete processes, in ancient mathematical discoveries.

That motivation was not mentioned in the 1952 paper on chemistry-based morphogenesis Turing (1952). There is, however, a one sentence clue in the 1950 paper: “In the nervous system chemical phenomena are at least as important as electrical”, though he doesn’t say why. I’ll offer a conjectured answer below.

Kenneth Craik (1943) made some relevant deep remarks in his discussion of mechanisms of perception, e.g. asking how a tangled network of neurons can represent straightness, though he is better known for the claim, later in the book, that brains build “models” of what they perceive in the environment, and use those models in planning and controlling actions. I have not been able to find out whether Turing and Craik, two of the deepest thinkers of the time, ever met, or knew of each others’ work. Perhaps the combination of Craik’s deep knowledge of brain physiology (leading him to wonder how a tangled mess of neurons could detect or represent straightness, for example) and Turing’s deep mathematical and computational insights might have produced important discoveries that are still waiting to be made seven decades later.

There is also a deep but less obvious connection with themes in Schrödinger’s extremely influential little book What is life? (1944), though I have no evidence that Turing had read that when writing about chemistry a few years later. There is also no evidence that Schrödinger thought chemical mechanisms in brains might be relevant to ancient mathematical discovery processes. Nevertheless, it is possible that the mixture of discrete chemical changes (e.g. catalytically enabled bond formation) discussed by Schrödinger and continuous spatial deformation that can occur to molecules suspended in a chemical soup, may turn out to be relevant to the kind of mathematical intuition supported by human brains, of which Turing thought computers were incapable.

It seems unlikely that Turing’s work on reaction/diffusion processes forming patterns on external surfaces of organisms was intended to be relevant to mathematical intuition. But perhaps he regarded that as an interesting preliminary investigation, to be followed later by a shift of focus from "macro" movements of continuous fluids close to the surface of an organism to sub-microscopic
molecular interactions within brain cells, processes involving both continuous processes (e.g. items folding, twisting, or moving together or apart) and discrete switching (e.g. forming and releasing chemical bonds) --- combinations that are not possible in Newtonian physics. The strongest evidence I have found for that is the remark about the importance of chemistry in brains in his Mind 1950 paper, quoted above.

Life, and even the existence of rocks (whose solidity and rigidity depend on chemical bonds) would be impossible in a purely Newtonian universe. I suggest that a full account of how evolution produced the mechanisms making possible ancient mathematical minds, and what those mechanisms are, will include as yet unknown exceedingly complex processes of evolution and individual development that make use of currently unknown sub-neural chemical mechanisms.

**Biological Mechanisms of Mathematical Discovery**

**The Meta-Morphogenesis project**

Until fairly recently (the first half of the 20th Century), Euclidean geometry was a standard part of a good mathematical education -- including finding geometric proofs or counter-examples, finding geometric constructions, proving properties or limitations of such constructions, etc. That gave students first-hand experience of replicating some of the achievements of great mathematicians of the past, including independently making discoveries presented in Euclid's *Elements*. In some cases they made discoveries that went beyond Euclid, for example the (re-)discovery, in the early 1970s, by Mary Pardoe (a young mathematics teacher) of a proof of the triangle sum theorem, using a construction not included in Euclid, and not derivable from Euclid's constructions and theorems, as explained here:

http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-sum.html

Hilbert's "logicisation" of Euclidean geometry Hilbert(1899) is often used to claim that ancient discoveries in geometry, reported in Euclid's *Elements*, are all based on purely logical derivations from a set of axioms presented by Euclid. However this does not explain how the axioms and constructions in Euclid were originally discovered. Moreover there were constructions and proofs known to ancient mathematicians, and some discovered more recently, that are not derivable from Euclid's axioms. This proves that both Euclid's axioms, and Hilbert's presentation of Euclid's axioms, do not allow all ancient mathematical discoveries regarding spatial structures to be derived using purely logical reasoning.

This refutes claims that Hilbert had shown that all geometrical reasoning is implicitly logical reasoning, a claim that is often used against Kant.

Ancient mathematicians (e.g. Archimedes) knew of and made use of geometric constructions that went beyond the power of Euclidean geometry, but for centuries those constructions were disparaged by mathematics teachers and not mentioned in textbooks on geometry, until very recently. An example is the *neusis* construction that makes it easy to trisect an arbitrary angle, which is not possible using only Euclid's constructions. The construction is described and discussed further in http://www.cs.bham.ac.uk/research/projects/cogaff/misc/trisect.html

As a result of reflecting on experiences in which necessary connections or impossibilities involving spatial structures and processes are discovered, Immanuel Kant (1781) claimed that there are kinds of mathematical discovery that produce knowledge with three features that had not previously been clearly distinguished, although David Hume came close with his division of kinds of
knowledge into two categories, roughly: *empirical* (labelled "Matters of fact" by Hume) and *conceptual* (labelled "Relations between ideas" by Hume). This is a shallow summary, not to be confused with deep Hume-scholarship! Kant was provoked into criticising Hume’s two-fold account by making three different binary distinctions, explained below. I suspect that Alan Turing’s investigation of chemistry-based morphogenesis was in part motivated by a belief (or hunch) that chemistry-based brain mechanisms could play a role in mathematical reasoning that cannot be replicated in digital computers. Whether he had that hunch or not, I think it was correct. But I also think Turing was looking at the wrong sorts of chemistry-based mechanisms.

**Note**

I have not found any evidence that Alan Turing was acquainted with, or approved of Kant’s work. However, in 1936 he mentioned a distinction between mathematical *ingenuity* and mathematical *intuition*, neither of which he attempted to define precisely, and suggested that human mathematicians have both ingenuity and intuition, whereas computers can only have mathematical ingenuity. Unfortunately, as far as I know, he nowhere explained precisely what he thought the differences were between mathematical ingenuity and intuition, and why he thought only brains, but not computers (e.g. Turing machines) could use mathematical intuition. The distinction is summarised very briefly in *Turing* (1938), based on his 1936 Thesis. I suspect he had had insights similar to those that led to Kant’s three distinctions described below. I also suspect that Turing had a hunch that some ancient mathematical abilities of human brains make use of sub-neural chemical processes involving a mixture of continuous and discrete processes. The rest of this paper uses examples of spatial mathematical reasoning that support the claims I have attributed to Kant and Turing, but does not attempt to demonstrate that they really were making these claims. The claims are of interest whether made by Kant or Turing or neither.

**Immanuel Kant’s three distinctions**

Kant, partly reacting against Hume’s two-fold distinction mentioned above, noticed that the following three binary distinctions are different and depend on different cognitive capabilities.

1. Discoveries may be *empirical* or *non-empirical*.

Most discoveries made by perceiving and interacting with objects in the environment are empirical. In contrast, mathematical discoveries triggered by experiences, may be *non-empirical* insofar as thinking and reasoning suffices to demonstrate the truth of the discovery without using observation of the world to check all possible cases, or to sample enough to compute probabilities with high confidence. Exactly what this ("thinking and reasoning suffices") means and how it works is very difficult to specify in the spirit that I am sure Kant intended. An example of a non-empirical discovery in this sense is "Spatial containment must be transitive". It is *possible* to discover that merely as a generalisation from experience (i.e. as empirical), but it is not *necessary* to do so. Moreover it is hard to see how any finite collection of examples could justify the generalisation that for all contained or containing shapes, no matter where they are, what their shapes are, how big or how small they are, etc. containment must be transitive. For example, that would include very complex and intricate containing or contained spatial regions with shapes nobody has ever encountered or thought of.

That spatial containment is transitive can be discovered merely by reflecting on the nature of containment, without conducting large surveys of examples and looking for counter-examples. However, there is nothing in current psychology or neuroscience that characterizes brain
mechanisms capable of making such discoveries except as empirical generalisations. Empirical
generalisations are all liable to be refuted by future examples. They cannot include *necessity* (what
*must be the case*) as their content.

Piaget, who had studied Kant, wrote two books, published posthumously exploring some of the
issues *Piaget*(1981,1983), though I believe he lacked a deep understanding of forms of
computation required for thinking about those issues. Related work, emphasising topological
aspects of spatial reasoning, is presented in *Sauvy and Sauvy*(1974), partly inspired by Piaget, but
without presenting mechanisms. Text-books of mathematics or logic include many more examples
of facts that can be discovered non-empirically. But Kant claims they have two additional features.

2. Propositions may be analytically true or false
Kant (possibly inspired by Hume's concept of truths that merely express "relations between ideas")
noted that some knowledge is *analytic*, i.e. derivable purely from definitions using logic. An
example (not used by Kant) is "No bachelor uncle is an only child". Anyone who understands the
words "bachelor" (referring to unmarried males, usually adults), "uncle" referring to the brother
(male sibling) of a mother or father, and the phrase "only child" referring to persons who have no
brother or sister, can work out that if someone is a bachelor uncle then he is not married (being a
bachelor) but must have a brother or sister who has at least one child, i.e. a niece or nephew. But
having a brother or sister entails, by definition, not being an only child. So using nothing but logic
and definitions one can establish the truth of "No bachelor uncle is an only child". Therefore it is
analytic. In contrast, "No bachelor uncle lives in a house made of gold" may be true, but it is not
analytic.

3. Propositions may be impossible, contingent or necessarily true.
Kant noted that there is another distinction that is different from both the empirical/non-empirical
and the synthetic/analytic distinction, namely the contingent/necessary distinction. A proposition
that is capable of being true in some possible situations and false in others is *contingent*. If it is true
in all possible situations it is *necessary*. If it is false in all, then it is *impossible*, i.e. *necessarily false.*
Necessity and impossibility are connected insofar as the negation of a necessarily true proposition
is necessarily false and vice versa. These categories are often referred to as "modalities". More
precisely, they are *alethic* modalities, concerned with what can or cannot exist or happen, as
opposed to *deontic* modalities that are concerned with what *ought to be/should be/should not be/
the case. Saying that someone ought to do X e.g. help someone in distress (a deontic necessity)
normally presupposes that it is possible not to do X, e.g. possible to refrain from helping an alethic
possibility. So deontic possibilities depend on alethic possibilities.

Kant pointed out that much mathematical knowledge is concerned with these alethic categories:
e.g. it is *possible* for the corners of a planar quadrilateral to lie on a circle, for example if the
quadrilateral is a rectangle, and also *possible* for a planar quadrilateral to exist whose corners do
not lie on any circle, e.g. a quadrilateral formed by pushing one corner of a rectangle inward
towards the opposite corner. Moreover for any triangle there *necessarily* exists a circle passing
through the vertices of the triangle (one of the theorems in Euclidean geometry). On a planar or
spherical surface containing a closed continuous line C (e.g. a circle or ellipse or an octagon) it is
*impossible* for another continuous line L to exist in the same surface joining a point in the interior of
C to a point outside C without any point in C coinciding with a point in L. This is just a tiny set of
examples: there are infinitely many different examples of different spatial possibilities, necessities
and impossibilities. Many of these were discovered by ancient mathematicians several thousand
years ago.
However, I know of no theory in psychology or neuroscience that explains what sorts of brain mechanisms make such discoveries possible. For example, psychological or neural theories that focus on learning by collecting statistical evidence and computing probabilities, are incapable of explaining how something is impossible, or necessarily true. So key features of ancient mathematical consciousness, which I suspect overlap with some non-human forms of intelligence --- in animals with good spatial reasoning abilities --- are not explained by anything in current psychology or neuroscience. Piaget understood the problem, as shown in Piaget(1983), but was unable to provide an explanation.

I am not talking about "possible worlds"

There are modern discussions of modal properties that I think are irrelevant to the points Kant was making. In particular, one modern notion starts from the concept of a set of possible worlds, and divides propositions into different categories according to which set of possible worlds (if any) they are true in. In this modern sense a proposition would be "necessary" if it is true in all possible worlds, "impossible" if it is false in all possible worlds", and "contingent" if it is true in some possible worlds but not all. However, I don't think these "possible world" concepts are relevant to the sense in which a child may discover that it is impossible to separate rigid linked rings made of impermeable materials, or that spatial containment is necessarily transitive, i.e. if some object, or region of space, X is wholly contained in a region of space Y and Y is wholly contained in a region of space Z, then X is necessarily contained in Z. As I pointed out in Sloman(1962), thinking about why that is true requires the ability think only about possible and impossible fragments of this world and their relationships, not the ability to think about total universes as assumed in most recent thinking about modal concepts. I see no evidence that young children, or other intelligent animals able to recognize spatial impossibilities are thinking about all possible universes.

We can however, correctly say that ancient discoveries about possible or impossible configurations and necessary truths of geometry and topology are concerned with possible and impossible configurations of possible fragments of this world.

How brains represent and reason about possible fragments of this universe is a deep question beyond the scope of this paper (and beyond current neuroscience).

The earliest discoveries, by ancient mathematicians, of truths and falsehoods concerning geometry made essential use of spatial reasoning, i.e. using diagrams and spatial operations on diagrams to prove that certain combinations of spatial properties were impossible, or to prove that having certain spatial properties necessarily implied having another spatial property. For example, it was known to ancient mathematicians, and proved in Euclid’s Elements, that in any triangle the sum of the three interior angles adds up to exactly half a rotation (i.e. 180 degrees) -- The triangle sum theorem.

Pythagoras’ theorem, also known to ancient mathematicians, states that if a triangle has an angle whose size is exactly 90 degrees (a quarter of a rotation), then the square on its longest side has an area equal to the sum of the squares on the other two sides. Ancient mathematicians discovered many different ways of proving this. But they were considering only possible planar triangles not all possible worlds.

A proof is not a physical object, and need not be purely logical

In all these cases of mathematical discovery and proof using diagrams, the diagrams did not have to be physically realised: it was enough to be able to imagine them and operations on them. The
ancient mathematicians who made discoveries in that way could not have used logical formalisms and techniques that were not developed until centuries later. There is no evidence that they were using the modern logical forms of reasoning unwittingly. Moreover, some of their discoveries, e.g. the neusis construction that allows arbitrary angles to be trisected go beyond what modern axiomatisations of Euclidean geometry can support, so it is implausible to argue that ancient mathematicians merely used logic to derive consequences from definitions.

Example: slice a vertex off a convex polyhedron
For readers who have not personally had the kind of discovery experience described by Kant here is an example. Try to answer this question: If there is a solid convex polyhedron, and exactly one vertex is sliced off with a single planar cut, e.g. using a very thin planar saw, how will the number of vertices, edges and faces (V, E and F) of the new resulting polyhedron be related to the original three numbers? Note that this question can be given a definite answer only if the polyhedron is convex: i.e. any straight line joining two points on the surface lies entirely inside the polyhedron, except for the end points.

Solid, opaque, polyhedron with partly visible faces, edges and vertices.
Use a planar cut that removes one vertex, e.g. the top one.
(There may be vertices, edges and faces not visible from a particular viewpoint)

Readers should attempt to answer the question without first reading on. If you find the question hard to understand, the answer sketched below should provide an understanding of the question. Try to find an answer without reading on!

Some hints
One way to think about the problem is to imagine a 2D projected view of the polyhedron, as in the picture. Some visible vertices have all their adjoining faces and edges visible in that view, like the labelled vertex on the left in the figure. That makes it fairly easy to think about what happens if such a fully visible vertex is sliced off. It is obvious (why?) where new edges and vertices will appear, and also obvious that a new face is produced by the cut. (The circle merely indicates where the sliced off vertex is, and does not imply that any new circular shape will be produced.)
Other vertices are only *partly* visible from a particular viewpoint: some of the edges and faces meeting at such vertex are not visible from that viewpoint. An example is shown at the top of the polyhedron in the picture. From what is visible in that view, it is impossible to infer the number of invisible edges and faces on the far side of the polyhedron, all meeting at the selected vertex.

However, we can infer (how?) that no matter how many invisible edges and faces there are that meet at the top vertex, the slice that removes the top vertex will produce a new convex polyhedron. In the process it will cut off a portion of each invisible face meeting the top vertex, since the top vertex will be a sliced-off corner for each such face. The slice will also shorten the invisible edges originally meeting the top vertex, as it does for the visible edges sliced.

Less obviously, the slice will introduce a number of new vertices where the slice-plane intersects sliced edges in the original polyhedron. And there will be new edges connecting pairs of those vertices.

From this view of the polyhedron it is not possible to determine how many edges will be shortened and how many faces will lose a vertex. So the answer to the question will have mention the numbers that change (numbers of edges, vertices, and faces), but will not be able to specify the exact old or new numbers.

**Steps toward a solution**

Because the cut is planar and the polyhedron is convex, the cut must intersect each of the pre-existing planar surfaces meeting at the sliced off vertex. Each of those planar surfaces will have a triangular portion removed, leaving a new straight edge. All the new edges, visible or not, will together form the boundary of a new planar surface bounded by new straight edges meeting at newly created vertices.

That new planar surface is in the plane of the cut: it separates the volume previously occupied by the sliced off material and the volume occupied by the remainder of the polyhedron.

Readers may find it interesting to think about how the reasoning might be affected by considering a different view of the polyhedron to be sliced, namely one looking down at the the vertex to be removed.

From that view all the edges and surfaces meeting at the vertex before it is sliced off will be visible, as will all the new edges and vertices and the new surface produced after slicing.

But given only the view shown in the picture above, it would be impossible to draw a picture of that "fully visible" view because the number of edges and surfaces originally meeting at the sliced off vertex is unspecified. Any drawing of the view from the top based on the original view depicted above, must therefore be incomplete.

Whichever views are considered, the process of thinking *spatially* about the problem is totally different from stating the problem using a logic-based axiom system for geometry and deducing the answer to the question using only logical reasoning, without any spatial reasoning.

Since visibility is nowhere mentioned in Euclid’s axioms, and they include no mention of a slicing operation, I believe it would not be possible to solve the problem by giving a logical proof starting from Hilbert’s axiom system for Euclidean geometry. It may be possible in an expanded version of Hilbert’s system, based on new concepts added to Euclid’s system. But the resulting geometric
reasoning system would not be able to model the original reasoning processes used by non-expert mathematicians who can answer the question.

For anyone who has not worked out the solution: after the cut, (a) one old vertex will have been removed (the one sliced off), (b) each edge meeting at the old vertex will be shortened, producing new vertices at their new ends, (c) new edges joining the new vertices will bound newly produced polygonal faces, each of which is a truncated portion of an old face and (d) finally a new face is created in the plane of the cut, bounded by the new edges, meeting at newly formed vertices.

This form of "visual" reasoning easily leads to the conclusion that one edge has gone, N new edges have been created, if there were N edges meeting at the removed vertex, and those new edges will bound a flat polygon with N edges and N vertices that did not exist before the cut.

**What brain mechanisms are required?**

Working out that answer does not require unusual mathematical genius. I find that many people who have never previously encountered the problem, but understand all the concepts (especially "convex") can work out what the changes must be, even if they have never previously studied Euclidean geometry. Times taken may be a few seconds to several minutes, though a few individuals require longer times or cannot answer the question.

What sorts of brain mechanisms make this form of reasoning possible? It is not done simply by applying syntactic operations to discrete components of sentences, as in a logical theorem prover. It is also not done by examining a large number of polyhedra, cutting off one vertex with a planar cut, counting the changes, and then using statistical reasoning to work out the probabilities of various answers, as might be done by a neural-net based reasoning system. Unfortunately, neural nets are limited to deriving probabilities from statistical evidence: they cannot reason about what is impossible or what necessarily is the case. So they cannot make mathematical discoveries with the features specified by Kan.

As far as I know, no other species can make such a discovery. Why not? Moreover, very young humans cannot make such discoveries. Why not? What has to change in their brains between not being able to make them and being able to? I don’t think anyone knows the answer at present. Unfortunately, most researchers haven’t even thought about the question, including researchers in AI, psychology and neuroscience, because they have not encountered Kant’s distinctions. I cannot offer an explanatory mechanism, except that it seems to require a reasoning medium that supports operations on spatial structures and the ability to abstract from the particular spatial structures and processes to produce generic answers to questions like ours.

I am not aware of any proposed mechanism in computer science, psychology, AI, neuroscience, or logic that can do the required reasoning. It could be done using logic if the statement of the problem included a rich description of the cutting process. But humans don’t need that: they can work out the details by imagining a planar cut removing one vertex. I suspect the explanation depends on the use of sub-neural chemical mechanisms that can produce both discrete and continuous changes in representational structures -- a feature of chemistry that is missing from digital computers, making use only of rules allowing discrete changes with discrete consequences. But that feature of chemistry, made possible only by aspects of quantum physics (since molecular structures and processes could not exist in a Newtonian universe consisting only of interacting elastic particles acting under gravitational forces), was shown by Schrödinger in 1944 to be essential for biological reproduction mechanisms.
**A meta-question**

However, there is a related question: when you have found the answer, i.e. you can describe the changes produced by the slice, are you merely reporting *a generalisation from a collection of examples you have found*? Or have you made a deeper discovery, namely that that is what the answer *must* be, i.e. no other answer is possible? In other words have you discovered a *necessary* truth that is applicable to all possible ways of slicing a single vertex off any possible convex polyhedron, with a single planar slice?

Of course, very young children cannot understand the question, and some who do may not be able to answer it until they are older. That change does not require any training in answering questions like this, or experience of sawing through convex polyhedra. It may require some experience of seeing and manipulating solid objects, and experience of cutting off portions, in order to enable the concepts required for understanding the question to be developed. But the ability to answer the question requires a deeper kind of change: development of the ability to make non-empirical discoveries about relationships between spatial properties and processes.

How do you know that there isn’t a convex polyhedron that you have never encountered, such that there is a way of removing exactly one of its vertices with a single planar cut that produces a different result?

I claim, inspired by Immanuel Kant’s discussion in his *Critique of Pure Reason* that my example is one of infinitely many examples of possible discoveries of truths that are *synthetic* (non-analytic, non-definitional, not based on pure logic), that are *non-empirical*, i.e. not mere generalisations from examples and subject to refutation by an example that will be discovered one day, and *non-contingent*, i.e. *necessarily true*. (These three concepts, identified by Kant, are summarised and distinguished in Sloman(1965).)

There are infinitely many different convex polyhedra, and for each one infinitely many different ways in which exactly one vertex can be sawn off. But there is a fairly simple answer to the question how that process necessarily changes the numbers of vertices, edges and faces, and I claim that millions of humans are capable of understanding the question and discovering the answer, including realising that it is not merely an empirical generalisation that could have exceptions at high altitudes, or in the depths of an ocean.

What sort of brain development can enable a young child to acquire the ability to grasp that something is *impossible*, or is *necessarily* the case -- not just accidental results of slicing a particular vertex off a particular polyhedron?

There are many examples of such discoveries. E.g. by the age of five or six years many children seem to understand that one-to-one-correspondence is *necessarily* transitive, i.e. if there is a one-to-one correspondence between the members of two sets of objects, S1 and S2, and also a one-to-one correspondence between S2 and a third set, S3, then there \(\exists k\) exists a one-to-one correspondence between S1 and S3.

That fact is one of the discoveries that enabled our distant ancestors to discover the great utility of counting systems using a memorised collection of symbols to be used in different one-to-one correspondences, as explained in Sloman[1978, revised] Chapter 8.
One consequence is that two collections of objects do not need to be adjacent and aligned for the existence of a 1-1 correspondence between them to be established. If both collections are in 1-1 correspondence with an initial sequence of a set of memorised numerals, then they must be in 1-1 correspondence with each other. This makes it unnecessary to take your whole family on a fishing trip to ensure that catch at least one fish for each member, as pointed out in Sloman(2016).

Answering the question why a certain answer is correct involves describing a form of spatial reasoning that is sufficiently precise to produce the correct answer yet is independent of the numbers of vertices, faces and edges (V, F, E) involved or the precise locations or orientations of slicing operations that remove one vertex.

In other words, people who work out the answer are able to reason spatially not merely about a particular polyhedron with particular numbers V, E and F, e.g. a tetrahedron or a cube, but in a general way that applies to all possible initial convex polyhedra, and to any vertex sliced off, using any planar cut that removes only one vertex. Thus answering the question (as in many cases of geometric reasoning) involves a mode of thinking that correctly applies to infinitely many different structures and processes. This cannot be achieved by any form or probabilistic inference from statistical data, and therefore cannot be achieved by neural net based AI mechanisms.

I am deliberately leaving working out the answer as an exercise for the reader. After finding the answer, try to describe the reasoning you use and explain why it works no matter how many edges meet at the vertex chosen for removal.

This example, and many others, illustrate Kant’s claim that it is possible to acquire mathematical knowledge about necessary (non-contingent) truths and falsehoods concerning possible types of structures and processes, including the spatial structures and processes investigated in Euclid’s Elements. The proofs show “how things must be, or cannot be”. When valid, they show why counter-examples to theorems are impossible.

For example, readers should be able to explain why their answer to the sliced polygon questions is true, by talking about the effects of the slice in a general way -- independent of the precise shape of the polygon. However, explaining how they do that, and what brain mechanisms make it possible, requires major advances in cognitive science/neuroscience/AI, based on (future!) deep theories about spatial cognition, its evolution, and its development in individual animals Sloman(2020).

There may be some people (e.g. students of David Hilbert?) who answer this and similar questions only by starting from a fully specified logical version of the problem, and then reason using only logic. But for most of the history of geometry that could not be done, until Hilbert (or a similar thinker) had produced a purely logical specification of Euclidean geometry. Moreover the mathematicians and non-mathematicians to whom I have presented the sliced polygon problem and other problems have all dealt with the problem by reasoning about spatial structures and processes, not logical structures and logical manipulations of formulae.

The third feature of mathematical knowledge/truth, i.e. necessity, is frequently omitted from summaries of Kant’s claims about mathematical knowledge. Unfortunately, his ideas are now either ignored by most philosophers of mathematics or badly misrepresented, e.g. in Hempel(1945).
During the 20th Century, mathematical and philosophical opinions on the nature of mathematical knowledge changed, partly under the influence of developments in formal logic (by Boole, Peirce, Dedekind, Peano, Frege, Russell and many others). Many mathematicians regarded Hilbert’s logic-based axiomatisation of Euclidean Geometry, Hilbert(1899) as removing the need for any non-logical forms of representation or reasoning in geometry.

Moreover, Euclidean geometry had already been dethroned by a combination of discovery of alternative geometries, Einstein’s theory of general relativity, claiming that physical space was not Euclidean, and Eddington’s confirmation of Einstein’s work based on observations of the solar eclipse in 1919.

After becoming friendly with philosophy graduate students in Oxford, around 1959, I learnt that philosophers thought that Immanuel Kant’s philosophy of mathematics e.g. as presented in Kant(1781) had been refuted. A typical view of Kant as mistaken was expressed in Hempel(1945). Yet Kant’s view, as I understood it, corresponded to my experience as a student learning about geometry, making discoveries and finding proofs. So, around 1959, I switched from mathematics to philosophy in order to defend Kant’s view of mathematical discovery, completing my thesis in 1962 Sloman(1962), partly summarised in Sloman(1965). I tried to show that, as Kant had claimed, the labels “Analytic”, “Non-empirical” and “Necessarily true (or false)” indicate importantly different distinctions that can be applied to different kinds of knowledge.

References


Immanuel Kant Critique of Pure Reason (1781) http://archive.org/details/immanuelkantscri032379mbp

Piaget(1981,1983). Much of Jean Piaget’s work is also relevant, especially his last two (closely related) books written with collaborators:

Possibility and Necessity
Vol 1. The role of possibility in cognitive development (1981)
Vol 2. The role of necessity in cognitive development (1983)
Tr. by Helga Feider from French in 1987

Erwin Schrödinger (1944), *What is life?* CUP, Cambridge, A partial transcript with some comments and questions is available at http://www.cs.bham.ac.uk/research/projects/cogaff/misc/kenneth-craik.html


http://www.cs.bham.ac.uk/research/projects/cogaff/62-80.html#1965-02

http://www.cs.bham.ac.uk/research/projects/cogaff/crp/


https://doi.org/10.1112/plms/s2-45.1.161

Note: Parts of the thesis, including the section on intuition vs ingenuity are replicated in many collections, including S. B. Cooper & J. van Leeuwen (Eds.), *Alan Turing - His Work and Impact* (p. 849-856) Amsterdam: Elsevier.


Compare the Turing Conversation web site:
https://www.turing.ethz.ch/the-turing-conversation.html