Vickrey auctions

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We tend to follow Maskin 2004, §3, defining restricted versions of the more general objects in Milgrom 2004. In particular, we consider sealed-bid auctions with independently and identically distributed private values – according to a commonly known distribution – for a single indivisible good.[1]

Definition 1. *$N = \{1, \dots, n\}$ is a set of* participants *(also referred to as* bidders*), often indexed by i.*[2]

Definition 2. *An* allocation *is a vector $\boldsymbol{x} \in \{0, 1\}^n$ such that $x_i = 1$ denotes participant i's award of the indivisible good to be auctioned, and $x_j = 0$ otherwise.*

Definition 3. *An* outcome*, $(\boldsymbol{x}, \boldsymbol{p})$, specifies both an allocation and a vector of payments, $\boldsymbol{p} \in \mathbb{R}^n$, made by each participant i.*

Definition 4. *Participant i's* payoff *is $u_i \equiv v_i \cdot x_i - p_i$, where $v_i \in \mathbb{R}_+$ is participant i's valuation of the good.*

Definition 5. *Let it be common knowledge that each v_i is an independent realization of a random variable, $\tilde{v}$, whose distribution is described by density function f. Then a* strategy *for bidder i is a mapping g_i such that $b_i = g_i(v_i, f) \geq 0$, where b_i is called i's* bid*. A* strategy profile *is the full vector of bids, $\boldsymbol{b} \in \mathbb{R}^n$*

Definition 6. *Given some n-vector $\boldsymbol{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$, let*

$$\overline{y} \equiv \max_{j \in N} y_j;$$
$$\overline{y}_{-i} \equiv \max_{j \in N \setminus \{i\}} y_j.$$

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[1] Definition 5 may be more general than is needed here but helps to understand the term "*strategy* profile".

[2] We start indexing at 1, as the seller may be added as $i = 0$ when it is useful to do so. For Vickrey's theorem it is not.

Definition 7. *Let* $M \equiv \left\{i \in N : b_i = \overline{b}\right\}$*. Then a* second-price auction *(or* Vickrey auction*) is an outcome,* $(\boldsymbol{x}, \boldsymbol{p})$ *satisfying:*

1. $\forall j \in N \backslash M, x_j = p_j = 0$*; and*
2. *for one* $i \in M$*, selected according to any randomization device,* $x_i = 1$ *and* $p_i = \overline{b}_{-i}$*, while,* $\forall j \in M \backslash \{i\}, x_j = p_j = 0$*.*

Definition 8. *In the single good case, an auction is* efficient *if* $x_1 = 1 \Rightarrow v_i = \overline{v}$*.*

Definition 9. *Given some auction, a strategy profile* $\boldsymbol{b}$ *supports an* equilibrium in weakly dominant strategies *if, for each* $i \in N$ *and any* $\hat{\boldsymbol{b}} \in \mathbb{R}^n$ *such that* $\hat{b}_i \neq b_i$*,*

$$u_i\left(\hat{b}_1, \ldots, \hat{b}_{i-1}, b_i, \hat{b}_{i+1}, \ldots, \hat{b}_n\right) \geq u_i\left(\hat{\boldsymbol{b}}\right). \tag{1}$$

That is, whatever others do, i *will not be better off by deviating from the original bid* b_i*.*

Remark 1. The notation $u_i(\boldsymbol{b})$ is standard within economics, but misleading for formal systems. A more careful notation is $u_i(x_i, v_i, p_i)$, where x_i and p_i depend on $\boldsymbol{b}$ and the auction type.

Theorem 1 (Vickrey 1961; Milgrom 2.1)**.** *In a second-price auction, the strategy profile* $\boldsymbol{b} = \boldsymbol{v}$ *supports an equilibrium in weakly dominant strategies. Furthermore, the auction is efficient.*

Proof. Suppose that participant i bids $b_i = v_i$, whatever bids $\hat{b}_j$ the others may submit. We abbreviate the overall bid vector $\left(\hat{b}_1, \ldots, \hat{b}_{i-1}, v_i, \hat{b}_{i+1}, \ldots, \hat{b}_n\right)$ as $\hat{\boldsymbol{b}}^{i \leftarrow v}$. There are two cases:

1. Participant i wins. From this follows $b_i = v_i = \overline{\hat{\boldsymbol{b}}^{i \leftarrow v}}$, $p_i = \overline{\hat{\boldsymbol{b}}^{i \leftarrow v}}_{-i}$, and $u_i(\hat{\boldsymbol{b}}^{i \leftarrow v}) = v_i - p_i = \hat{\boldsymbol{b}}^{i \leftarrow v}_i - \overline{\hat{\boldsymbol{b}}^{i \leftarrow v}}_{-i} \geq 0$. Now consider i submitting an arbitrary bid $\hat{b}_i \neq b_i$, i.e. assume an overall bid vector $\hat{\boldsymbol{b}}$. This has two sub-cases:
   a) i wins with the new bid, that is, $u_i(\hat{\boldsymbol{b}}) = u_i(\hat{\boldsymbol{b}}^{i \leftarrow v})$, since the second highest bid has not changed.
   b) i loses with the new bid, that is, $u_i(\hat{\boldsymbol{b}}) = 0 \leq u_i(\hat{\boldsymbol{b}}^{i \leftarrow v})$.
2. Participant i loses. From this follows $p_i = 0$, $u_i(\hat{\boldsymbol{b}}^{i \leftarrow v}) = 0$, and $b_i \leq \overline{\hat{\boldsymbol{b}}^{i \leftarrow v}}_{-i}$; otherwise i would have won. This yields again two cases:
   a) i wins with the new bid, that is, $u_i(\hat{\boldsymbol{b}}) = v_i - \overline{\hat{\boldsymbol{b}}}_{-i} = b_i - \overline{\hat{\boldsymbol{b}}^{i \leftarrow v}}_{-i} \leq 0 = u_i(\hat{\boldsymbol{b}}^{i \leftarrow v})$
   b) i loses with the new bid, that is, $u_i(\hat{\boldsymbol{b}}) = 0 = u_i(\hat{\boldsymbol{b}}^{i \leftarrow v})$.

Applying this reasoning to all bidders establishes that $\boldsymbol{b} = \boldsymbol{v}$ supports an equilibrium in weakly dominant strategies.

Efficiency is immediate: when $\boldsymbol{b} = \boldsymbol{v}$, the highest bid belongs to the bidder with the highest valuation. □

References

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