Around Hilbert’s 13th Problem

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1. Introduction — Hilbert’s 13th Problem
   - The 13th Problem. Motivation.
   - Answers: Vitushkin, Kolmogorov, Arnold

2. Kolmogorov’s Superposition Theorem
   - The KST and its Proof
   - New Results on the KST

3. Further Results
   - A New Cardinal Invariant, basic ($X$)
   - ‘Real World’ Applications
   - Strong Approximation and Universal PDEs
The 13th Problem

Question:
Can every continuous function of 3 variables be written as a composition of continuous functions of 2 variables?

Example

\[ f(x, y, z) = \frac{e^{x^2} \sqrt[3]{\sin(y + z) + 1}}{1 + x^2 + z^2}. \]

Composition of: exp, squaring, \times, \sqrt[3] \cdot, \sin, +, /. 
The 13th Problem

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Superposition of: \(\exp, \) squaring, \(\times, \sqrt[3]{\cdot}, \) sin, ++,

Solving Functions for Polynomials

Quadratic  \[ az^2 + bz + c \rightarrow z^2 + pz + q \]
Solution map:
\[ z(a, b, c) \rightarrow z(p, q) = \frac{1}{2}(-p + \sqrt{p^2 - 4q}), \]
\[ p = b/a, q = c/a. \]

Quintic  General Quintic  \[ z^5 + pz + q \]
Solution map:  \[ z(a, b, c, d, e, f) \rightarrow z(p, q) \]
\[ p, q \text{ arithmetic functions of coefficients.} \]

Degree 7  General Degree 7  \[ z^7 + pz^3 + qz^2 + rz + 1 \]
Solution map:  \[ z(p, q, r), \]
\[ p, q, r \text{ arithmetic functions of coefficients.} \]
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Solution map: \[ z(p, q, r), \]
\[ p, q, r \text{ arithmetic functions of coefficients.} \]

Arithmetic functions are compositions of maps of 2 variables.
Solving Functions for Polynomials

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Solution map:
\[ z(a, b, c) \rightarrow z(p, q) = \frac{1}{2}(-p + \sqrt{p^2 - 4q}), \]
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Degree 7 General Degree 7 \[ \rightarrow z^7 + pz^3 + qz^2 + rz + 1 \]
Solution map: \[ \rightarrow z(p, q, r), \]
\[ p, q, r \text{ arithmetic functions of coefficients.} \]
Hilbert Thought No
Vitushkin’s Negative Answer

Theorem (Vitushkin, 1954)

There is a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ of $n$–variables which has continuous $q$th order derivatives but cannot be expressed as a superposition of functions of $n'$–variables which have continuous $q'$th order derivatives, if $n/q > n'/q'$.

Example

- There are continuously differentiable functions of 3 variables which are not the superposition of continuously differentiable functions of 2 variables.
Vitushkin’s Negative Answer

**Theorem (Vitushkin, 1954)**

There is a function \( f : I^n \rightarrow \mathbb{R} \) of \( n \)–variables which has continuous \( q \)th order derivatives but can not be expressed as a superposition of functions of \( n' \)–variables which have continuous \( q' \)th order derivatives, if \( n/q > n'/q' \).

**Example**

- There are continuously differentiable functions of 3 variables which are not the superposition of continuously differentiable functions of 2 variables.
Theorem (Kolmogorov, 1956)

Every $f \in C(I^n)$ for $n \geq 4$ is the superposition of continuous functions with $\leq 3$ variables.

Theorem (Arnold, 1957)

Every $f \in C(I^3)$ is the superposition of continuous functions with $\leq 2$ variables.
YES!
Every continuous function of 3 variables, from I, can be written as a superposition of continuous functions of \( \leq 2 \) variables.

Applies to:

- Solution function \( z(p, q, r) \) to \( z^7 + pz^3 + qz^2 + rz + 1 = 0 \).
New Results

A space $X$ satisfies Hilbert 13

- every continuous function of $n$ variables from $X$ is a superposition of continuous functions of $\leq 2$ variables

Example

Every continuous function of $n$ real variables can be written as a superposition of continuous functions of $\leq 2$ real variables.
New Results

A space $X$ satisfies Hilbert 13

- every continuous function of $n$ variables from $X$ is a superposition of continuous functions of $\leq 2$ variables

if and only if

$X$ is locally compact, finite dimensional and separable metric

(homeomorphic to a closed subspace of Euclidean space).

Example

Every continuous function of $n$ real variables can be written as a superposition of continuous functions of $\leq 2$ real variables.
New Results

A space $X$ satisfies Hilbert 13

- every continuous function of $n$ variables from $X$ is a superposition of continuous functions of $\leq 2$ variables

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(*homeomorphic to a closed subspace of Euclidean space*).

Example

Every continuous function of $n$ real variables can be written as a superposition of continuous functions of $\leq 2$ real variables.
Kolmogorov’s Superposition Theorem (KST)

Theorem (Kolmogorov, 1957)

Step 1  \textit{There exist} $\phi_1 \ldots, \phi_{2n+1}$ \textit{in} $C(I^n)$ \textit{such that}

$$\forall f \in C(I^n) \quad f = \sum_{i=1}^{2n+1} g_i \circ \phi_i,$$

\textit{for some} $g_i \in C(\mathbb{R})$.

Step 2  \textit{Further, can choose the} $\phi_1, \ldots, \phi_{2n+1}$ \textit{such that:}

$$\phi_i(x_1, \ldots x_n) = \sum_{j=1}^{n} \psi_{ij}(x_j)$$
Theorem (KST: $n=2$)

Step 1 *There exist* $\phi_1 \ldots, \phi_5$ *in* $C(I^2)$ *such that*

$$\forall f \in C(I^2) \quad f = \sum_{i=1}^{5} g_i \circ \phi_i,$$

*for some* $g_i \in C(\mathbb{R})$.

Step 2 *Further, can choose the* $\phi_1, \ldots, \phi_5$ *such that:*

$$\phi_i(x_1, x_2) = \psi_{i1}(x_1) + \psi_{i2}(x_2)$$
Ideas in Proof of KST: The $\psi_{pq}$ and $\phi_q$
Proof of KST: Contour Map of $\phi$

$\phi = \phi(x_1, x_2)$
Proof of KST: Looking Deeper at ψ

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Proof of KST: Looking Deeper at $\psi$
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Proof of KST: Looking Deeper at $\psi$

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Proof of KST: Approximate $f$

**$f$**

**$\phi$**
Proof of KST: Approximate $f$
Proof of KST: Approximate $f$

$g \circ \phi, \quad (\|g \circ \phi\| \leq \|f\|)$

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Proof of KST: Replicate, Shift . . .

\[ \phi_1, \phi_2, \phi_3, \phi_4, \phi_5 \]

\[ g_1 \circ \phi_1, g_2 \circ \phi_2, g_3 \circ \phi_3, g_4 \circ \phi_4, g_5 \circ \phi_5 \]
Proof of KST: ... and Average

\[ f \]

\[ \sum_{q=1}^{5} \frac{1}{3} g_q \circ \phi_q \]
Proof of KST: Correct and Try Again

Let \( f_0 = f \).

Then \( f_0 \sim \sum_q \frac{1}{3} g_q^0 \circ \phi_q \).  
Get \( \| f_0 - \sum_q \frac{1}{3} g_q^0 \circ \phi_q \| \leq \frac{5}{6} \| f_0 \| \).

Set \( f_1 = f_0 - \sum_q \frac{1}{3} g_q^0 \circ \phi_q \).

And repeat \( f_1 \sim \sum_q \frac{1}{3} g_q^1 \circ \phi_q \).  
Get \( \| f_1 - \sum_q \frac{1}{3} g_q^1 \circ \phi_q \| \leq \frac{5}{6} \| f_1 \| \).

And repeat ...  

So that \( f = \sum_q g_q \circ \phi_q \),  
where \( g_q = \sum_k \frac{1}{3} g_q^k \).
Extending the KST

Smoothness

Fridman  Lipschitz inner functions
Vitushkin & Khenkin  Inner functions can not be $C^1$

Simplicity of Representation

Lorentz  Single outer function
Sprecher  Number of inner functions

Domain

Ostrand  Compact, finite dimensional and metric
Doss  Step 1 (basic functions) for $\mathbb{R}^n$
Theorem (KST for $\mathbb{R}$)

There exist $\psi_{pq} \in C(\mathbb{R})$, for $1 \leq q \leq 2n + 1$ and $1 \leq p \leq n$, such that for any $f \in C(\mathbb{R}^n)$, there is a $g \in C(\mathbb{R})$ such that:

$$f(x) = \sum_{q=1}^{2n+1} g(\phi_q(x)),$$

where $\phi_q(x_1, \ldots, x_n) = \sum_{p=1}^{n} \psi_{pq}(x_p)$.

The $\psi_{pq}$ can be taken to be Lipschitz.
New Results on Hilbert 13 (II)

Theorem (General KST)

Let $X$ be

locally compact, finite dimensional and separable metrizable.

There exist $\psi_{pq} \in C(X)$, for $1 \leq q \leq m$ and $1 \leq p \leq n$, such that

for any $f \in C(X^n)$, there is a $g$ in $C(\mathbb{R})$ such that:

$$f(x) = \sum_{q=1}^{m} g(\phi_q(x)),$$

where $\phi_q(x_1, \ldots, x_n) = \sum_{p=1}^{n} \psi_{pq}(x_p)$.
Theorem (Necessary Condition for KST for $X$)

If a space $X$ has a finite basic family then $X$ is

locally compact, finite dimensional and separable metrizable.

Definition

$\phi_1, \ldots, \phi_m$ in $C(X)$ are a finite basic family for $X$ if for any $f \in C(X)$, there are $g_1, \ldots, g_m$ in $C(\mathbb{R})$ such that:

$$f(x) = \sum_{q=1}^{m} g_q(\phi_q(x)).$$
$\phi = \phi(x_1, x_2) = \psi_1(x_1) + \psi_2(x_2)$. 

$\psi_2 = \psi_2(x_2)$ 

$\psi_1 = \psi_1(x_1)$
Related Problems

- Minimal size of basic family on $C(X)$
  Connections to set theory and Banach Algebra.
- Constructive proof of KST on $\mathbb{R}$
  Possible applications in “real world”.
- ‘Fixed-formed’ Strong Analytic Approximation theorem
  Applications to Universal PDE.
Related Problems

- Minimal size of basic family on $C(X)$
  Connections to set theory and Banach Algebra.

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- ‘Fixed-formed’ Strong Analytic Approximation theorem
  Applications to Universal PDE.
• Minimal size of basic family on $C(X)$
  Connections to set theory and Banach Algebra.

• Constructive proof of KST on $\mathbb{R}$
  Possible applications in “real world”.

• ‘Fixed-formed’ Strong Analytic Approximation theorem
  Applications to Universal PDE.
A family $\Phi \subseteq C(X)$ is basic if for any $f \in C(X)$, there are $\phi_1, \ldots, \phi_m$ in $\Phi$ and $g_1, \ldots, g_m$ in $C(\mathbb{R})$ such that:

$$f(x) = \sum_{q=1}^{m} g_q(\phi_q(x)).$$

Definition

$$\text{basic}(X) = \min\{|\Phi| : \Phi \text{ basic for } X\}.$$
Finite and Countable Basic Families

**Theorem**

\[
\text{basic}(X) \leq 2n + 1 \text{ if and only if } X \text{ is locally compact, separable metrizable and } \dim X \leq n.
\]

*Note:* Sternfeld.

**Theorem**

\[
\text{basic}(X) \geq \aleph_0 \text{ if and only if } \text{basic}(X) \geq \aleph_1.
\]

**Example**

\[
\text{basic}(\omega_1) = \aleph_1.
\]
Expected:

Definition

\[ ck(X) = \text{cof} (\mathcal{K}(X), \subseteq) = \min \{ |\mathcal{K}| : \mathcal{K} \text{ cofinal in } \mathcal{K}(X) \}, \]

where \( \mathcal{K}(X) = \) all the compact subsets of \( X \).

Examples

\[ ck(\mathbb{R}) = \aleph_0, \quad ck(\mathbb{P}) = \emptyset = ck(\mathbb{Q}). \]
Let $X$ be separable metrizable. Then either $\text{basic}(X)$ finite which happens iff $X$ locally compact, finite dimensional or $\text{basic}(X) = \mathfrak{c}$. 
Lemma (Shelah’s Observation)

\[ \kappa^\omega = \text{cof}\left( [\kappa]^\omega, \subseteq \right) \times |\mathcal{P}(\omega)|. \]

Theorem (Shelah)

\[ \aleph_\omega < \text{cof}\left( [\aleph_\omega]^\aleph_0, \subseteq \right) < \aleph_{\omega_4}. \]

Making \( \text{cof}\left( [\aleph_\omega]^\aleph_0, \subseteq \right) > \aleph_{\omega_1} \) requires large cardinals.
Theorem

For any finite dimensional $K$: $\text{basic}(K) \leq \cof([\omega(K)]^{\aleph_0}, \subseteq)$.

For any infinite dimensional $K$: $\text{basic}(K) \geq \mathfrak{c}$.

For any $K$ with a discrete subspace $D$, $|D| = \omega(K)$:

$$\text{basic}(K) \geq \cof([\omega(K)]^{\aleph_0}, \subseteq).$$

Hence, for such $K$:

either $K$ is finite dimensional, and

$$\text{basic}(K) = \cof([\omega(K)]^{\aleph_0}, \subseteq),$$

or $K$ is infinite dimensional, and

$$\text{basic}(K) = |\mathcal{C}(K)| = \omega(K)^{\aleph_0}.$$
Theorem

Let \( n \) be a natural number. Take any \( \gamma \geq 2n + 2 \), and let \( D = \{ k/\gamma^\ell : k, \ell \in \mathbb{Z} \} \) be the set of all rationals base \( \gamma \).

There are \( \psi_{pq} \) for \( 1 \leq q \leq 2n + 1 \) and \( 1 \leq p \leq n \) which are defined constructively on \( D \), such that for any \( f \in C(\mathbb{R}^n) \), there is a constructive algorithm for computing \( g_1, \ldots, g_{2n+1} \) in \( C(\mathbb{R}) \) such that:

\[
f(x) = \sum_{q=1}^{2n+1} g_q(\phi_q(x)), \quad \text{where} \quad \phi_q(x_1, \ldots, x_n) = \sum_{p=1}^{n} \psi_{pq}(x_p).
\]

to within a specified error \( \epsilon > 0 \) on any specified compact subset \( K \) of \( \mathbb{R}^n \).
Possible Applications

- Function reconstruction
- Statistical Pattern Recognition
- Image Compression
- Multidimensional Signal Processing
- Neural networks
- Analog computing devices

etc...
The strong topology on $C(\mathbb{R}^n)$ has basic open sets:

$$B(f, \epsilon) = \{g \in C(\mathbb{R}^n) : |f(x) - g(x)| < \epsilon(x) \ \forall x \in \mathbb{R}^n\},$$

where $\epsilon \in C(\mathbb{R}^n, (0, \infty))$.

**Example**

$C_S(\mathbb{R}^n)$ is Baire, not first countable.

The polynomials are *not* dense.

But the analytic functions *are* dense.

Frequently used in differential topology, geometric analysis.
The **strong topology** on $C(\mathbb{R}^n)$ has basic open sets:

$$B(f, \epsilon) = \{ g \in C(\mathbb{R}^n) : |f(x) - g(x)| < \epsilon(x) \ \forall x \in \mathbb{R}^n \},$$

where $\epsilon \in C(\mathbb{R}^n, (0, \infty))$.

**Example**

$C_S(\mathbb{R}^n)$ is Baire, not first countable.

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Frequently used in differential topology, geometric analysis.
Theorem \((n = 2)\)

Given \(f \in C(\mathbb{R}^2)\) and \(\epsilon \in C(\mathbb{R}^2, (0, \infty))\), \(\exists\) analytic functions of 1–variable \(\alpha, \beta, \tilde{g}_q^i\) and \(\tilde{\psi}_{pq}^i\) for \(i = 1, 2, \ p = 1, 2\) and \(q = 1, \ldots, 5\) such that

\[
|f(x, y) - \tilde{f}(x, y)| < \epsilon(x, y),
\]

where,

\[
\tilde{f}(x, y) = a(x, y)\tilde{f}_1(x, y) + (1 - a(x, y))\tilde{f}_2(x, y),
\]

\[
a(x, y) = \alpha((\beta(x) + \beta(y))y),\quad \text{and}
\]

\[
\tilde{f}_i(x, y) = \sum_{q=1}^{5} \tilde{g}_q^i \left((\tilde{\psi}_{1q}^i((x + 1)^2 + (y - (-1)^i))^2) + \tilde{\psi}_{2q}^i((x - 1)^2 + (y - (-1)^i))^2\right).
\]
**Theorem \((n = 2)\)**

*Given* \(f \in C(\mathbb{R}^2)\) *and* \(\epsilon \in C(\mathbb{R}^2, (0, \infty))\), \(\exists\) analytic functions of 1–variable \(\alpha, \beta, \tilde{g}_q^i\) and \(\tilde{\psi}_pq^i\) for \(i = 1, 2, p = 1, 2\) *and* \(q = 1, \ldots, 5\) *such that* \(|f(x, y) - \tilde{f}(x, y)| < \epsilon(x, y)\), *where,*

\[
\tilde{f}(x, y) = a(x, y)\tilde{f}_1(x, y) + (1 - a(x, y))\tilde{f}_2(x, y),
\]

\[
a(x, y) = \alpha (\beta(x) + \beta(y))y, \quad \text{and}
\]

\[
\tilde{f}_i(x, y) = \sum_{q=1}^{5} \tilde{g}_q^i \left( \tilde{\psi}_{1q}^i ((x + 1)^2 + (y - (-1)^i))^2 \right) \\
+ \tilde{\psi}_{2q}^i ((x - 1)^2 + (y - (-1)^i)^2).
\]
Every $f$ in $C(\mathbb{R}^n)$ can be strongly approximated by functions which are:

- the superposition of analytic functions of 1–variable, along with $+$ and $\times$,
- in a $\text{Fixed}$ form.
Strong Universal PDE

**Theorem**

For every \( n \geq 2 \), there is an algebraic \( n \)-variable PDE

\[
Q \left( \ldots, \frac{\partial^{\alpha} f}{\partial x^{\alpha}}, \ldots \right) = 0,
\]

whose analytic solutions are strongly dense in \( C(\mathbb{R}^n) \).

(Here \( Q \) is a polynomial with integer coefficients.)
Prior Work

- (Rubel): there is an algebraic ODE whose $C^\infty$ solutions are dense in $C_S(\mathbb{R})$.
- (Boshernitzan): there is an algebraic ODE whose analytic solutions are dense in $C_k(\mathbb{R})$.
- (Buck): for every $n \geq 2$, there is an algebraic PDE of functions of $n$ variables whose $C^\infty$ solutions are dense in $C(I^n)$.
Thank you!
We will show that:

every function which is

the superposition of analytic functions of 1–variable,

along with $\pm$ and $\times$, in a fixed form,

satisfies an algebraic PDE (depending only on the form).
Consider functions of the form \( f(x, y) = g(\psi_1(x) + \psi_2(y)) \)
where \( g, \psi_1, \psi_2 \) are analytic.

We show that any such \( f \) satisfies a (fixed) algebraic PDE.

Set
\[
\begin{align*}
    u_1 &= g'(\psi_1(x) + \psi_2(y)), \\
    u_2 &= \psi_1'(x), \\
    u_3 &= \psi_2'(y), \\
    u_4 &= g''(\psi_1(x) + \psi_2(y)), \\
    u_5 &= \psi_1''(x), \\
    u_6 &= \psi_2''(y), \text{ and} \\
    u_7 &= g^{(3)}(\psi_1(x) + \psi_2(y)).
\end{align*}
\]

Differentiate \( f \) relentlessly! Simplify using the \( u_i \)'s.
A Simplified Example

Consider functions of the form $f(x, y) = g(\psi_1(x) + \psi_2(y))$

where $g, \psi_1, \psi_2$ are analytic.

We show that any such $f$ satisfies a (fixed) algebraic PDE.

Set

$u_1 = g'(\psi_1(x) + \psi_2(y))$, $u_2 = \psi'_1(x)$, $u_3 = \psi'_2(y)$,

$u_4 = g''(\psi_1(x) + \psi_2(y))$, $u_5 = \psi''_1(x)$, $u_6 = \psi''_2(y)$, and

$u_7 = g^{(3)}(\psi_1(x) + \psi_2(y))$.

Differentiate $f$ relentlessly! Simplify using the $u_i$'s.

$$P(1) = \frac{\partial f}{\partial x} = \psi'_1(x) \cdot g'(\psi_1(x) + \psi_2(y)) = u_2 u_1.$$
Consider functions of the form $f(x, y) = g(\psi_1(x) + \psi_2(y))$
where $g, \psi_1, \psi_2$ are analytic.

We show that any such $f$ satisfies a (fixed) algebraic PDE.

Set
\[u_1 = g'(\psi_1(x) + \psi_2(y)), \quad u_2 = \psi_1'(x), \quad u_3 = \psi_2'(y),\]
\[u_4 = g''(\psi_1(x) + \psi_2(y)), \quad u_5 = \psi_1''(x), \quad u_6 = \psi_2''(y),\]
and
\[u_7 = g^{(3)}(\psi_1(x) + \psi_2(y)).\]

Differentiate $f$ relentlessly! Simplify using the $u_i$’s.

\[P_{(2)} = \frac{\partial f}{\partial y} = \psi_2'(y) \cdot g'(\psi_1(x) + \psi_2(y)) = u_3 u_1.\]
Consider functions of the form $f(x, y) = g(\psi_1(x) + \psi_2(y))$

where $g, \psi_1, \psi_2$ are analytic.

We show that any such $f$ satisfies a (fixed) algebraic PDE.

Set

$u_1 = g'(\psi_1(x) + \psi_2(y)), \quad u_2 = \psi_1'(x), \quad u_3 = \psi_2'(y),$

$u_4 = g''(\psi_1(x) + \psi_2(y)), \quad u_5 = \psi_1''(x), \quad u_6 = \psi_2''(y),$ and

$u_7 = g^{(3)}(\psi_1(x) + \psi_2(y)).$

Differentiate $f$ relentlessly! Simplify using the $u_i$’s.

$$P_{(21)} = \frac{\partial f}{\partial y \partial x} = \psi_1'(x) \cdot \psi_2'(y) \cdot g''(\psi_1(x) + \psi_2(y)) = u_2 u_3 u_4.$$
Consider functions of the form $f(x, y) = g(\psi_1(x) + \psi_2(y))$
where $g, \psi_1, \psi_2$ are analytic.

We show that any such $f$ satisfies a (fixed) algebraic PDE.

Set
\[
\begin{align*}
    u_1 &= g'(\psi_1(x) + \psi_2(y)), \\
    u_2 &= \psi_1'(x), \\
    u_3 &= \psi_2'(y), \\
    u_4 &= g''(\psi_1(x) + \psi_2(y)), \\
    u_5 &= \psi_1''(x), \\
    u_6 &= \psi_2''(y), \text{ and} \\
    u_7 &= g^{(3)}(\psi_1(x) + \psi_2(y)).
\end{align*}
\]

Differentiate $f$ relentlessly! Simplify using the $u_i$’s.

\[
P_{(11)} = \frac{\partial^2 f}{\partial x \partial x} = u_2^2 u_4 + u_5 u_1.
\]
Consider functions of the form $f(x, y) = g(\psi_1(x) + \psi_2(y))$

where $g, \psi_1, \psi_2$ are analytic.

We show that any such $f$ satisfies a (fixed) algebraic PDE.

Set

$u_1 = g'(\psi_1(x) + \psi_2(y)), \quad u_2 = \psi_1'(x), \quad u_3 = \psi_2'(y),$

$u_4 = g''(\psi_1(x) + \psi_2(y)), \quad u_5 = \psi_1''(x), \quad u_6 = \psi_2''(y),$ and

$u_7 = g^{(3)}(\psi_1(x) + \psi_2(y)).$

Differentiate $f$ relentlessly! Simplify using the $u_i$’s.

$$P_{(22)} = \frac{\partial f}{\partial y \partial y} = u_3^2 u_4 + u_6 u_1.$$
Consider functions of the form \( f(x, y) = g(\psi_1(x) + \psi_2(y)) \)

where \( g, \psi_1, \psi_2 \) are analytic.

We show that any such \( f \) satisfies a (fixed) algebraic PDE.

Set
\[
\begin{align*}
  u_1 &= g'(\psi_1(x) + \psi_2(y)), \\
  u_2 &= \psi'_1(x), \\
  u_3 &= \psi'_2(y), \\
  u_4 &= g''(\psi_1(x) + \psi_2(y)), \\
  u_5 &= \psi''_1(x), \\
  u_6 &= \psi''_2(y), \text{ and} \\
  u_7 &= g^{(3)}(\psi_1(x) + \psi_2(y)).
\end{align*}
\]

Differentiate \( f \) relentlessly! Simplify using the \( u_i \)'s.

\[
P_{(121)} = \frac{\partial f}{\partial x \partial y \partial x} = u_3(u_5 u_4 + u_2^2 u_7).
\]
Consider functions of the form \( f(x, y) = g(\psi_1(x) + \psi_2(y)) \)
where \( g, \psi_1, \psi_2 \) are analytic.

We show that any such \( f \) satisfies a (fixed) algebraic PDE.

Set
\[
\begin{align*}
    u_1 &= g'(\psi_1(x) + \psi_2(y)), \\
    u_2 &= \psi_1'(x), \\
    u_3 &= \psi_2'(y), \\
    u_4 &= g''(\psi_1(x) + \psi_2(y)), \\
    u_5 &= \psi_1''(x), \\
    u_6 &= \psi_2''(y), \text{ and} \\
    u_7 &= g^{(3)}(\psi_1(x) + \psi_2(y)).
\end{align*}
\]

Differentiate \( f \) relentlessly! Simplify using the \( u_i \)'s.

\[
P_{(212)} = \frac{\partial f}{\partial y \partial x \partial y} = u_2(u_6 u_4 + u_3^2 u_7).
\]
Consider functions of the form $f(x, y) = g(\psi_1(x) + \psi_2(y))$ where $g, \psi_1, \psi_2$ are analytic.

We show that any such $f$ satisfies a (fixed) algebraic PDE.

Set
\[
\begin{align*}
  u_1 &= g'(\psi_1(x) + \psi_2(y)), & u_2 &= \psi_1'(x), & u_3 &= \psi_2'(y), \\
  u_4 &= g''(\psi_1(x) + \psi_2(y)), & u_5 &= \psi_1''(x), & u_6 &= \psi_2''(y) \text{, and} \\
  u_7 &= g^{(3)}(\psi_1(x) + \psi_2(y)).
\end{align*}
\]

Differentiate $f$ relentlessly! Simplify using the $u_i$'s.

\[
P_{(122)} = \frac{\partial f}{\partial x \partial y \partial x} = u_3^2 u_2 u_7 + u_6 u_4.
\]
\[
P_{(1)} = u_2 u_1
\]
\[
P_{(2)} = u_3 u_1
\]
\[
P_{(21)} = u_2 u_3 u_4
\]
\[
P_{(11)} = u_2^2 u_4 + u_5 u_1
\]
\[
P_{(22)} = u_3^2 u_4 + u_6 u_1
\]
\[
P_{(121)} = u_3 (u_5 u_4 + u_2^2 u_7)
\]
\[
P_{(212)} = u_2 (u_6 u_4 + u_3^2 u_7)
\]
\[
P_{(122)} = u_2^2 u_2 u_7 + u_6 u_4
\]

8 equations, but only 7 variables \((u_1, \ldots, u_7)\).

**Eliminate** them!
To get a (rational) polynomial $Q$ such that

$$Q(\ldots, P_\alpha, \ldots) = 0.$$ 

In fact:

$$P_{(1)}^2 P_{(2)} P_{(122)} - P_{(1)} P_{(2)}^2 P_{(212)} - P_{(1)}^2 P_{(12)} P_{(22)} + P_{(2)}^2 P_{(12)} P_{(11)} = 0.$$ 

And remember: $P_\alpha = (\frac{\partial}{\partial x})^\alpha f$.

So $f$ of the given form satisfies the PDE:

$$Q(\ldots, \left(\frac{\partial}{\partial x}\right)^\alpha f, \ldots) = 0.$$
Let $f(x, y)$ be a superposition of analytic functions of 1–variable, along with $+$ and $\times$, in the fixed form of the previous theorem. Fix $k$. Differentiate $f$ repeatedly wrt $x, y$ up to order $k$. This gives $N$ equations of the form:

$$\left( \frac{\partial}{\partial x} \right)^{\alpha} f = P_{\alpha}(u_1, \ldots, u_m),$$

the $u_i$ ($i = 1, \ldots, m$) are the functions arising by differentiating the given representation of $f$, and $P_{\alpha}$ are polynomials.

$$N = O(k^2), \quad m = O(k) \quad \implies \quad \exists k : N > m.$$  

Then there is a polynomial $Q$ so that $Q(\ldots, P_{\alpha}, \ldots) = 0$. And $f$ satisfies the algebraic PDE: $Q\left(\ldots, \left( \frac{\partial}{\partial x} \right)^{\alpha} f, \ldots \right) = 0.$