Constructive analysis in univalent type theory

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Related work

P. Schuster and H. Schwichtenberg. *Constructive Solutions of Continuous Equations.* 2003

R. O'Connor. "Incompleteness & Completeness: Formalizing Logic and Analysis in Type Theory". PhD thesis. Radboud Universiteit Nijmegen, 2009

R. Krebbers and B. Spitters. "Type classes for efficient exact real arithmetic in Coq". In: *Logical Methods in Computer Science* 9.1:1 (2013), pp. 1–27. DOI: 10.2168/LMCS-9(1:01)2013

D. Lešnik. "Unified Approach to Real Numbers in Various Mathematical Settings". In: *ArXiv e-prints* (Feb. 2014). arXiv: 1402.6645 [math.GM]

A. Mahboubi, G. Melquiond, and T. Sibut-Pinote. "Formally Verified Approximations of Definite Integrals". In: *Interactive Theorem Proving - 7th International Conference, ITP 2016, Nancy, France, August 22-25, 2016, Proceedings.* 2016, pp. 274–289. DOI: 10.1007/978-3-319-43144-4_17

Part I

Constructive analysis in type theory

An intuitionistic theory of types

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The theory of types with which we shall be concerned is intended to be a full scale system for formalizing intuitionistic mathematics as developed, for example, in the book by Bishop 1967. The language of the theory is richer than the language of first order predicate logic. This makes it possible to strengthen the axioms for existence and disjunction. In the case of existence, the possibility of strengthening the usual elimination rule seems first to have been indicated by Howard 1969, whose proposed axioms are special cases of the existential elimination rule of the present theory. Furthermore, there is a reflection principle which links the generation of objects and types and plays somewhat the same role for the present theory as does the replacement axiom for Zermelo-Fraenkel set theory.

Constructive analysis in type theory

- Martin-Löf style type theories, c.f. Agda and Coq
- Constructions as programs: Agda to Haskell, Coq to OCaml

Dependent type theory

$$\lambda(x:\mathbb{N}).x+x:\mathbb{N}\to\mathbb{N}$$

$$\star:\mathbf{1} \quad , \quad 0:\mathbb{N} \quad , \quad S:\mathbb{N}\to\mathbb{N}$$

$$\lambda(f:A\to A).\lambda(a:A).f(f(a))$$
$$:(A\to A)\to A\to A$$

$A, B : \mathcal{U}$ then get $A+B:\mathcal{U}$.
For $a:A$, get $\mathrm{inl}(a):A+B$.
For $b:B$, get $\mathrm{inr}(b):A+B$.
For $a:A$ and $b:B(a)$ (i.e. $b:B[a/x]$), get $(a,b):\sum_{x:A}B(x)$.

$$\frac{\Gamma, x:A\vdash b:B}{\Gamma\vdash\lambda(x:A).b:\prod_{x:A}B}\ \Pi\text{-intro}$$

$$\frac{\Gamma, x:\mathbf{0}\vdash C:\mathcal{U} \qquad \Gamma\vdash a:\mathbf{0}}{\Gamma\vdash \mathrm{ind}_{\mathbf{0}}(\lambda(x:\mathbf{0}).C, a):C[a/x]}\ \mathbf{0}\text{-elim}$$

$$\frac{\Gamma, x:\mathbf{1}\vdash C:\mathcal{U} \qquad \Gamma\vdash c_\star:C[\star/x] \qquad \Gamma\vdash n:\mathbf{1}}{\Gamma\vdash \mathrm{ind}_{\mathbf{1}}(\lambda x.C, c_\star, \lambda x.\lambda y.c_s, n):C[n/x]}\ \mathbf{1}\text{-elim}$$

$$\frac{\Gamma, x:\mathbb{N}\vdash C:\mathcal{U} \qquad \Gamma\vdash c_0:C[0/x] \qquad \Gamma, x:\mathbb{N}, y:C\vdash c_s:C[Sx/x] \qquad \Gamma\vdash n:\mathbb{N}}{\Gamma\vdash \mathrm{ind}_{\mathbb{N}}(\lambda x.C, c_0, \lambda x.\lambda y.c_s, n):C[n/x]}\ \mathbb{N}\text{-elim}$$

Dedekind reals in Coq[1]

```
(** A Dedekind cut is represented by the
    predicates [lower] and [upper],
    satisfying a number of conditions. *)
Structure R := {
  (* The cuts are represented as propositional
     functions, rather than subsets,
     as there are no subsets in type theory. *)
  lower : Q -> Prop;
  upper : Q -> Prop;
  (* The cuts respect equality on Q. *)
  lower_proper : Proper (Qeq ==> iff) lower;
  upper_proper : Proper (Qeq ==> iff) upper;
  (* The cuts are inabited. *)
  lower_bound : {q : Q | lower q};
  upper_bound : {r : Q | upper r};
  (* The lower cut is a lower set. *)
  lower_lower : forall q r,
      q < r -> lower r -> lower q;
  (* The lower cut is open. *)
  lower_open : forall q,
      lower q -> exists r, q < r /\ lower r;
  (* The upper cut is an upper set. *)
  upper_upper : forall q r,
      q < r -> upper q -> upper r;
  (* The upper cut is open. *)
  upper_open : forall r,
      upper r -> exists q,  q < r /\ upper q;
  (* The cuts are disjoint. *)
  disjoint : forall q, ~ (lower q /\ upper q);
  (* There is no gap between the cuts. *)
  located : forall q r,
      q < r -> lower q \/ upper r
}.
```

```
(** Strict order. *)
Definition Rlt (x y : R) :=
    exists q : Q, upper x q /\ lower y q.

(** Non-strict order. *)
Definition Rle (x y : R) :=
    forall q, lower x q -> lower y q.

(** Equality. *)
Definition Req (x y : R) :=
    Rle x y /\ Rle y x.
```

[1] Andrej Bauer, https://github.com/andrejbauer/dedekind-reals

Logic in MLTTUTT

$P, Q : \mathcal{U}$Prop

$$
\begin{aligned}
\top &:= \mathbf{1} \\
\bot &:= \mathbf{0} \\
P \wedge Q &:= P \times Q \\
P \Rightarrow Q &:= P \to Q \\
P \Leftrightarrow Q &:= (P \to Q) \times (Q \to P) \quad P = Q \\
\neg P &:= P \to \mathbf{0} \\
P \vee Q &:= P + Q \quad \| P + Q \| \\
\forall (x : A). P(x) &:= \prod_{x:A} P(x) \\
\exists (x : A). P(x) &:= \sum_{x:A} P(x) \quad \left\| \sum_{x:A} P(x) \right\|
\end{aligned}
$$

MLTT	$\to$	UTT
Setoids $(X, \sim)$	$\to$	Identity types $\mathrm{Id}_X(x, y)$, also written $x =_X y$ or $x = y$
Propositions as types $P : \mathcal{U}$	$\to$	Propositions as (h)props, see next slide
Equivalence relation of function types $X \to Y$ induced by equivalence relations of X and Y	$\to$	Function extensionality $\left(\prod_{x:X} fx =_Y gx\right) \to f =_{X \to Y} g$
Quotients by setoids	$\to$	Quotient types by higher inductive types

(H)Propositions

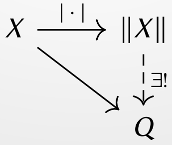
For $P : \mathcal{U}$:

$$\mathrm{isProp}(P) := \prod_{p,q:P} p =_P q \qquad \mathrm{Prop} := \sum_{P:\mathcal{U}} \mathrm{isProp}(P)$$

Any $X : \mathcal{U}$ can be *truncated* to a proposition:

$$X \quad \rightsquigarrow \quad \|X\|$$

The universal property says that for any Q : Prop we have:



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\end{aligned}
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Logic in MLTTUTT

$P, Q : \mathcal{U}\mathrm{Prop}$

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P \Leftrightarrow Q &:= (P \to Q) \times (Q \to P) P = Q \\
\neg P &:= P \to \mathbf{0} \\
P \vee Q &:= P + Q \| P + Q \| \\
\forall (x : A). P(x) &:= \prod_{x:A} P(x) \\
\exists (x : A). P(x) &:= \sum_{x:A} P(x) \left\| \sum_{x:A} P(x) \right\|
\end{aligned}
$$

Types of numbers

$\mathbb{N}$: inductively, i.e. as the type freely generated by $0 : \mathbb{N}$ and $S : \mathbb{N} \to \mathbb{N}$

$\mathbb{Z}$: e.g. as a quotient of $\mathbb{N} \times \mathbb{N}$, or as the coproduct $\mathbb{N} + \mathbb{N}$, or as a higher-inductive type[2] generated by $0 : \mathbb{Z}$, a map $S : \mathbb{Z} \to \mathbb{Z}$, and equations that make S into an isomorphism.

$\mathbb{Q}$: e.g. as a quotient of $\mathbb{Z} \times \mathbb{N}_{>0}$, or by an explicit enumeration

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[2] Thorsten Altenkirch at HoTT/UF 2017, Oxford

Cauchy approximations

$$\mathbb{Q}_+ \coloneqq \{q : \mathbb{Q} \mid q > 0\}$$

A *Cauchy approximation* $x. : C_F$ in an ordered field F is a map $x. : \mathbb{Q}_+ \to F$ such that

$$\forall(\varepsilon, \theta : \mathbb{Q}_+).|x_\varepsilon - x_\theta| < \varepsilon + \theta.$$

Equivalently, a Cauchy approximation is a Cauchy sequence with modulus.

More generally: A *premetric*[3] on a type X is a ternary relation (namely a map $X \times X \times \mathbb{Q}_+ \to \mathrm{Prop}$) written as $x \sim_\varepsilon y$ for $x, y : X$ and $\varepsilon : \mathbb{Q}_+$. If $x \sim_\varepsilon y$ then we say that x *and* y *are* ε*-close*. Then a *Cauchy approximation* $x. : C_X$ in a premetric space X is a map $x. : \mathbb{Q}_+ \to X$ such that

$$\forall(\varepsilon, \theta : \mathbb{Q}_+).x_\varepsilon \sim_{\varepsilon+\theta} x_\theta.$$

[3] c.f. Richman 2007, “Real numbers and other completions”

Types of reals

$\mathbb{R}_{\mathbf{C}}$: quotient type of the type $C_{\mathbb{Q}}$ of $\mathbb{Q}$-valued Cauchy approximations. Not necessarily Cauchy complete![4]

$\mathbb{R}_{\mathbf{H}}$: HoTT reals. The free Cauchy completion of the rationals. Assuming a small type of propositions, an interval in $\mathbb{R}_{\mathbf{H}}$ forms an Escardó-Simpson interval object.[5]

$\mathbb{R}_{\mathbf{D}}$: Dedekind reals. (see next slides)

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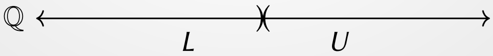
[4] Lubarsky 2015, "On the Cauchy Completeness of the Constructive Cauchy Reals"

[5] B. 2017, "The HoTT reals coincide with the Escardó-Simpson reals"

Dedekind reals (1/2)

Let $q, r : \mathbb{Q}$ and $x = (L, U)$ a pair of predicates on $\mathbb{Q}$, that is, $L, U : \mathbb{Q} \to \text{Prop}$, then we write

$$(q < x) :\equiv (q \in L) \qquad \text{and}$$
$$(x < r) :\equiv (r \in U).$$



Dedekind reals (2/2)

$x = (L, U)$ is a *Dedekind cut* or *Dedekind real* if it satisfies the following conditions.

1. *bounded:* $\exists(q : \mathbb{Q}).q < x$ and $\exists(r : \mathbb{Q}).x < r$.
2. *rounded:* For all $q : \mathbb{Q}$,

$$q < x \Leftrightarrow \exists(q' : \mathbb{Q}).(q < q') \wedge (q' < x) \qquad \text{and}$$
$$x < r \Leftrightarrow \exists.(r' : \mathbb{Q}).(r' < r) \wedge (x < r').$$

3. *transitive:* $(q < x) \wedge (x < r) \Rightarrow (q < r)$ for all $q : \mathbb{Q}$.
4. *located:* $(q < r) \Rightarrow (q < x) \vee (x < r)$ for all $q, r : \mathbb{Q}$.

We let $\mathrm{isCut}(L, U)$ denote the conjunction of these conditions. The type of *Dedekind reals* is

$$\mathbb{R}_{\mathbf{D}} \coloneqq \{(L, U) : (\mathbb{Q} \to \mathrm{Prop}) \times (\mathbb{Q} \to \mathrm{Prop}) \mid \mathrm{isCut}(L, U)\}.$$

$$x < y \coloneqq \exists(q : \mathbb{Q}).x < q < y$$

Reals in normal form

```
Build_R (fun q : Q => Qlt q (Qmake Z0 xH))
(fun r : Q => Qlt (Qmake Z0 xH) r)
(fun (x y : Q) (E : Qeq x y) =>
 @trans_co_eq_inv_impl_morphism Prop iff iff_Transitive
   (Qlt x (Qmake Z0 xH)) (Qlt y (Qmake Z0 xH))
   (@Qminmax.Q.OT.lt_compat x y E (Qmake Z0 xH)
      (Qmake Z0 xH)
      (@reflexive_proper_proxy Q Qeq
         (@Equivalence_Reflexive Q Qeq
            Qminmax.Q.OT.eq_equiv)
         (Qmake Z0 xH))) (Qlt y (Qmake Z0 xH))
   (Qlt y (Qmake Z0 xH))
   (@eq_proper_proxy Prop (Qlt y (Qmake Z0 xH)))
   (@conj
      (forall _ : Qlt y (Qmake Z0 xH), Qlt y (Qmake Z0 xH))
      (forall _ : Qlt y (Qmake Z0 xH), Qlt y (Qmake Z0 xH))
      (fun H : Qlt y (Qmake Z0 xH) => H)
      (fun H : Qlt y (Qmake Z0 xH) => H)))
(fun (x y : Q) (E : Qeq x y) =>
 @trans_co_eq_inv_impl_morphism Prop iff iff_Transitive
   (Qlt (Qmake Z0 xH) x) (Qlt (Qmake Z0 xH) y)
   (@Reflexive_partial_app_morphism Q
      (forall _ : Q, Prop) Qeq
      (@respectful Q Prop Qeq iff) Qlt
      Qminmax.Q.OT.lt_compat (Qmake Z0 xH)
      (@reflexive_proper_proxy Q Qeq
         (@Equivalence_Reflexive Q Qeq
            Qminmax.Q.OT.eq_equiv)
         (Qmake Z0 xH)) x y E) (Qlt (Qmake Z0 xH) y)
   (Qlt (Qmake Z0 xH) y)
   (@eq_proper_proxy Prop (Qlt (Qmake Z0 xH) y))
   (@conj
      (forall _ : Qlt (Qmake Z0 xH) y, Qlt (Qmake Z0 xH) y)
      (forall _ : Qlt (Qmake Z0 xH) y, Qlt (Qmake Z0 xH) y)
      (fun H : Qlt (Qmake Z0 xH) y => H)
      (fun H : Qlt (Qmake Z0 xH) y => H)))
                         .
                         .
                         .
```

$$0.000000\ldots$$

$$\bar{1}.111111\ldots$$

$$1.\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}\ldots$$

Signed-digit representations

How to compute $x \mapsto 3x$ in *unsigned* decimal representations?

- Suppose we read 10 digits off the input: 0.3333333333
- Still can't print a single output digit: both 0. and 1. may be possible.
- But the 11th digit may make one of 0. and 1. impossible (or leave it undecided):

$$0.33333333332$$

$$0.33333333334$$

Instead consider *signed* digit representations.

$$n.a_1 a_2 a_3 \ldots$$

$$n : \mathbb{Z} \quad , \quad a_i \in \left\{\bar{1}, 0, 1\right\} \quad , \quad \bar{1} := -1$$

Signed-bit representation, representing the value:

$$n + \sum_{i=1}^{\infty} a_i \cdot 2^{-i}$$

Reals and observable data

Brouwer: A real-valued function $f(x)$ which is defined everywhere on a closed interval of the continuum is uniformly continuous on that interval.[6]

For an appropriate subcategory $\mathbf{T} \subseteq \mathrm{Top}$, in the topos of sheaves $\mathrm{Sh}(\mathbf{T})$ on $\mathbf{T}$, the statement[7] that "every function on the Dedekind reals is continuous" holds.[8]

In the effective topos, the statement that "every function on the Cauchy reals is continuous" holds.[9]

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[6] Heyting 1956, "Intuitionism. An introduction."
[7] As expressed e.g. in the Mitchell-Bénabou internal language of a topos.
[8] MacLane and Moerdijk 1994, "Sheaves in Geometry and Logic"
[9] van Oosten 2008, "Realizability: An Introduction to its Categorical Side"

Part II

Locators

Locators: definition

Recall that $x = (L, U)$ is *located* if

$$\forall (q, r : \mathbb{Q}).(q < r) \Rightarrow (q < x) \vee (x < r).$$

A *locator* for $x : \mathbb{R}_{\mathbf{D}}$ is the data/structure (*not* unique for a given x)

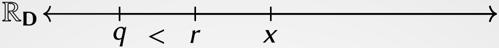
$$\mathrm{locator}(x) :\equiv \prod_{q, r : \mathbb{Q}} q < r \to (q < x) + (x < r).$$

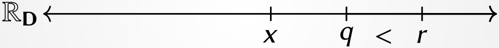
The set of all Dedekind reals with locators:

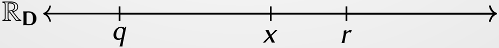
$$\mathbb{R}_{\mathbf{D}}{}^{\mathfrak{L}} :\equiv \sum_{x : \mathbb{R}_{\mathbf{D}}} \mathrm{locator}(x)$$

Locators: visualization

Suppose f : locator(x). What is $f(q, r)$?







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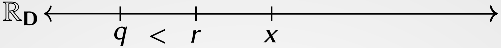
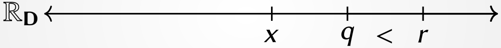
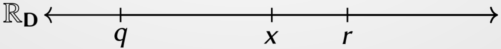


A: The locator must have answered $q < x$, because $x < r$ is false. In other words, we know that isLeft($f(q, r)$).





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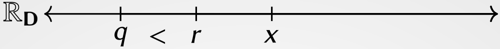

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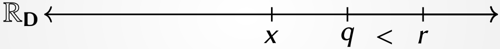

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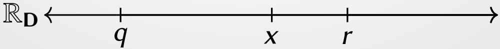

Locators: visualization

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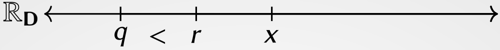

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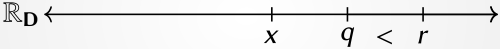

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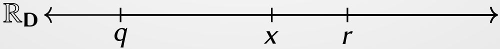


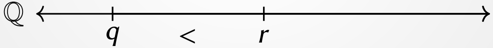
A: The locator can answer both $q < x$ and $x < r$. Both isLeft($f(q, r)$) and isRight($f(q, r)$) are possible.

Locators: examples (1/2)

By trichotomy of the rationals, namely for all $s, t : \mathbb{Q}$,

$$(s < t) + (s = t) + (s > t),$$

the rationals have locators.



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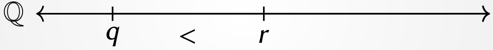

- $s < q$: then $s < q < r$ so answer $s < r$ (right)
- $s = q$: then $s = q < r$ so answer $s < r$ (right)
- $s > q$: then answer $q < s$ (left)

Locators: examples (1/2)

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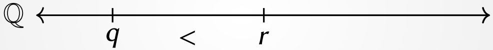


- $s < r$: then answer $s < r$ (right)
- $s = r$: then $q < r = s$ so answer $q < s$ (left)
- $s > r$: then $q < r < s$ so answer $q < s$ (left)

Locators: examples (2/2)

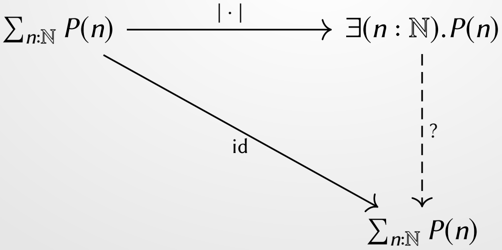
- 0 and 1 are have locators.
- If x and y have locators, then so do $-x$, $x + y$, $x \cdot y$, x/y, $\min(x, y)$ and $\max(x, y)$.
- If all values of a Cauchy approximation come equipped with locators, then the limit of the approximation has a locator, too.

(As we'll see later, in some sense, these are the only examples.)

Structure from property: constructive indefinite description

Let $P : \mathbb{N} \to \mathrm{Prop}$ be a *decidable* proposition, namely $\forall (n : \mathbb{N}). P(n) + \neg P(n)$.

Recall that $\exists (n : \mathbb{N}). P(n) = \| \sum_{n:\mathbb{N}} P(n) \|$.
Objective: to obtain $\sum_{n:\mathbb{N}} P(n)$.



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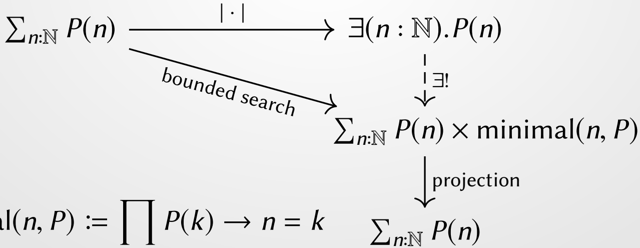
Recall that $\exists (n : \mathbb{N}). P(n) = \| \sum_{n:\mathbb{N}} P(n) \|$.
Objective: to obtain $\sum_{n:\mathbb{N}} P(n)$.



$$\text{minimal}(n, P) :\equiv \prod_{k \leq n} P(k) \to n = k$$

Structure from property: archimedean principle

By definition of $<$, all $x, y : \mathbb{R}_{\mathbf{D}}$ satisfy the archimedean *property*

$$x < y \Rightarrow \exists(q : \mathbb{Q}).x < q < y.$$

Theorem: Reals with locators have the archimedean *structure*

$$x < y \to \sum_{q:\mathbb{Q}} x < q < y.$$

Proof.

Take any bijection $\phi : \mathbb{Q} \times \mathbb{Q}_+ \to \mathbb{N}$. Let $(x, f), (y, g) : {\mathbb{R}_{\mathbf{D}}}^{\mathfrak{L}}$. By applying the archimedean property several times, there *exist* $q : \mathbb{Q}$ and $\varepsilon : \mathbb{Q}_+$ with $x < q - \varepsilon < q + \varepsilon < y$.
For a given pair $(q', \varepsilon') : \mathbb{Q} \times \mathbb{Q}_+$, the following property is decidable

$$P(q', \varepsilon') :\equiv \text{isRight}(f(q' - \varepsilon', q')) \times \text{isLeft}(g(q', q' + \varepsilon')).$$

If $P(q', \varepsilon')$ then $x < q' < y$. But *necessarily* $P(q, \varepsilon)$, and hence by constructive indefinite description we can find a ϕ-minimal pair (q', ε'). □

Structure from property: Dedekind data

Let $x : \mathbb{R}_{\mathbf{D}}{}^{\mathfrak{L}}$ be a real equipped with a locator. Then we also obtain:

1. *strongly bounded:* $\sum_{q:\mathbb{Q}} q < x$ and $\sum_{r:\mathbb{Q}} x < r$.
2. *strongly rounded:* For all $q, r : \mathbb{Q}$,

$$q < x \leftrightarrow \sum_{q':\mathbb{Q}} (q < q') \wedge (q' < x) \qquad \text{and}$$
$$x < r \leftrightarrow \sum_{r':\mathbb{Q}} (r' < r) \wedge (x < r').$$

4. *strongly located:* $(q < r) \to (q < x) + (x < r)$ for all $q, r : \mathbb{Q}$.

Proof.
E.g. for strong boundedness: apply the strong archimedean structure to $x - 1 < x$ resp. $x < x + 1$.
strongly located = locator □

Structure from property: field structure

For $x, y : {\mathbb{R}_{\mathbf{D}}}^{\mathfrak{L}}$, the definitions of the ring structure on the dedekind reals can be strengthened. For $q, r : \mathbb{Q}$, we obtain:

$$
\begin{aligned}
q < x + y &\quad\leftrightarrow\quad \sum_{s, t : \mathbb{Q}} (q = s + t) \wedge s < x \wedge t < y \\
x + y < r &\quad\leftrightarrow\quad \sum_{s, t : \mathbb{Q}} (r = s + t) \wedge x < s \wedge y < t \\
q < -x &\quad\leftrightarrow\quad \sum_{r : \mathbb{Q}} q = -r \wedge x < r \\
-x < r &\quad\leftrightarrow\quad \sum_{q : \mathbb{Q}} r = -q \wedge q < x \\
q < x \cdot y &\quad\leftrightarrow\quad \sum_{a, b, c, d : \mathbb{Q}} q < \min(a \cdot c, a \cdot d, b \cdot c, b \cdot d) \\
&\qquad\qquad \wedge\, a < x < b \wedge c < y < d \\
&\quad\vdots
\end{aligned}
$$

Structure from property: strong cotransitivity

Cotransitivity $x < y \Rightarrow x < z \vee z < y$ strengthens to $x < y \to (x < z) + (z < y)$ for $x, y, z : \mathbb{R}_{\mathbf{D}}{}^{\mathfrak{L}}$.

Collectively, these "structure from property" results suggest that many constructions on the Dedekind reals will lift to reals with locators.

Position of truncation

Let $x : \mathbb{R}_{\mathbf{D}}$.

$$\prod_{q,r:\mathbb{Q}} q < r \to \|(q < x) + (x < r)\|$$

This expresses that x is *located* (property) in the sense of Dedekind cuts. It is equivalent to:

$$\prod_{q,r:\mathbb{Q}} \|q < r \to (q < x) + (x < r)\|.$$

It is *not* equivalent to

$$\left\| \prod_{q,r:\mathbb{Q}} q < r \to (q < x) + (x < r) \right\|$$

which expresses that there *exists* a locator (structure) for x.

Reals with locators and Cauchy reals

Suppose we have a locator for $x : \mathbb{R}_{\mathbf{D}}$.
Define a Cauchy approximation $x_{\cdot} : \mathbb{Q}_{+} \to \mathbb{Q}$ by letting x_{ε} to be the rational obtained from the strong archimedean structure applied to $x - \varepsilon < x + \varepsilon$. In other words, obtain rationals x_{ε} with $x - \varepsilon < x_{\varepsilon} < x + \varepsilon$.
Indeed:

$$-\varepsilon - \theta < x_{\varepsilon} - x_{\theta} < \varepsilon + \theta$$

and hence

$$|x_{\varepsilon} - x_{\theta}| < \varepsilon + \theta$$

Conversely, given a rationally-valued Cauchy approximation $x_{\cdot} : \mathbb{Q}_{+} \to \mathbb{Q}$ whose limit is $x : \mathbb{R}_{\mathbf{D}}$, we can construct a locator for x.

- Dedekind reals for which there *exists* a locator are in the image of the inclusion of the Cauchy reals into the Dedekind reals.
- "x is a Cauchy real" is logically equivalent to "there *exists* a locator for x".

Part III

Case study

Lifting locators

Let $f : \mathbb{R}_{\mathbf{D}} \to \mathbb{R}_{\mathbf{D}}$.

Say f is *lifts locators* if, whenever x comes equipped with a locator, we can construct a locator for $f(x)$. “Lifting locators” itself is structure.

NB: no coherence conditions.

$$\begin{array}{ccc} {\mathbb{R}_{\mathbf{D}}}^{\mathfrak{L}} & \longrightarrow & {\mathbb{R}_{\mathbf{D}}}^{\mathfrak{L}} \\ \downarrow{\scriptstyle \mathrm{pr}_1} & \circ & \downarrow{\scriptstyle \mathrm{pr}_1} \\ \mathbb{R}_{\mathbf{D}} & \xrightarrow{f} & \mathbb{R}_{\mathbf{D}} \end{array}$$

Continuity

$f : \mathbb{R}_{\mathbf{D}} \to \mathbb{R}_{\mathbf{D}}$ is *continuous at* $x : \mathbb{R}_{\mathbf{D}}$ if

$$\forall(\varepsilon : \mathbb{Q}_+).\exists(\delta : \mathbb{Q}_+).\forall(y : \mathbb{R}_{\mathbf{D}}).|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

f is *pointwise continuous* if it is continuous at all $x : \mathbb{R}_{\mathbf{D}}$.

f is *uniformly continuous on* $[a, b]$ if it has a modulus of continuity $\omega : \mathbb{Q}_+ \to \mathbb{Q}_+$ for $[a, b]$, i.e.:

$$\forall(x, y \in [a, b]).|x - y| < \omega(\varepsilon) \Rightarrow |f(x) - f(y)| < \varepsilon$$

Riemann integration

Suppose:

- $f : \mathbb{R}_{\mathbf{D}} \to \mathbb{R}_{\mathbf{D}}$ is uniformly continuous on $[a, b]$, and
- a and b have locators, and
- f lifts locators

then $\int_a^b f(x)\,\mathrm{d}x$ has a locator.

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Proof.

$\int_a^b f(x)\,\mathrm{d}x$ is defined as

$$\mathrm{lim}_{n\to\infty} \frac{b-a}{n} \sum_{k=0}^{n-1} f\left(a + k \cdot \frac{b-a}{n}\right)$$

which has a locator. □

Exact IVT (1/2)

$f : \mathbb{R}_{\mathbf{D}} \to \mathbb{R}_{\mathbf{D}}$ is *locally nonconstant* if for all $x < y$ and $t : \mathbb{R}_{\mathbf{D}}$, there exists $z : \mathbb{R}_{\mathbf{D}}$ with $x < z < y$ and $f(z) \# t$ (namely, f is *apart* from t, meaning $f(z) > t \vee f(z) < t$).

Theorem: Suppose

- f is pointwise continuous, and
- f is locally nonconstant, and
- x, y and t have locators, and
- f lifts locators

then we can find $r : \mathbb{Q}$ with $x < r < y$ and $f(r) \# t$.
("strong local nonconstancy")

Proof.

Since f is locally nonconstant, there *exist* $z : \mathbb{R}_{\mathbf{D}}$ and $\varepsilon : \mathbb{Q}_+$ with $|f(z) - t| > \varepsilon$. Since f is continuous at z, there exists $q : \mathbb{Q}$ with $|f(q) - t| > \varepsilon/2$. Since $\mathbb{Q}_+$ and $\mathbb{Q}$ are denumerable, we can find $r : \mathbb{Q}$ such that there exists $\eta : \mathbb{Q}_+$ with $|f(r) - t| > \eta$. In particular r satisfies $|f(r) - t| > 0$, that is, $f(r) \# t$. □

Exact IVT (2/2)

Theorem: Suppose

- f is pointwise continuous, and
- f is locally nonconstant, and
- x and y have locators, and
- f lifts locators, and
- $f(x) \leq 0 \leq f(y)$,

then we can *find* a root of f.

Proof.
From the previous slide, we obtain strong local nonconstancy of f.
Then apply an appropriate form of bisection. □

Part IV

Classical axioms via locators

Do Dedekind reals have signed digit expansions?

Theorem: If, for every Dedekind real x, there exists a signed-digit expansion, then the Cauchy reals and the Dedekind reals coincide.

Do Dedekind reals have signed digit expansions?

Theorem: If, for every Dedekind real x, there exists a signed-digit expansion, then the Cauchy reals and the Dedekind reals coincide.

Proof: x has a locator iff it has a signed digit expansion. A signed digit expansion is a Cauchy approximation whose elements are of a certain form. Conversely, if x is the limit of a rationally-valued Cauchy approximation, then it has a locator.

Unsigned binary expansion, dyadic locators, LLPO

Theorem: x has a binary expansion iff x has the structure

$$\prod_{p:\mathbb{Q}_2} (x \le p) + (x \ge p)$$

(with $\mathbb{Q}_2$ the dyadic rationals).

[10] LLPO: For every binary sequence $(a_n)_n$, if there is at most one a_n that is true, then either all even entries are false, or all odd entries are false.

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For $p = 0$, we get strong dichotomy of the reals: $(x \leq 0) + (x \geq 0)$.

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For $p = 0$, we get strong dichotomy of the reals: $(x \le 0) + (x \ge 0)$.

Theorem: if every Cauchy real has a binary expansion, then LLPO[10] holds.

[10] LLPO: For every binary sequence $(a_n)_n$, if there is at most one a_n that is true, then either all even entries are false, or all odd entries are false.

Part V

Questions, comments…

Part VI

Appendix

Unsigned digit representations

$x : \mathbb{R}_{\mathbf{D}}$ has a *signed* bit representation iff it has a locator

$$\prod_{q,r:\mathbb{Q}} q < r \to (q < x) + (x < r)$$

$x : \mathbb{R}_{\mathbf{D}}$ has an *unsigned* bit representation iff it comes equipped with the data

$$\prod_{q:\mathbb{Q}_2} (q \leq x) + (x \leq q)$$

where $\mathbb{Q}_2$ is the type of dyadic rationals.

The existence of unsigned bit representations for all Cauchy reals is equivalent to LLPO.

Ordered prefields

An *ordered prefield* is a set F together with constants 0, 1, a unary operation $-$, binary operations $+$, $\cdot$, min, max, and a binary relation $<$ such that:

(i) the strict order $<$ is transitive, irreflexive and cotransitive;

(ii) where we define for $x, y : F$:

$$x \leq y :\equiv \neg(y < x)$$
$$x \# y :\equiv (x < y) \vee (y < x)$$
$$x \approx y :\equiv (x \leq y) \wedge (y \leq x)$$

(iii) $(F, \leq, \min, \max)$ is a prelattice (this states that min and max are meets and joins with respect to $\leq$, but $\leq$ need not be antisymmetric in the ordinary sense);

(iv) $(F, 0, 1, -, +, \cdot)$ is a commutative ring with respect to the equivalence relation $\approx$;

(v) we can find a multiplicative inverse of $x : F$ with respect to $\approx$ if and only if $x \# 0$;

(vi) for all $x, y, z : F$:

$$x < y \Leftrightarrow x + z < y + z,$$
$$0 < x + y \Rightarrow 0 < x \vee 0 < y,$$
$$0 < z \Rightarrow (x < y \Leftrightarrow xz < yz),$$
$$0 < 1.$$

Strong archimedean structure

A *strong archimedean prefield* is an ordered prefield F with an additional structure

$$\prod_{x,y:F} x < y \to \sum_{q:\mathbb{Q}} x < q < y.$$

- The type ${\mathbb{R}_{\mathbf{D}}}^{\mathfrak{L}}$ of Dedekind reals with locators is a strong archimedean prefield.
- There is a canonical map from any ordered prefield to the Dedekind reals.
- Any element of a strong archimedean prefield automatically has a locator. Combined with the previous fact, this says that elements of strong archimedean prefields can canonically be seen as Dedekind reals with locators.

Univalence refresher

If $g : X \to Y$ is an isomorphism (i.e. g has a two-sided inverse h), then g is an equivalence. Write:

$$g : X \simeq Y$$

Univalence axiom[11]: from an equivalence $g : X \simeq Y$ we may derive an identity $\mathrm{ua}(g) : X =_{\mathcal{U}} Y$.

Recall Prop: those $P : \mathcal{U}$ with $x =_P y$ for all $x, y : P$.

Given $p : P$, can define equivalence $P \simeq \mathbf{1}$ (by mapping everything in P to $\star : \mathbf{1}$, and mapping everything in $\mathbf{1}$ to $p : P$).
($\mathbf{1}$ is the type whose only point is $\star : \mathbf{1}$)

[11]The actual univalence axiom also gives us computation rules for this conversion from equivalences to identity paths.

Contractible types and propositions

$$\mathrm{isContr}(A) :\equiv \Sigma_{a:A}\Pi_{x:A} a =_A x$$

$$\mathrm{isProp}(A) :\equiv \Pi_{x,y:A} x =_A y$$

In A is a proposition, inhabited $a : A$, we can take a to be the center of contraction.

$\mathbf{1}$ is contractible. If A is contractible, then $A \simeq \mathbf{1}$.