Logical and computational aspects of Gleason's theorem in probability theory

Willem L. Fouché

We explore the identification of the computational content, and implications of, having constructive proofs of the important Gleason theorem in quantum logic [1] (1957). A constructive proof of a suitable classical reformulation of Gleason's theorem appears in Richman and Bridges [4] (1999).

Write E^n for the Hilbert space of dimension n over the reals and write S^{n-1} for the unit sphere in E^n .

In this case Gleason's theorem states that, if $p : S^{n-1} \to [0, 1]$ is such that if p is in fact a function on the *rays* in H, (meaning that $p(-x) = p(x)$, for all relevant x), and for for each frame (orthonormal basis) $f = (e_i) \subset S^{n-1}$, we have that $\sum_{\alpha \in f} p(\alpha) = 1$; then there is some density matrix (a quantum state) ρ on H such that $p(x) = (x, \rho x)$, for all $x \in S^{n-1}$.

A *(quantum) state* ρ or a *density matrix*, is a Hermitian, positive operator on H of trace 1. Being positive means that $(\rho x, x) \geq 0$ for all $x \in H$.

We shall discuss the computational ramifications and physical implications of the following statement. This statement is a consequence of the arguments in the paper HP [3].

Theorem 1 *We can algorithmically and uniformly and construct for every natural number n, a first-order statement Π_n in the theory $\mathbf{R}$ of real closed fields which is* `classically` *equivalent to Gleason's theorem for E^n. An analogous result holds for the first order theory $\mathbf{C}$ for the field $\mathbb{C}$ of complex numbers and the version of Gleason's theorem for finite dimensional Hilbert spaces over $\mathbb{C}$.*

References

[1] Gleason, A. M. (1957). Measures on the closed subspaces of a Hilbert space. Journal of Mathematics and Mechanics, 6, 885-893.

[2] Hellman, G. (1993). Gleason's theorem is not constructively provable. Journal of Philosophical Logic, 22, 193-203.

[3] Hrushovski, E., & Pitowski I. (2004). Generalizations of Kochen and Specker's theorem and the effectiveness of Gleason's theorem. Studies in History and Philosophy of Modern Physics, 35, 177-194.

[4] Richman, F., & Bridges, D. (1999). A constructive proof of Gleason's theorem. Journal of Functional Analysis, 162, 287-312.