Logical and computational aspects of Gleason’s theorem in probability theory

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Write $E^n$ for the Hilbert space of dimension $n$ over the reals and write $S^{n-1}$ for the unit sphere in $E^n$.

In this case Gleason’s theorem states that, if $p : S^{n-1} \rightarrow [0,1]$ is such that if $p$ is in fact a function on the rays in $H$, (meaning that $p(-x) = p(x)$, for all relevant $x$), and for each frame (orthonormal basis) $f = (e_i) \subset S^{n-1}$, we have that $\sum_{\alpha \in f} p(\alpha) = 1$; then there is some density matrix (a quantum state) $\rho$ on $H$ such that $p(x) = (x, \rho x)$, for all $x \in S^{n-1}$.

A (quantum) state $\rho$ or a density matrix, is a Hermitian, positive operator on $H$ of trace 1. Being positive means that $(\rho x, x) \geq 0$ for all $x \in H$.

We shall discuss the computational ramifications and physical implications of the following statement. This statement is a consequence of the arguments in the paper HP [3].

**Theorem 1** We can algorithmically and uniformly and construct for every natural number $n$, a first-order statement $\Pi_n$ in the theory $\mathbb{R}$ of real closed fields which is classically equivalent to Gleason’s theorem for $E^n$. An analogous result holds for the first order theory $\mathbb{C}$ for the field $\mathbb{C}$ of complex numbers and the version of Gleason’s theorem for finite dimensional Hilbert spaces over $\mathbb{C}$.

**References**

