

Strictly associative and unital higher category theory

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(With Christoph Dorn, Christopher Douglas, and David Reutter)

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Working with higher structures

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- ▶ How can we use computers to help us achieve these goals, not only theoretically, but practically?

Proof assistants for higher algebra

Several proof assistants for higher category theory exist.

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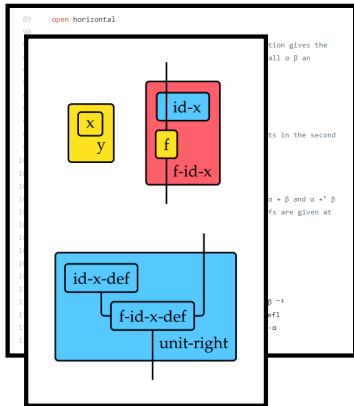
- ▶ Homotopy type theory in Agda, Coq (2010)

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89 open horizontal
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91 -- The shows that either ways of horizontal composition gives the
92 -- same higher homotopy. Here we need to induct on all  $\alpha \beta$  an
93  $***' : \{p : a = a\}(r : a = a)$ 
94    $(\alpha : p \equiv \text{refl}_1) (\beta : \text{refl}_1 \equiv r)$ 
95      $\rightarrow \alpha \cdot \beta \equiv \alpha \cdot r$ 
96  $***' \text{refl refl} = \text{refl}$ 
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98 -- The specialisation of the above lemma for elements in the second
99 -- loop space.
100  $***'-\text{refl} : \{a \beta : D^2(A, a)\} \rightarrow \alpha \cdot \beta \equiv \alpha \cdot r$ 
101  $***'-\text{refl} (\alpha) (\beta) = ***' \alpha \beta$ 
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103 -- When every path is refl horizontal compositions  $\alpha \cdot \beta$  and  $\alpha \cdot r$ 
104 -- reduce to  $\alpha \cdot \beta$  and  $\beta \cdot \alpha$  respectively. The proofs are given at
105 -- the end.
106  $\alpha\beta\text{Refl} : \{a \beta : D^2(A, a)\} \rightarrow \alpha \cdot \beta \equiv \alpha \cdot \beta$ 
107  $\alpha\beta\text{Refl} (\alpha) (\beta) = \alpha \cdot \beta$ 
108
109 -- From which we get the Eckmann-Hilton theorems.
110 Eckmann-Hilton :  $\{a \beta : D^2(A, a)\} \rightarrow \alpha \cdot \beta \equiv \beta \cdot \alpha$ 
111 Eckmann-Hilton  $\alpha \beta = \text{begin } \alpha \cdot \beta = \alpha \cdot \beta \text{ by } \alpha\beta\text{Refl}^{-1}$ 
112    $= \alpha \cdot r \text{ by } ***'-\text{refl}$ 
113    $= \beta \cdot \alpha \text{ by } \alpha\beta\text{Refl}$ 
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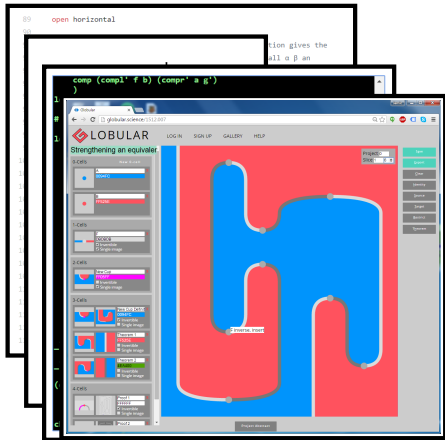
The screenshot shows a code editor with the following text:

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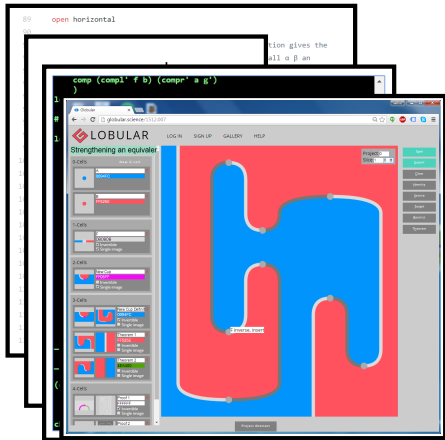
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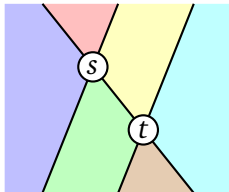
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Our goal: easy construction, visualization and manipulation of complex proofs in arbitrary dimension. None of these achieve this.

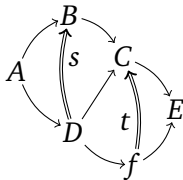
Graphical calculus

It is conjectured that n -categories have an n -dimensional graphical calculus, which is the dual of the ordinary 'commutative diagrams'.



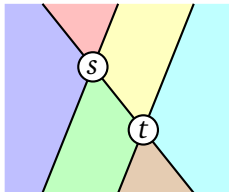
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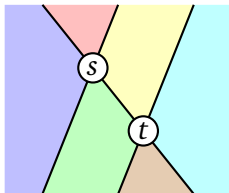
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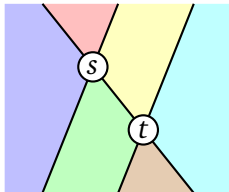
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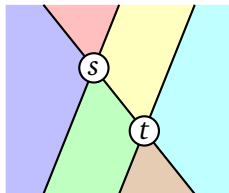
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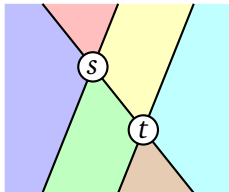
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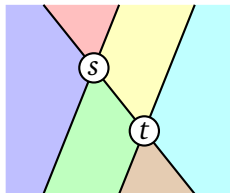
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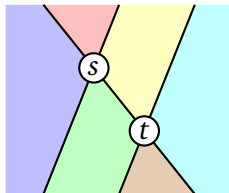
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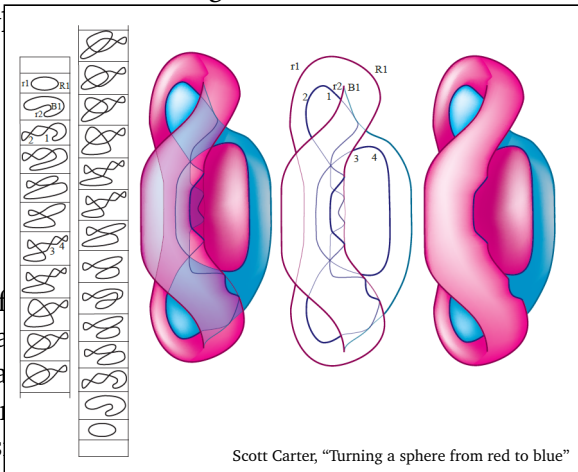
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Graphical calculus

It is conjectured that n -categories have an n -dimensional graphical calculus, with 'diagrams'.



Scott Carter, "Turning a sphere from red to blue"

- ▶ Idea of
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- ▶ Joyal a
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At the opposite end of the spectrum, the ‘weakest’ definitions allow not only these manipulations, but far more besides.

Weak



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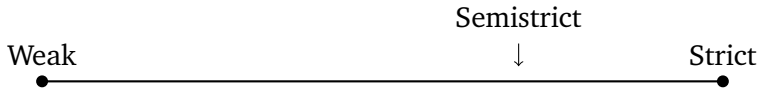


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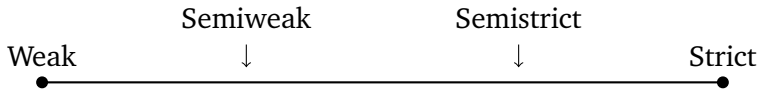
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A model of higher categories is *semistrict* if it is as *strict* as possible, while still allowing arbitrary homotopy. However, yields long proofs.

A definition is *semiweak* if it is as *weak* as possible, while still being strictly associative and unital; i.e. composites are unique.

Associative n -categories

We propose a new semiweak approach to higher category theory called *associative n -categories*.

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- ▶ Amenable to computer implementation.

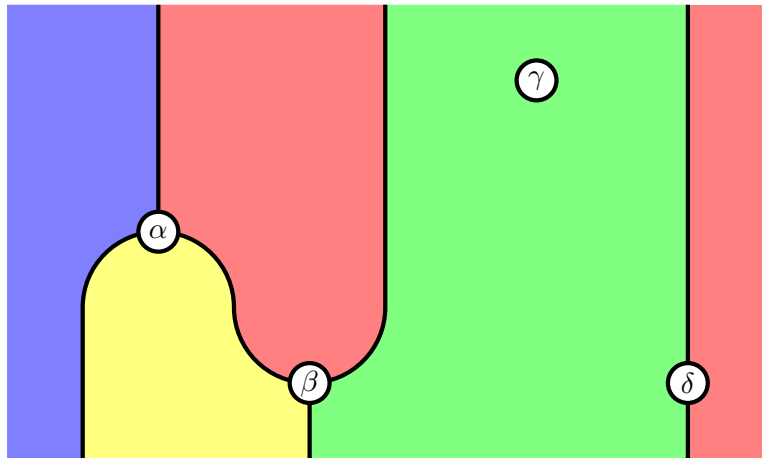
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- ▶ All the weak structure is in homotopies of composites.
- ▶ Amenable to computer implementation.
- ▶ High-level methods allow easy homotopy construction.

Monotone functions

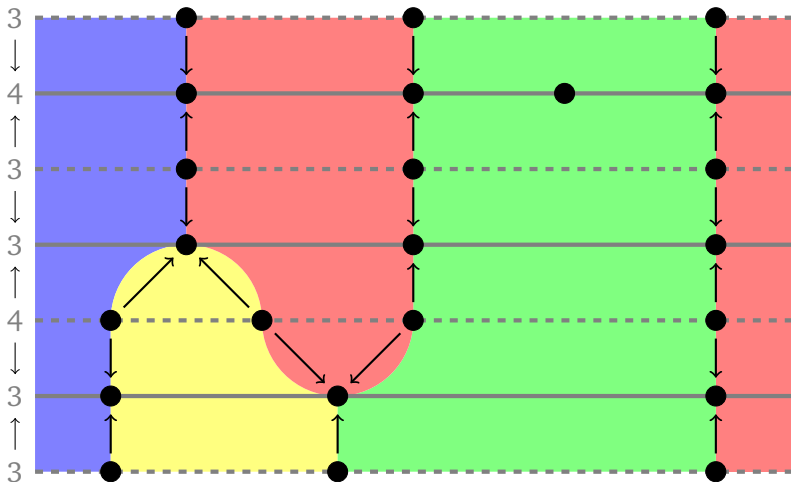
Consider the following string diagram in a bicategory.



Monotone functions

Consider the following string diagram in a bicategory.

It gives rise to a sequence of *cospan*s of monotone functions.



Diagrams from iterated cospans

Definition. Given a category \mathcal{C} , with $A, B \in \text{Ob}(\mathcal{C})$, we define the *category of iterated cospans* $IC_{\mathcal{C}}(A, B)$ as follows:

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- ▶ an object is a sequence of cospans in \mathcal{C} , from A to B ;

$$A \longrightarrow P_0 \longleftarrow V_1 \longrightarrow P_1 \longleftarrow V_2 \longrightarrow P_2 \longleftarrow V_3 \longrightarrow P_3 \longleftarrow B$$

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- ▶ a morphism is:

$$A \longrightarrow P_0 \longleftarrow V_1 \longrightarrow P_1 \longleftarrow V_2 \longrightarrow P_2 \longleftarrow V_3 \longrightarrow P_3 \longleftarrow B$$

$$A' \longrightarrow P'_0 \longleftarrow V'_1 \longrightarrow P'_1 \longleftarrow V_2 \longrightarrow P'_2 \longleftarrow B'$$

Diagrams from iterated cospans

Definition. Given a category \mathcal{C} , with $A, B \in \text{Ob}(\mathcal{C})$, we define the *category of iterated cospans* $IC_{\mathcal{C}}(A, B)$ as follows:

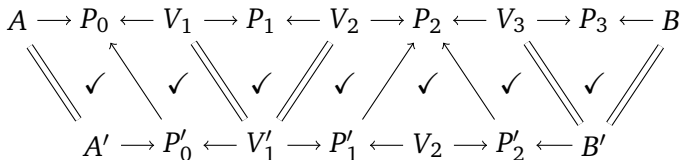
- ▶ an object is a sequence of cospans in \mathcal{C} , from A to B ;
- ▶ a morphism is:
 - a monotone function between peaks,

$$\begin{array}{ccccccccccc} A & \longrightarrow & P_0 & \longleftarrow & V_1 & \longrightarrow & P_1 & \longleftarrow & V_2 & \longrightarrow & P_2 & \longleftarrow & V_3 & \longrightarrow & P_3 & \longleftarrow & B \\ & & & & \nearrow & & & & \nwarrow & & \nwarrow & & \nwarrow & & & & \\ & & & & & & A' & \longrightarrow & P'_0 & \longleftarrow & V'_1 & \longrightarrow & P'_1 & \longleftarrow & V_2 & \longrightarrow & P'_2 & \longleftarrow & B' \end{array}$$

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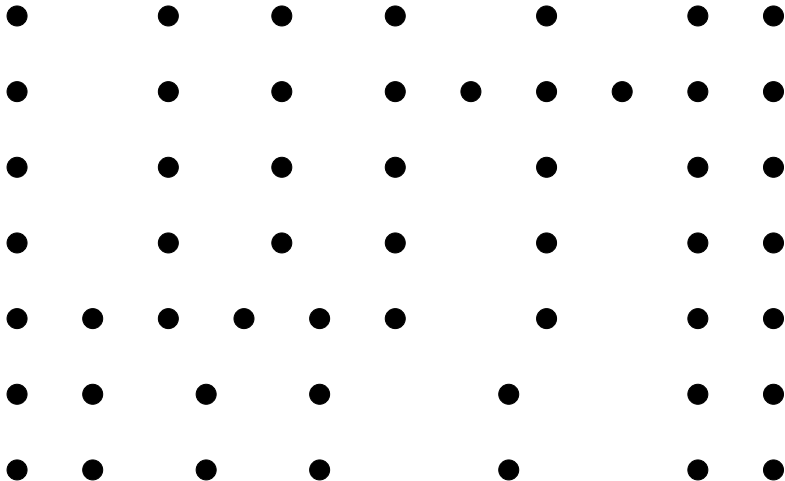
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 - built from morphisms of \mathcal{C} ,
 - with identities between valleys,
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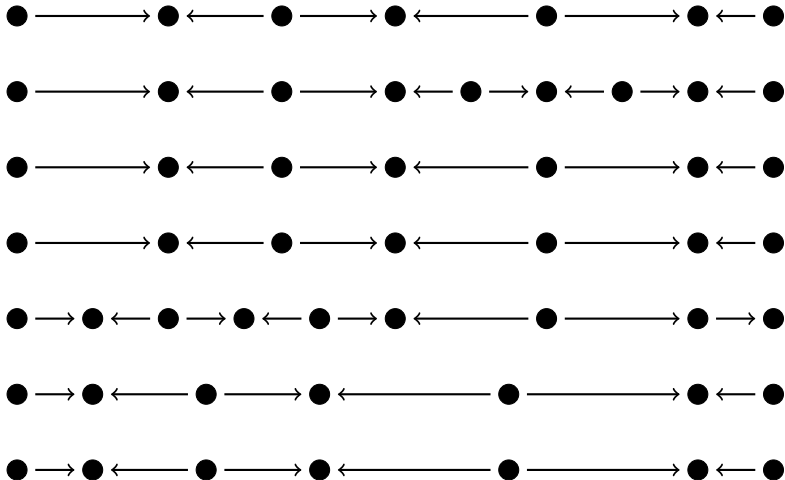
Diagrams from iterated cospans

Here are 53 examples of 0-diagrams, all objects of 1:



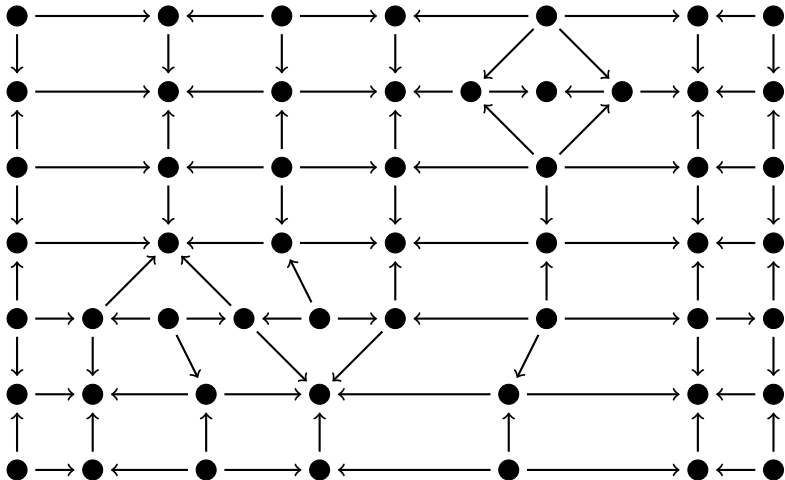
Diagrams from iterated cospans

Here are 7 examples of 1-diagrams, all objects of $IC_1(\bullet, \bullet) = \Delta$:



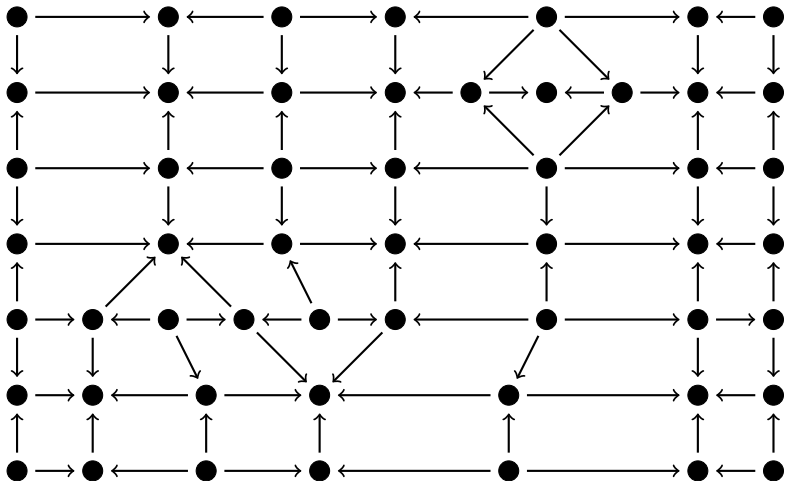
Diagrams from iterated cospans

Here is 1 example of a 2-diagram, an object of $IC_{\Delta}(3,3)$:



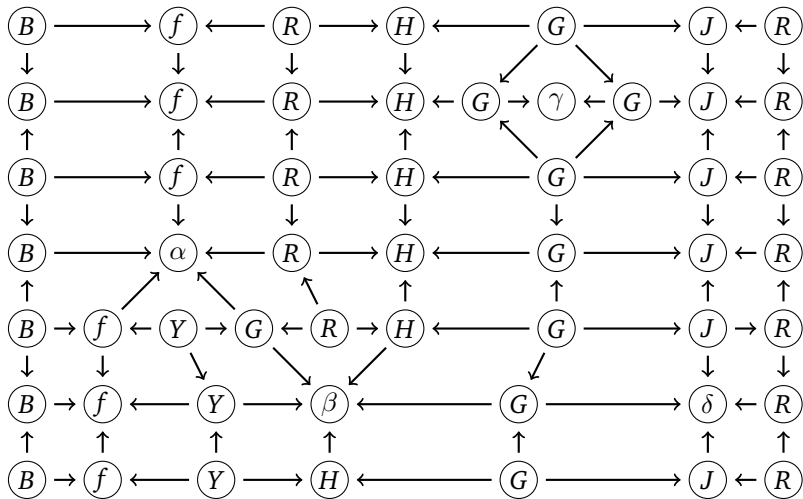
Types

These diagrams are *untyped*.



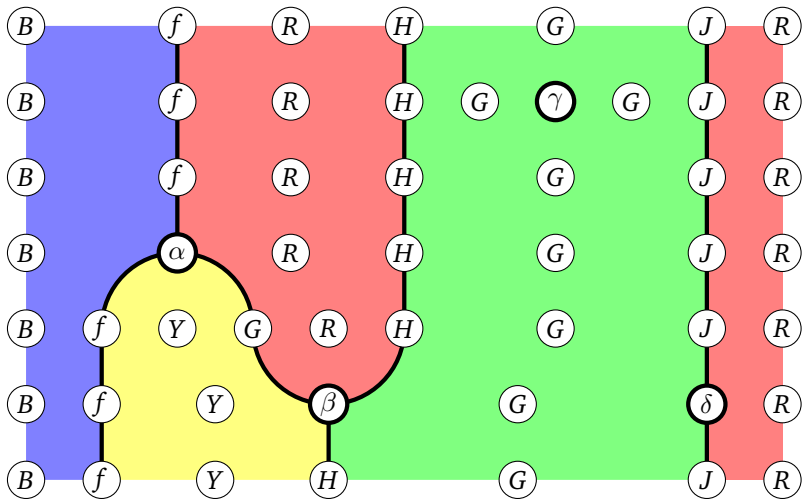
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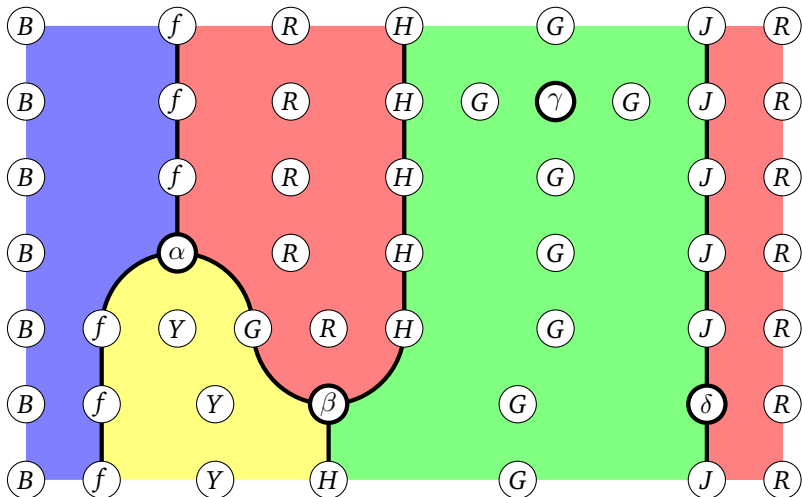
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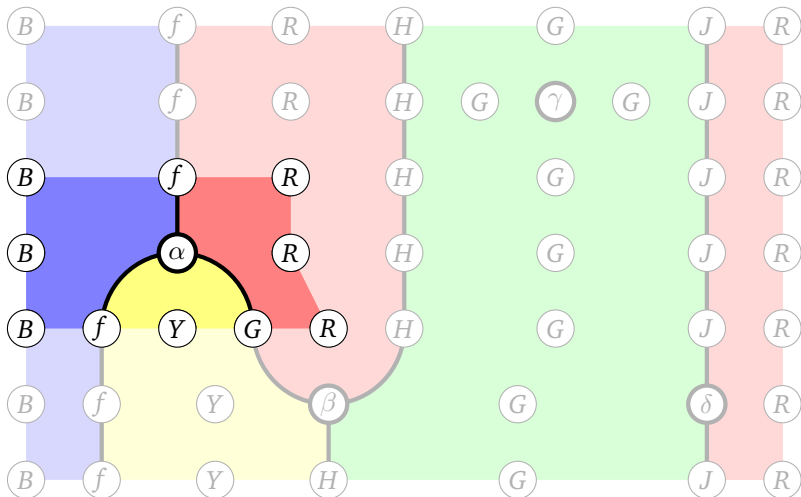
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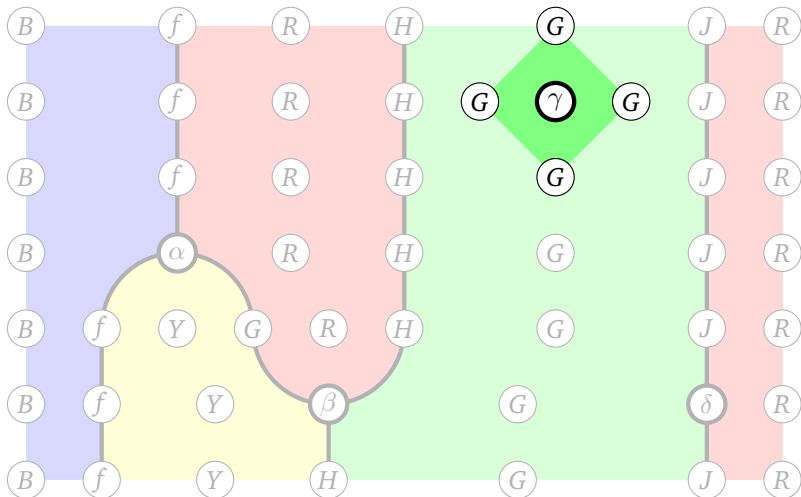
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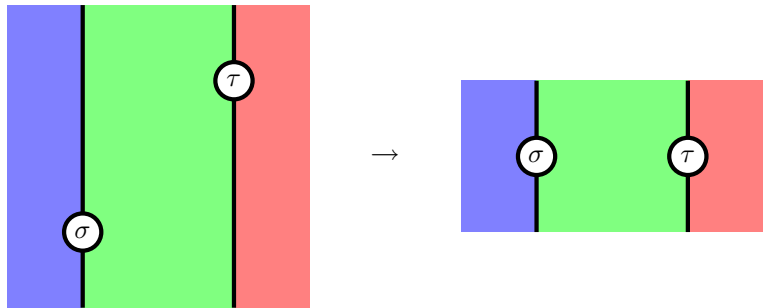
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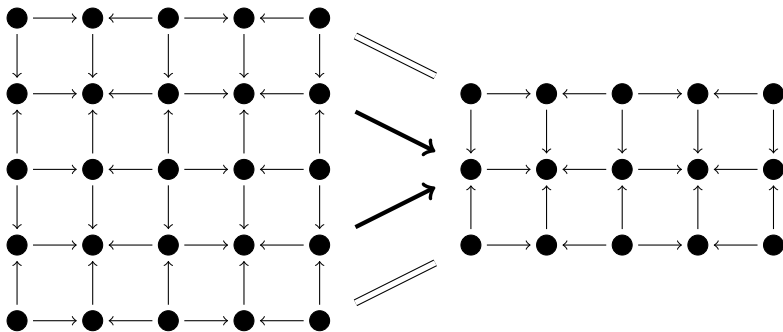
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Homotopies

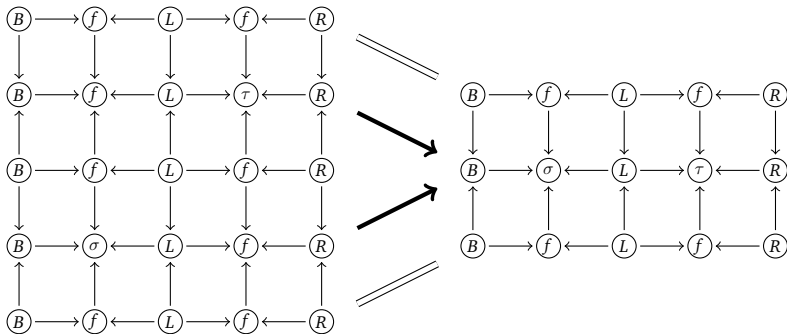
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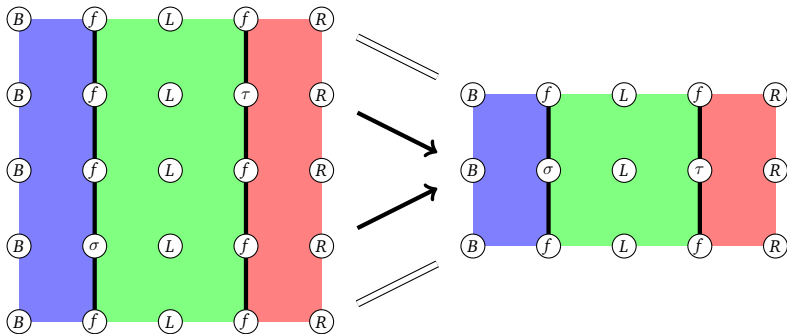


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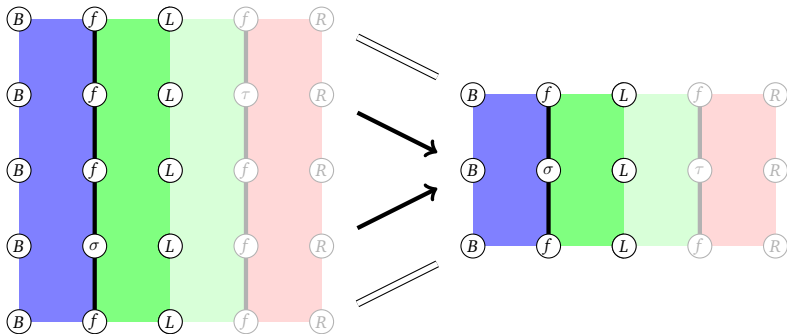


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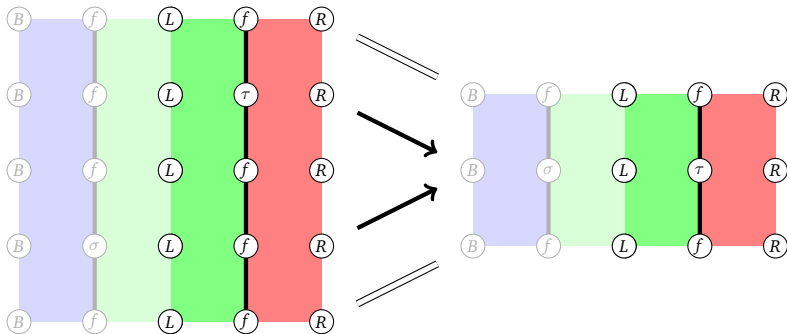
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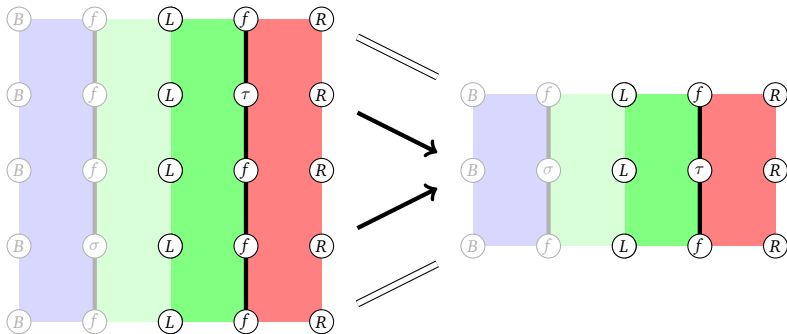
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So homotopies are built in to associative n -categories.

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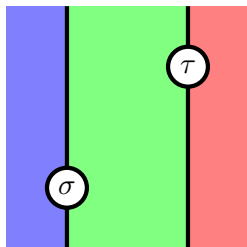
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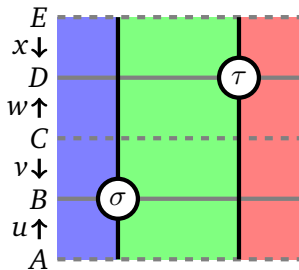
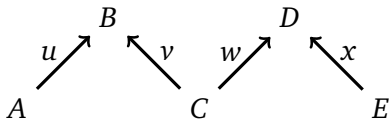


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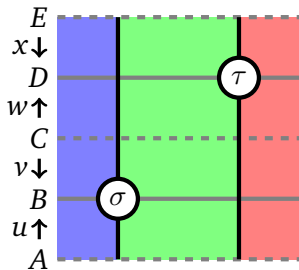
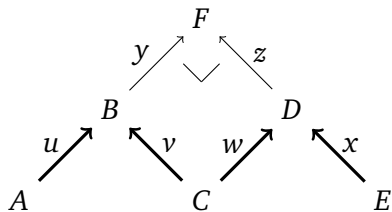
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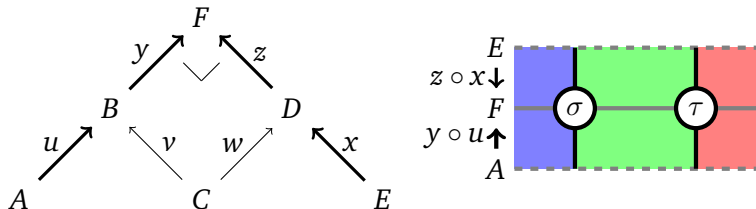
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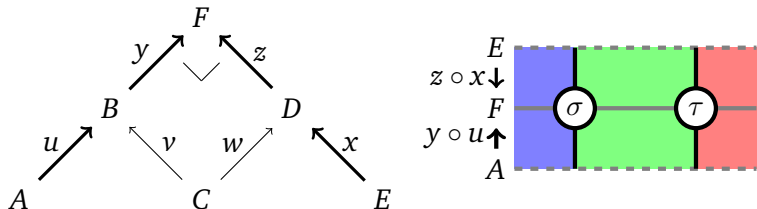
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Gives an insight into the relationship with virtual n -categories.

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