Strictly associative and unital higher category theory

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(With Christoph Dorn, Christopher Douglas, and David Reutter)

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- How can we communicate it to collaborators and readers, and learn more about it ourselves?
- How can we modify it, discover its properties, and prove theorems about it?
- How can we use computers to help us achieve these goals, not only theoretically, but practically?

Proof assistants for higher algebra

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open horizontal

-- The shows that either ways of horizontal composition gives the
-- same higher homotopy. Here we need to induct on all α β an
⋆≡⋆′ : {p : a ≡ a}{r : a ≡ a}
      (α : p ≡ reflₐ) (β : reflₐ ≡ r )
     → α ⋆ β ≡ α ⋆′ β
⋆≡⋆′ refl refl = refl

-- The specialisation of the above lemma for elements in the second
-- loop space.
⋆≡⋆′-refl : {α β : Ω² (A , a)} → α ⋆ β ≡ α ⋆′ β
⋆≡⋆′-refl {α} {β} = ⋆≡⋆′ α β

-- When every path is refl horizontal compositions α ⋆ β and α ⋆′ β
-- reduce to α · β and β · α respectively. The proofs are given at
-- the end.
α⋆β≡α·β : {α β : Ω² (A , a)} → α ⋆ β ≡ α · β
α⋆′β≡β·α : {α β : Ω² (A , a)} → α ⋆′ β ≡ β · α

-- From which we get the Eckmann-Hilton theorem.
Eckmann-Hilton : (α β : Ω² (A , a)) → α · β ≡ β · α
Eckmann-Hilton α β = begin α · β ≡ α ⋆  β by α⋆β≡α·β ⁻¹
                               ≡ α ⋆′ β by ⋆≡⋆′-refl
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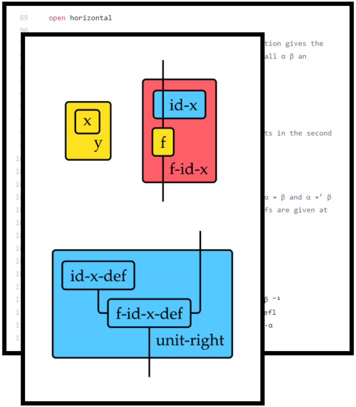

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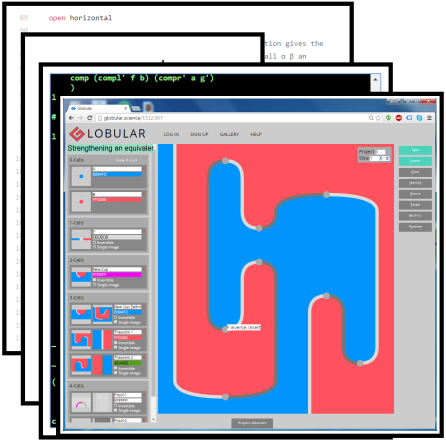

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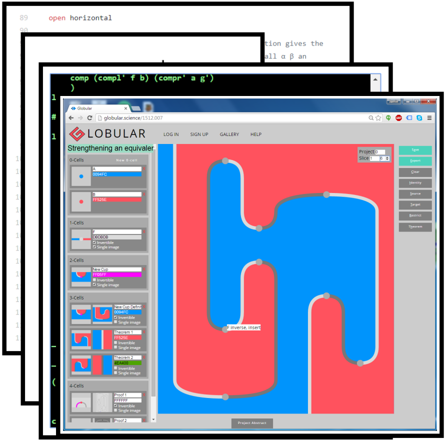
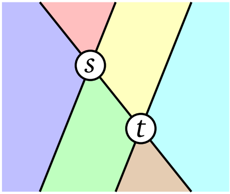


Our goal: easy construction, visualization and manipulation of complex proofs in arbitrary dimension. None of these achieve this.

Graphical calculus

It is conjectured that n-categories have an n-dimensional graphical calculus, which is the dual of the ordinary 'commutative diagrams'.



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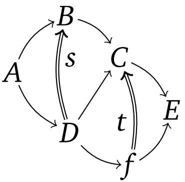

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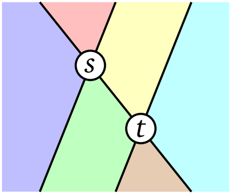

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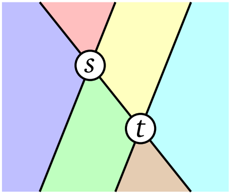

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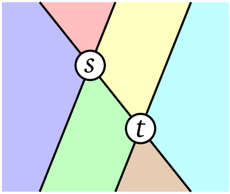

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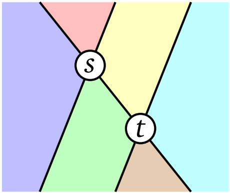

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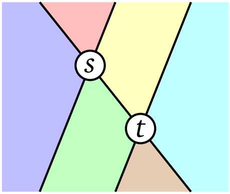

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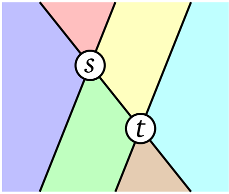

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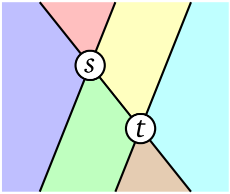

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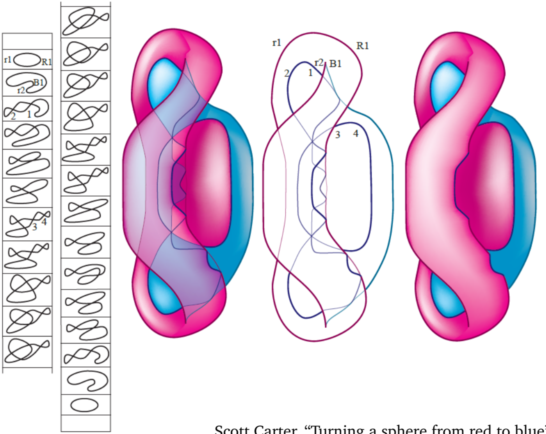

Scott Carter, "Turning a sphere from red to blue"

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A model of higher categories is *semistrict* if it is as *strict* as possible, while still allowing arbitrary homotopy. However, yields long proofs.

A definition is *semiweak* if it is as *weak* as possible, while still being strictly associative and unital; i.e. composites are unique.

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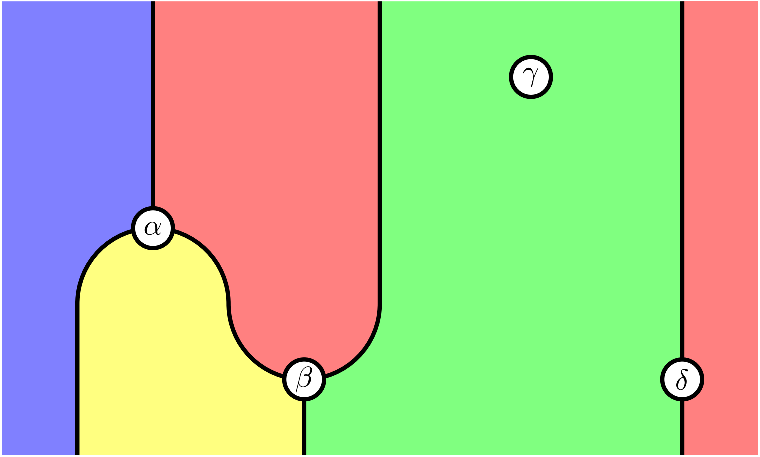
- Amenable to computer implementation.
- High-level methods allow easy homotopy construction.

Monotone functions

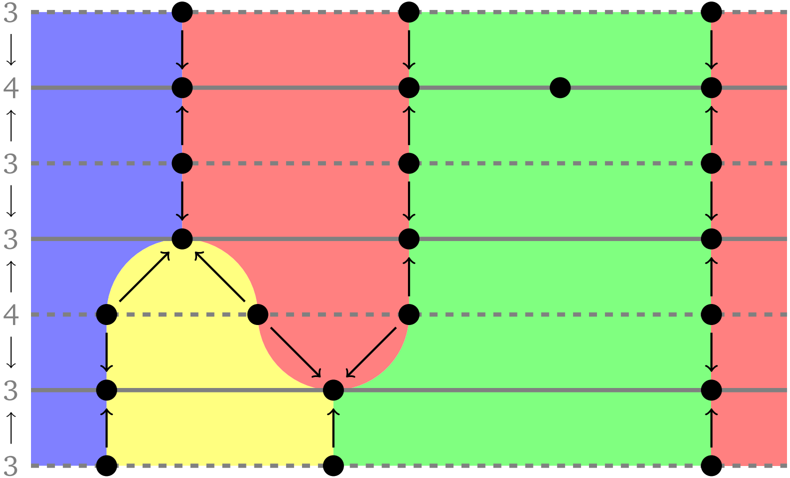
Consider the following string diagram in a bicategory.



Monotone functions

Consider the following string diagram in a bicategory.

It gives rise to a sequence of *cospans* of monotone functions.



Diagrams from iterated cospans

Definition. Given a category $\mathcal{C}$, with $A, B \in \mathrm{Ob}(\mathcal{C})$, we define the *category of iterated cospans* $IC_{\mathcal{C}}(A, B)$ as follows:

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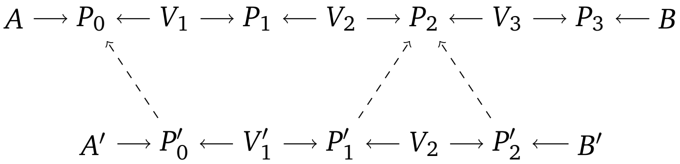

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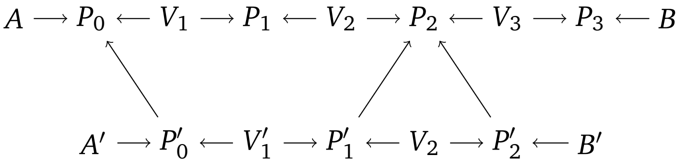

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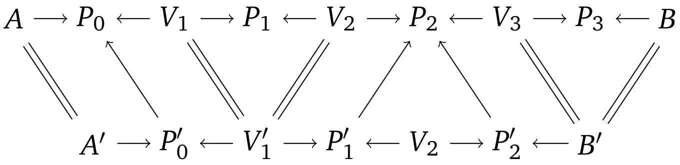

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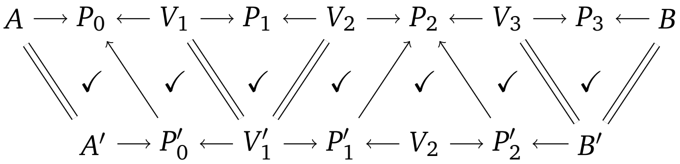

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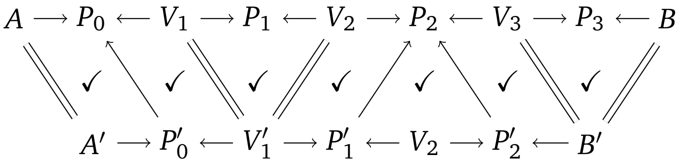
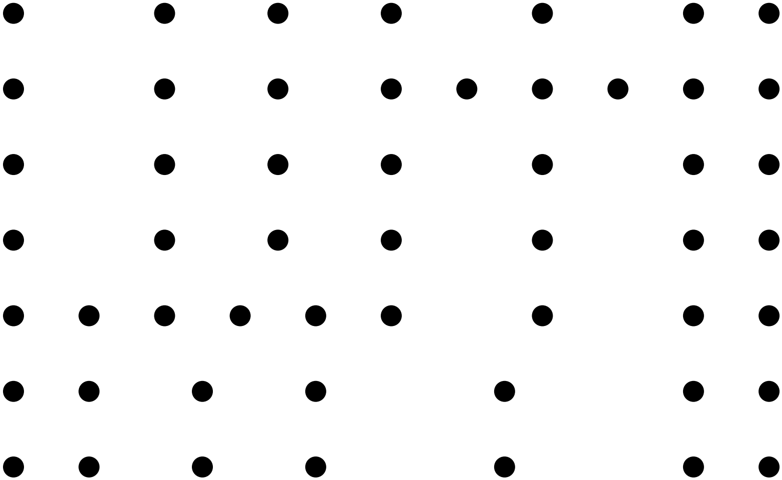


Definition. An *n-diagram* is an object of some category obtained by starting with 1, and applying the IC construction n times.

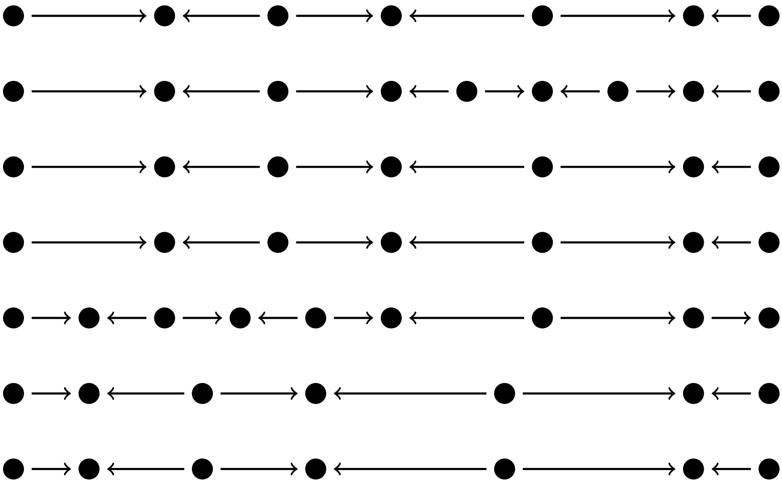
Diagrams from iterated cospans

Here are 53 examples of 0-diagrams, all objects of 1:



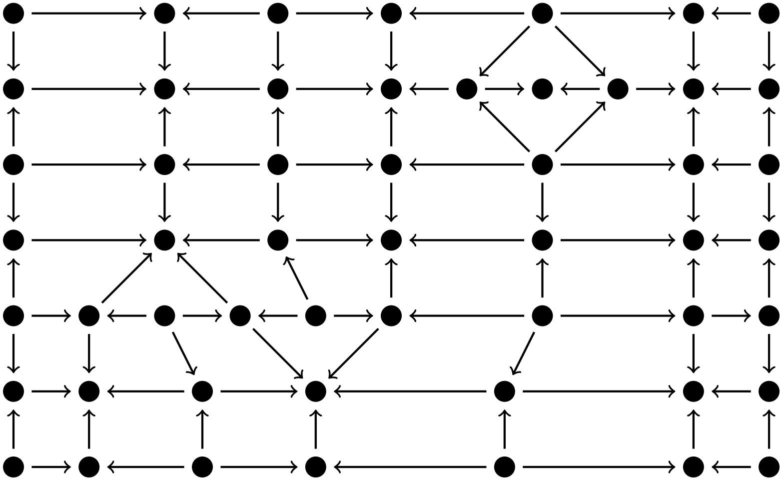
Diagrams from iterated cospans

Here are 7 examples of 1-diagrams, all objects of $IC_1(\bullet, \bullet) = \Delta$:



Diagrams from iterated cospans

Here is 1 example of a 2-diagram, an object of $IC_{\Delta}(3,3)$:



Types

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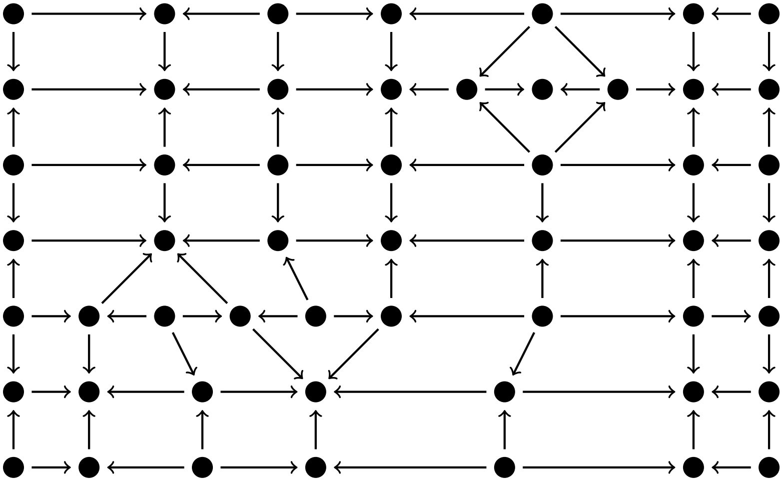

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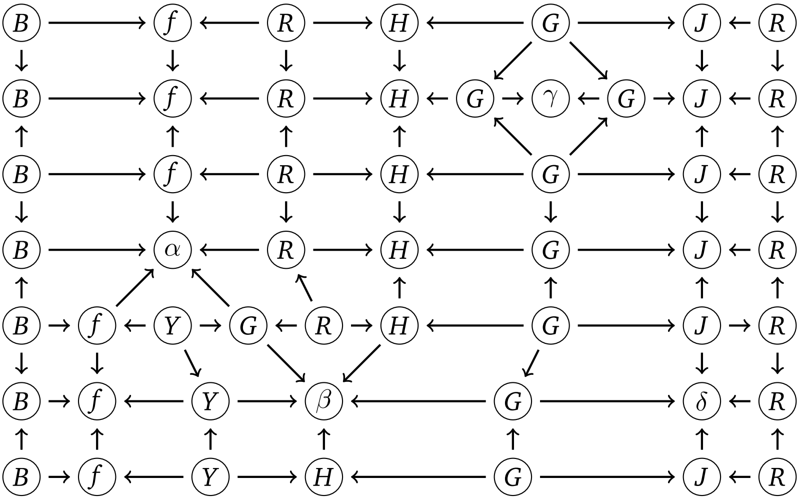

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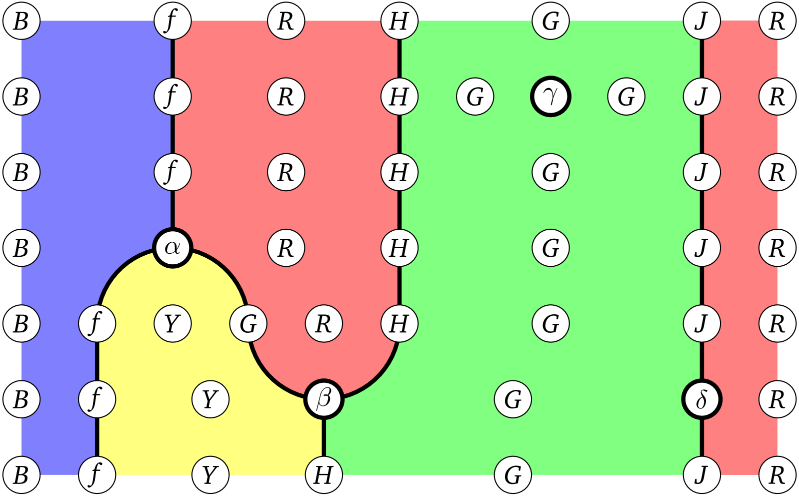

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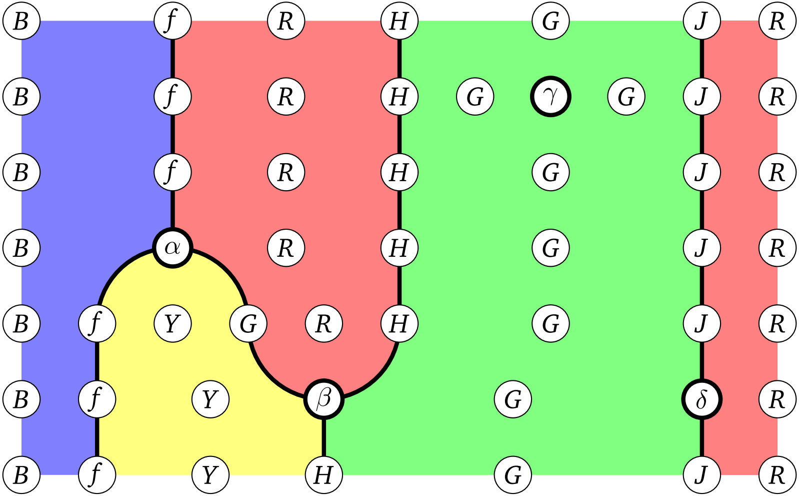

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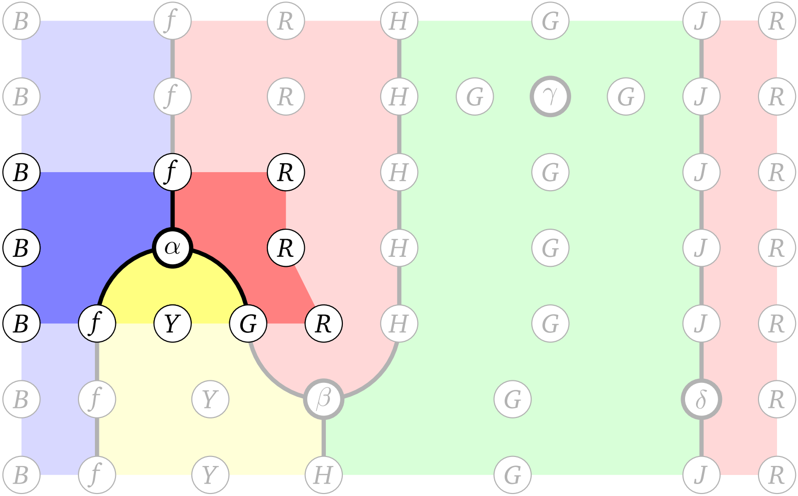

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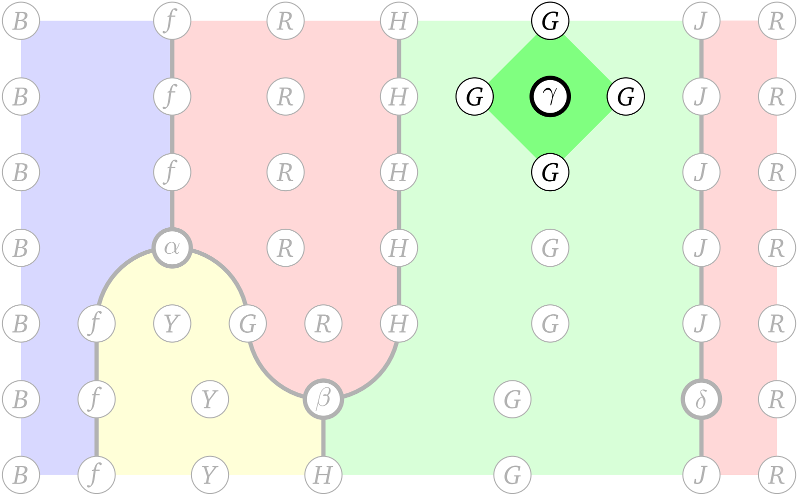

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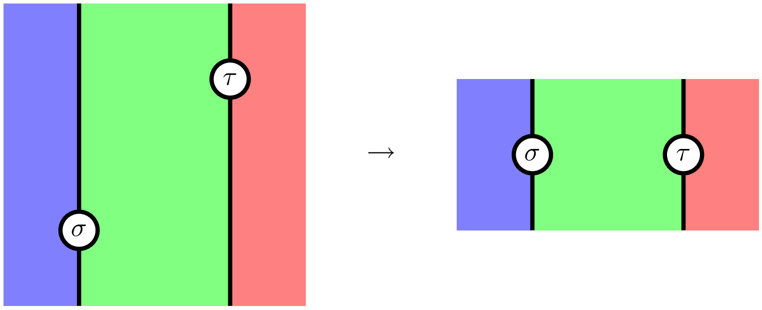

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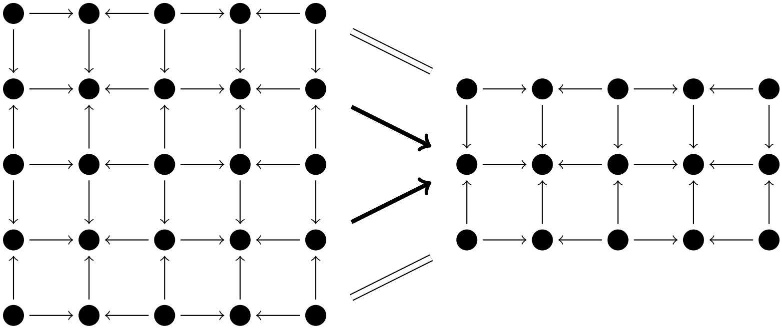

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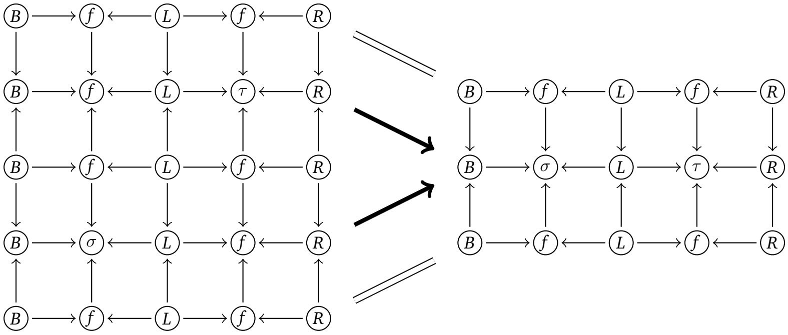

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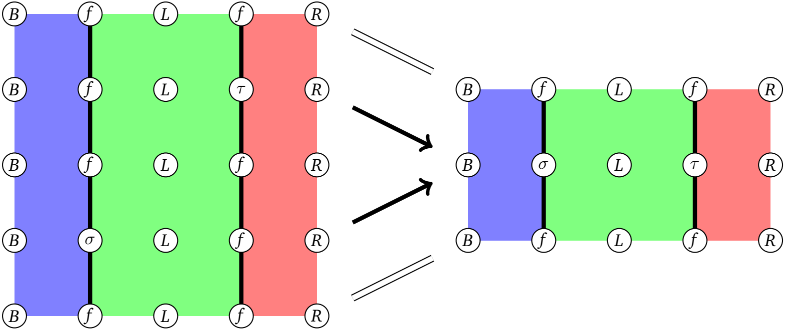

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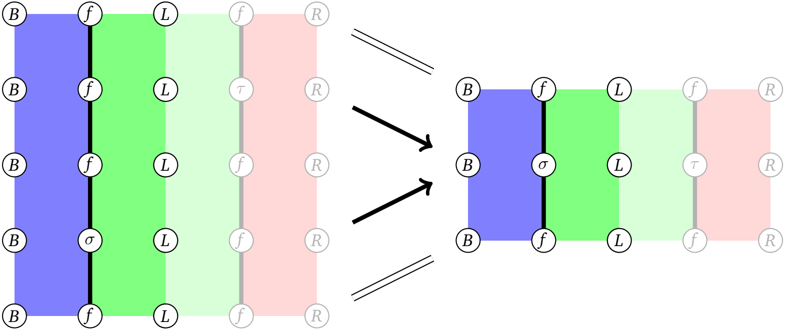

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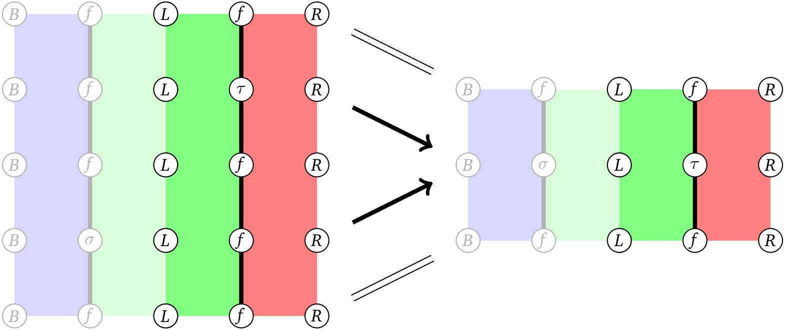

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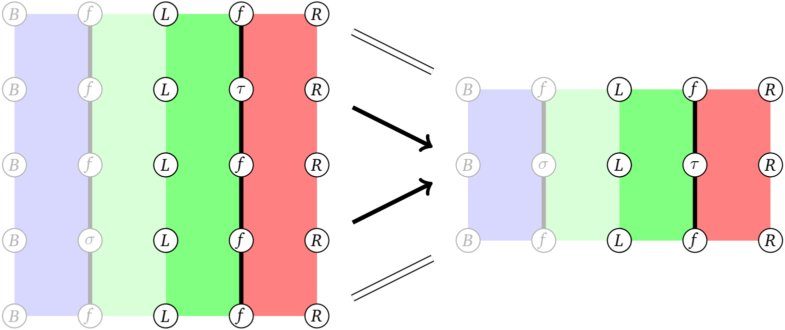

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So homotopies are built in to associative n-categories.

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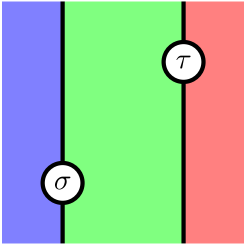

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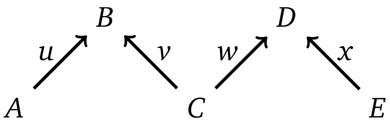

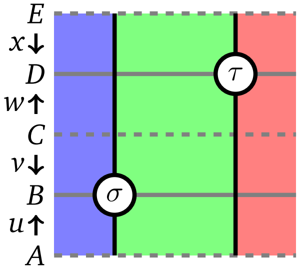

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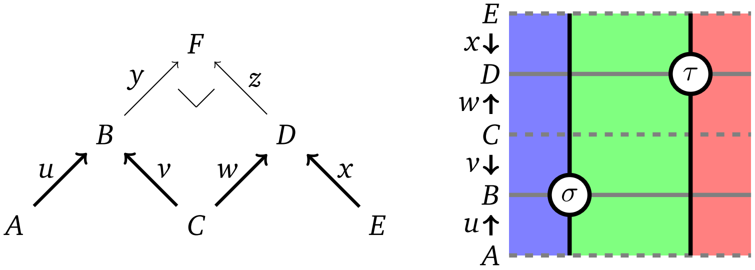

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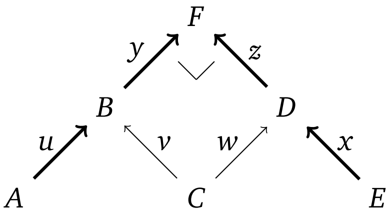

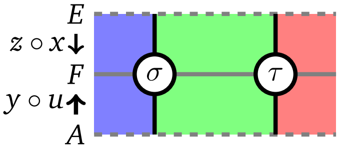

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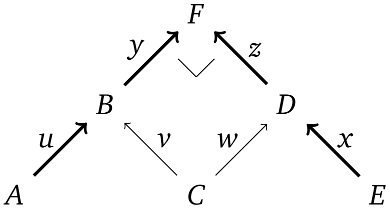

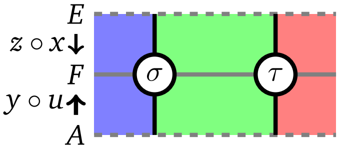

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Gives an insight into the relationship with virtual *n*-categories.

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- Associative n-categories give a new definition of free (∞, ∞)-categories which are strictly associative and unital.

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Thank you!