Formal Topology in Univalent Foundations

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Topology is a mathematical theory of observable properties.[1]

[1] as pointed out by Scott [5], Smyth [6], Abramsky [1], Vickers [9], Escardó [2], and Taylor [7], among others.

Frames

Frames

A frame is a poset $\mathcal{O}$ such that

- finite subsets of $\mathcal{O}$ have meets,
- arbitrary subsets of $\mathcal{O}$ have joins, and
- binary meets distribute over arbitrary joins:

$$a \wedge \left(\bigvee_{i \in I} b_i \right) = \bigvee_{i \in I} (a \wedge b_i),$$

for any $a \in \mathcal{O}$ and family $\{b_i \mid i \in I\}$ over $\mathcal{O}$.

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In type theory, the quantification over arbitrary subsets is problematic.

Frames — a prime example

Given a poset

$$\begin{aligned} A &: \mathsf{Type}_m \\ \sqsubseteq &: A \to A \to \mathsf{hProp}_m \end{aligned}$$

the type of downwards-closed subsets of A is:

$$\sum_{(U \,:\, \mathcal{P}(A))} \prod_{(x\ y \,:\, A)} x \in U \to y \sqsubseteq x \to y \in U,$$

where

$$\begin{aligned} \mathcal{P} &: \mathsf{Type}_m \to \mathsf{Type}_{m+1} \\ \mathcal{P}(X) &:\equiv X \to \mathsf{hProp}_m. \end{aligned}$$

Frames — a prime example

This forms a frame defined as:

$$\begin{aligned} \top &:\equiv \lambda\_.\ \mathsf{Unit} \\ A \wedge B &:\equiv \lambda x.\ (x \in A) \times (x \in B) \\ \bigvee_{i\,:\,I} B_i &:\equiv \lambda x.\ \left\| \sum_{(i\,:\,I)} x \in B_i \right\|. \end{aligned}$$

Frames — a prime example

Question: can we get any frame out of a poset in this way?

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One way is to employ the notion of a nucleus on a frame.

For this, we need to enrich the notion of a poset with a structure that gives rise to an appropriate nucleus (on its frame of downwards-closed subsets).

That structure is a formal topology.

Formal Topology

Formal topologies — as interaction systems

An interaction structure [4] on some type A comprises three functions:

$$
\begin{aligned}
B &: A \to \mathrm{Type} && (1),\\
C &: \prod_{(a\,:\,A)} B(a) \to \mathrm{Type} && (2), \text{ and}\\
d &: \prod_{(a\,:\,A)} \prod_{(b\,:\,B(a))} C(a,b) \to A && (3).
\end{aligned}
$$

An interaction system is a type A equipped with an interaction structure.

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basic opens

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Formal topologies — as interaction systems

A formal topology is an interaction system (B, C, d) on some poset P that satisfies the following two conditions.

1. **Monotonicity**:

$$\prod_{(a\,:\,A)} \prod_{(b\,:\,B(a))} \prod_{(c\,:\,C(a,b))} d(a, b, c) \sqsubseteq a.$$

2. **Simulation**:

$$\prod_{(a'\ a\,:\,A)} a' \sqsubseteq a \to$$
$$\prod_{(b\,:\,B(a))} \sum_{(b'\,:\,B(a'))} \prod_{(c'\,:\,C(a',b'))} \sum_{(c\,:\,C(a,b))} d(a', b', c') \sqsubseteq d(a, b, c).$$

An example: the formal Cantor topology

We write down the experimentational structure of the Cantor topology.

$$
\begin{aligned}
A &:\equiv \mathsf{SnocList}(\mathsf{Bool}) && \text{(stages)} \\
B(bs) &:\equiv \mathsf{Unit} && \text{(experiments)} \\
C(bs, \_) &:\equiv \mathsf{Bool} && \text{(outcomes)} \\
d(bs, \star, b) &:\equiv bs \frown b && \text{(revision)}
\end{aligned}
$$

The *monotonicity* and *simulation* properties are easily verified.

Nuclei

A nucleus on a frame F is an endofunction $\mathbf{j} : |F| \to |F|$ such that:

$$\prod_{(x\ y\ :\ |F|)} \mathsf{j}\,(x \wedge y) \quad = \quad \mathsf{j}\,(x) \wedge \mathsf{j}\,(y) \qquad \text{[meet preservation],}$$

$$\prod_{(x\ :\ |F|)} x \quad \sqsubseteq \quad \mathsf{j}\,(x) \qquad \text{[inflation], and}$$

$$\prod_{(x\ :\ |F|)} \mathsf{j}\,(\mathsf{j}\,(x)) \quad \sqsubseteq \quad \mathsf{j}\,(x) \qquad \text{[idempotence].}$$

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This is a meet-preserving, idempotent monad!

Nuclei

Let F be a frame, and $\mathbf{j} : |F| \to |F|$ a nucleus on it.

The set

$$\sum_{(x \,:\, |F|)} \mathbf{j}\,(x) = x$$

of fixed points for $\mathbf{j}$ is itself a frame:

$$\begin{aligned} \top &\quad :\equiv \quad \top_F \\ \_ \wedge \_ &\quad :\equiv \quad \_ \wedge_F \_ \\ \bigvee_i x_i &\quad :\equiv \quad \mathbf{j}\left(\bigvee_i^F x_i\right). \end{aligned}$$

We denote this $\mathfrak{fix}\,(F, \mathbf{j})$.

Nuclei

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In the posetal case, our *modality* will be the covering relation induced by the structure of a formal topology.

The covering nucleus — naive attempt

Let

- $\mathcal{F}$ be a formal topology with underlying poset P,
- $a : |P|$, and
- $U : \mathcal{P}(|P|)$, a downwards-closed subset of P.

$a \lhd U$ is inductively defined via two rules.

$$\frac{a \in U}{a \lhd U}\ \text{dir} \qquad \frac{b : B(a) \qquad \prod_{(c\ :\ C(a,b))} d(a,b,c) \lhd U}{a \lhd U}\ \text{branch}$$

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Notice: $a \lhd U$ is a ***structure*** *and not a* ***property****.*

The covering nucleus — naive attempt

▷ could be shown to be a nucleus, **if it had the type**

$$\begin{aligned} \lhd &: |P| \to \mathcal{P}(|P|) \to \mathsf{hProp} \\ \rhd &: \mathcal{P}(|P|) \to \mathcal{P}(|P|), \end{aligned}$$

but its type is

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Idea: use propositional truncation:

$$\begin{aligned} \|\_ \lhd \_\| : |P| \to \mathcal{P}(|P|) \to \text{hProp} \\ \|\_ \rhd \_\| : \mathcal{P}(|P|) \to \mathcal{P}(|P|). \end{aligned}$$

The covering nucleus — naive attempt

Need to show: $\|\_ \lhd \_\|$ is a nucleus.

This involves showing it is idempotent:

$$\|\_ \lhd \|\_ \lhd U\|\| \quad \subseteq \quad \|\_ \lhd U\|,$$

for which we need to prove a lemma stating:

$$\|a \lhd U\| \times \left(\prod_{(a' \,:\, |P|)} a' \in U \to \|a' \lhd V\| \right) \to \|a \lhd V\|,$$

for every formal topology $\mathcal{F}$ with underlying poset P, $a : |P|$, and downwards-closed subsets $U, V : \mathcal{P}(|P|)$.

The covering nucleus — naive attempt

In the branch case of an attempted proof, the inductive hypothesis gives us

$$\prod_{(c\,:\,C(a,b))} \|d(a,b,c) \lhd V\|\,,$$

but what we need is:

$$\left\| \prod_{(c\,:\,C(a,b))} d(a,b,c) \lhd V \right\|.$$

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This inference would require (a form of) the axiom of choice.

In fact, the form of choice needed is provably false [8, Lemma 3.8.5].

The covering nucleus — fixed

As we cannot truncate, we *revise* the inductive definition of $\lhd$ to be a higher inductive type.

$$\frac{a \in U}{a \lhd U}\ \text{dir} \qquad \frac{b : B(a) \qquad \prod_{(c\,:\,C(a,b))} d(a,b,c) \lhd U}{a \lhd U}\ \text{branch}$$

$$\frac{p : a \lhd U \qquad q : a \lhd U}{p = q}\ \text{squash}$$

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The mentioned lemma is now provable without choice *and* the type is propositional!

Generating frames from formal topologies

1. Start with formal topology $\mathcal{F}$ with underlying poset P.
2. Take the frame of downwards-closed subsets of P, denoted $P\downarrow$.
3. $\triangleright : P\downarrow \to P\downarrow$ is a nucleus.
4. The generated frame is the frame of fixed points of this nucleus (denoted $\mathfrak{fix}\,(P\downarrow, \triangleright)$).

Formal topologies present

Flat monotonic maps

To state the presentation theorem, we will have to talk about meet-preserving monotonic maps.

However, we are working with posets which may or may not have meets.

The solution is to consider those monotonic maps preserving latent meets: these are called flat monotonic maps.

Let $f : P \to F$ be a **monotonic map** from a poset P to the underlying poset of a frame F. We say that it is flat if:

$$\begin{aligned} \top_F &= \bigvee \{f(a) \mid a : |P|\}, \quad \text{and} \\ \prod_{(a_0\ a_1\ :\ |P|)} f(a_0) \wedge f(a_1) &= \bigvee \{f(a_2) \mid a_2 \sqsubseteq a_0 \text{ and } a_2 \sqsubseteq a_1\}. \end{aligned}$$

Representation

Let

- $\mathcal{F}$ be a formal topology,
- R, a frame, and
- $f : |\mathcal{F}| \to |R|$, a function.

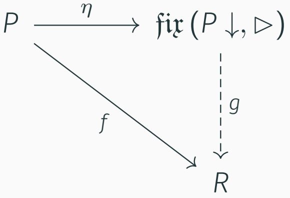
We say that f represents $\mathcal{F}$ in R if:

$$\prod_{(a\ :\ A)} \prod_{(b\ :\ B(a))} f(a) \sqsubseteq \bigvee_{c:C(a,b)} f(d(a,b,c)).$$

Theorem. Given

- a formal topology $\mathcal{F}$ with underlying poset P,
- a frame R, and
- a **flat** monotonic map $f : P \to R$;

if f represents $\mathcal{F}$ in R, then there exists a **unique** frame homomorphism g making the following diagram commute:



where $\eta(a) :\equiv \_ \lhd \{a' \mid a' \sqsubseteq a\}$.

Compact formal topologies

Compactness

We can easily pin down what it means for a formal topology to be compact.

Let $\mathcal{F} = (A, B, C, d)$ be a formal topology. We say that $\mathcal{F}$ is compact if

$$\prod_{(a\,:\,A)} \prod_{(U\,:\,\mathcal{P}(A))} \text{isDownwardsClosed}(U) \times a \lhd U \to \left\| \sum_{(as\,:\,\text{List}(A))} a \lhd \text{down}\,(as) \times \text{down}\,(as) \subseteq U \right\|,$$

where

$$\begin{aligned} \text{down} \quad &: \quad \text{List}\,(A) \to \mathcal{P}\,(A) \\ \text{down}\,([\,]) \quad &:\equiv \quad \lambda\_. \quad \bot \\ \text{down}\,(x :: xs) \quad &:\equiv \quad \lambda y. \quad \|(y \sqsubseteq x) + (y \in \text{down}\,(xs))\|\,. \end{aligned}$$

Compactness of the formal Cantor topology

Theorem. The formal Cantor topology is compact.

Proof outline.

- Let bs : SnocList (Bool), $U : \mathcal{P}$ (Bool) a downwards-closed subset, and assume $bs \lhd U$
- Induction (higher) on the derivation of $bs \lhd U$.
  - dir case: bs gives what we want.
  - branch case: we obtain by induction finite covers for $bs \frown$ `true` and $bs \frown$ `false`. We combine these to get a finite cover for bs.
  - squash case: this case is the reason we have to use truncation in the definition of compactness.

Conclusion

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In summary, this thesis development features:

- a reconstruction of the notion of covering within the univalent doctrine as an HIT,
- a *sketch* of the beginnings of an approach for carrying out formal topology in univalent type theory, and
- no postulates, no impredicativity (everything typechecks with `--safe`); no setoids either.

Further work

- Develop more topology using this approach!
- Define the reals as the *formal points* of the formal topology of the real numbers.
- What is the category of formal topologies?
- How can the presentation theorem be stated as an adjunction?

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