Computational counting and quantum information theory

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Outline

Background: computational counting problems

Basic quantum information theory

Exact evaluation of $\textsc{Holant}^c$

Approximating conservative holant (joint work with Leslie Goldberg)

Conclusions

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Computational counting

problems where the goal is to count number of solutions or compute total weight of all weighted solutions, e.g.

- number of satisfying assignments of some constraint satisfaction problem
- number of perfect matchings of some graph
- number of vertex covers
- strong classical simulation of quantum computations

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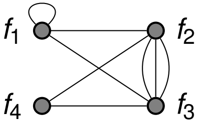
Complexity class $\#$P

- counting number of accepting paths of a polynomial-time Turing machine
- i.e. “counting complexity equivalent” of NP

Holant problems

signature grid $\Omega = (G, \mathcal{F}, \pi)$

- $G = (V, E)$ is a (pseudo-)graph
- $\mathcal{F}$ is a set of algebraic complex-valued functions of Boolean inputs, called signatures
- $\pi : V \to \mathcal{F}$



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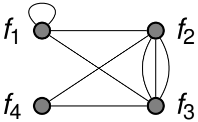


HOLANT $(\mathcal{F})$

- Input: a signature grid Ω over $\mathcal{F}$
- Output:

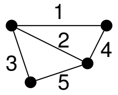
$$\text{HOLANT}_{\Omega} := \sum_{\sigma: E \to \{0,1\}} \prod_{v \in V} f_v \left(\sigma|_{E(v)} \right)$$

Example holant problem

Let $\mathcal{F} = \{\mathrm{ONE}_k \mid 1 \leq k \leq n\}$ for some positive integer n, where

$$\mathrm{ONE}_k : \{0,1\}^k \to \{0,1\} :: \mathbf{x} \mapsto \begin{cases} 1 & \text{if } |\mathbf{x}| = 1 \\ 0 & \text{otherwise} \end{cases}$$

and consider the holant for the following signature grid Ω:



$$\textsc{Holant}_\Omega = \sum_{x_1,\ldots,x_5 \in \{0,1\}} \mathrm{ONE}_3(x_1, x_2, x_3)\mathrm{ONE}_2(x_1, x_4)\mathrm{ONE}_3(x_4, x_2, x_5)\mathrm{ONE}_2(x_5, x_3)$$

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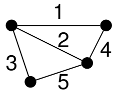

$$\begin{aligned}
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&= \sum_{x_2,\ldots,x_5 \in \{0,1\}} \mathrm{ONE}_3(0,x_2,x_3)\mathrm{ONE}_2(0,x_4)\mathrm{ONE}_3(x_4,x_2,x_5)\mathrm{ONE}_2(x_5,x_3) \\
&\quad + \sum_{x_4,x_5 \in \{0,1\}} \mathrm{ONE}_3(1,0,0)\mathrm{ONE}_2(1,x_4)\mathrm{ONE}_3(x_4,0,x_5)\mathrm{ONE}_2(x_5,0)
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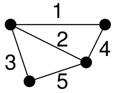

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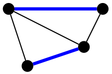
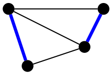
&= 2
\end{aligned}$$





Holant variants and some existing complexity classifications

Exact evaluation	Holant $(\mathcal{F})$	Holant$^c(\mathcal{F})$ pinning functions	Holant$^*(\mathcal{F})$ arbitrary unaries
symmetric	[Cai, Guo, Williams 2012]	[Cai, Huang, Lu 2012]	
non-negative	[Lin, Wang 2017]		
real		[Cai, Lu, Xia 2017]	
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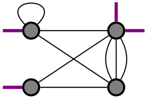
Approximation

- partial result for HOLANT$^*(\mathcal{F})$ [Yamakami 2012]
- numerous results in #CSP framework (i.e. holant with equality functions)
- full result for HOLANT$^*(\mathcal{F})$ here

Gadgets and holant clones

A signature grid with k "dangling edges" and m internal edges defines a function

$$g(y_1,\ldots,y_k) := \sum_{x_1,\ldots,x_m \in \{0,1\}} F(x_1,\ldots,x_m,y_1,\ldots,y_k)$$



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$$\textsc{Holant}\,(\mathcal{F} \cup \{g\}) \equiv_T \textsc{Holant}\,(\mathcal{F})$$

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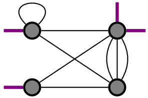

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The holant clone $\langle \mathcal{F} \rangle_h$ is the set of all functions that can be realised by gadgets using functions from $\mathcal{F}$.

The vector perspective and holographic reductions

Let $\{|x\rangle\}_{x\in\{0,1\}^n}$ be an orthonormal basis for $\mathbb{C}^{2^n}$, then there exists a bijection:

$$f : \{0,1\}^n \to \mathbb{C} \qquad \leftrightarrow \qquad |f\rangle = \sum_{x\in\{0,1\}^n} f(x)\,|x\rangle \quad \in \mathbb{C}^{2^n}$$

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For any invertible 2×2 matrix M, define a holographic transformation:

- $M \circ f$ is the function corresponding to $M^{\otimes arity(f)}\,|f\rangle$, and
- $M \circ \mathcal{F} = \{M \circ f \mid f \in \mathcal{F}\}$.

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Theorem (Valiant's Holant Theorem)

Let $\mathcal{F}$ and $\mathcal{G}$ be two sets of functions, and M an invertible 2×2 matrix. Then:

$$\textsc{Holant}\,(\mathcal{F} \mid \mathcal{G}) \equiv_T \textsc{Holant}\left(M \circ \mathcal{F} \,\middle|\, (M^{-1})^T \circ \mathcal{G}\right).$$

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Quantum states and their vector description

system	state space	computational basis
1 qubit	$\mathbb{C}^2$	$\{\|0\rangle, \|1\rangle\}$
2 qubits	$\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$	$\{\|0\rangle \otimes \|0\rangle, \|0\rangle \otimes \|1\rangle, \|1\rangle \otimes \|0\rangle, \|1\rangle \otimes \|1\rangle\} = \{\|00\rangle, \|01\rangle, \|10\rangle, \|11\rangle\}$
$\vdots$	$\vdots$	$\vdots$
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n-ary signature $\leftrightarrow$ n-qubit state

Quantum entanglement

A quantum state of multiple qubits is entangled if it cannot be written as a tensor product of single-qubit states.

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A quantum state is fully entangled if there is no way of writing it as a tensor product:

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Multipartite entanglement refers to states in which at least 3 qubits are fully entangled with each other, for example:

$$(i\,|000\rangle - 3\,|111\rangle) \otimes |101\rangle$$

Correspondences

Counting problems	Quantum information theory
signature	quantum state
degenerate signature	product state
non-degenerate signature	entangled state
affine signature	stabiliser state
equality signature	generalised GHZ state
holographic transformation	stochastic local operation with classical communication (SLOCC)
(exact) Holant problem	(strong) classical simulation of quantum circuits

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Idea

$\text{HOLANT}^c(\mathcal{F}) := \text{HOLANT}(\mathcal{F} \cup \{\delta_0, \delta_1\})$ $\delta_0(x) = 1 - x$ and $\delta_1(x) = x$

know complexity classifications for

- $\text{HOLANT}(\{f\} \mid \{g\})$ for symmetric f, g with $\mathsf{arity}(f) = 3$ and $\mathsf{arity}(g) = 2$ [Cai, Huang, Lu 2012]
- $\#\mathrm{CSP}_2^c(\mathcal{F}) := \text{HOLANT}(\mathcal{F} \cup \{\delta_0, \delta_1\} \cup \{\mathrm{EQ}_{2n} \mid n \in \mathbb{N}_{>0}\})$ [Cai, Lu, Xia 2017]

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Approach

- if $\textsc{Holant}(\mathcal{F} \cup \{\delta_0, \delta_1\})$ known to be polynomial-time computable: done
- in particular, this includes the case $\mathcal{F} \subseteq \langle \Upsilon_1, \Upsilon_2 \rangle_{\mathrm{h}}$
- so assume there exists $h \in \mathcal{F} \setminus \langle \Upsilon_1 \cup \Upsilon_2 \rangle_{\mathrm{h}}$ with $\mathsf{arity}(h) \geq 3$, build gadgets of decreasing arity by pinning and self-loops
- either get function of arity 3 and reduce from $\textsc{Holant}(\{f\} \mid \{g\})$
- or get specific function or arity 4, interpolate EQ_4, and reduce from $\#\mathrm{CSP}_2^c$

Arity reduction

For any function h with $n := \mathrm{arity}(h) > 3$, consider the functions

- $h_j^a(x_1, \ldots, x_{n-1}) = \sum_{y \in \{0,1\}} \delta_a(y) h(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_{n-1})$ for $a \in \{0, 1\}$
- $h_{jk}(x_1, \ldots, x_{n-2}) = \sum_{y \in \{0,1\}} h(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_{k-1}, y, x_{k+1}, \ldots, x_{n-2})$

If one of these contains a non-decomposable tensor factor of arity > 3, replace h by that factor and repeat. If the arity of the non-decomposable tensor factor is 3, stop and proceed to symmetrisation.

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If one of these contains a non-decomposable tensor factor of arity > 3, replace h by that factor and repeat. If the arity of the non-decomposable tensor factor is 3, stop and proceed to symmetrisation.

The above process always works unless $n = 4$ and h has the following property (up to permutations of the input variables)

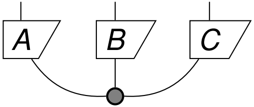
$$h(\mathbf{x}) = 0 \text{ unless } \mathbf{x} \in \{0000, 0011, 1100, 1111\}$$

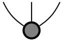
In that case, can realise or interpolate EQ_4, then realise $\{\mathrm{EQ}_{2n} \mid n \in \mathbb{N}_{>0}\}$, thus problem is as hard as $\#\mathrm{CSP}_2^c$.

A different perspective on ternary signatures

From quantum theory we know that, up to non-symmetric holographic transformations, there are only two distinct non-decomposable ternary signatures.

It can be convenient to think of signatures as gadgets even if they are not defined that way, e.g. for a non-decomposable ternary signature:

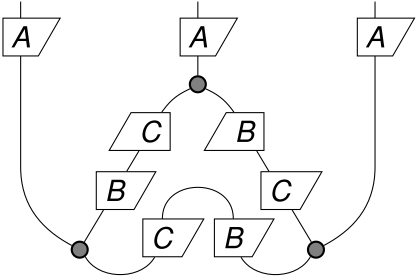


where  is assigned either

$$\mathrm{EQ}_3(\mathbf{x}) = \begin{cases} 1 & \text{if } x_1 = x_2 = x_3 \\ 0 & \text{otherwise} \end{cases} \quad \text{or} \quad \mathrm{ONE}_3(\mathbf{x}) = \begin{cases} 1 & \text{if } x_1 + x_2 + x_3 = 1 \\ 0 & \text{otherwise} \end{cases}$$

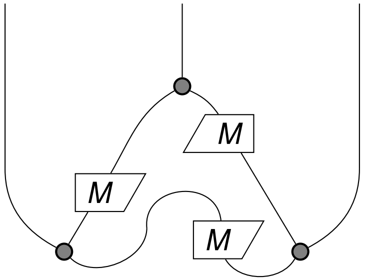
and A, B, C are non-decomposable binary signatures

Symmetrising ternary signatures

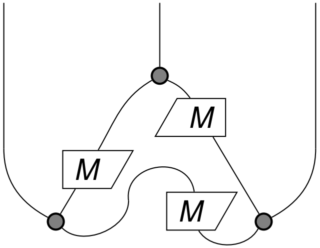


Symmetrising ternary signatures

Up to a holographic transformation, the following gadget is determined by 2 cases with 4 parameters each:



Gadgets for non-decomposable symmetric ternary signatures



Given a set $\mathcal{F}$ containing a non-decomposable ternary signature, can show:

- either it is possible to realise a symmetric non-decomposable ternary signature using the above gadget (among others),
- or $\mathcal{F} \cup \{\delta_0, \delta_1\}$ is one of the known polynomial-time computable families.

Can similarly realise non-decomposable symmetric binary g, and then get hardness by reduction from HOLANT $(\{f\} \mid \{g\})$.

The HOLANT^c dichotomy

Theorem

Let $\mathcal{F}$ be a set of algebraic complex-valued functions of Boolean inputs. Then HOLANT^c $(\mathcal{F})$ *is polynomial time computable if:*

- HOLANT* $(\mathcal{F})$ *can be solved in polynomial time, or*
- $\mathcal{F}$ *contains only affine functions under certain holographic transformations, or*
- $\mathcal{F}$ *contains only local affine functions.*

In all other cases, HOLANT^c $(\mathcal{F})$ *is* $\#$P*-hard.*

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Approximation vs exact evaluation

HOLANT $(\mathcal{F})$	determine holant value exactly
HolantNorm$(\mathcal{F}; 1.01)$	approximate norm to within multiplicative error of 1.01
HolantArg$(\mathcal{F}; \pi/3)$	approximate argument to within additive error of $\pi/3$

- specific values of accuracy parameters not relevant
- cannot use technique of polynomial interpolation when analysing approximation complexity, hence many existing results cannot be used

Exact dichotomy for conservative holant (aka HOLANT*)

Theorem ([Cai, Lu, Xia 2011])

Suppose $\mathcal{F}$ is a finite set of functions. If

1. $\mathcal{F} \subseteq \langle \Upsilon_1 \cup \Upsilon_2 \rangle_{\mathrm{h}}$, *or*
2. *there exists an orthogonal matrix O such that* $\mathcal{F} \subseteq O \circ \langle \Upsilon_1, \mathrm{EQ}_3, \mathrm{NEQ} \rangle_{\mathrm{h}}$, *or*
3. $\mathcal{F} \subseteq \left(\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right) \circ \langle \Upsilon_1, \mathrm{EQ}_3, \mathrm{NEQ} \rangle_{\mathrm{h}}$, *or*
4. *there exists a matrix $K \in \left\{ \left(\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ -i & i \end{smallmatrix}\right) \right\}$ such that $\mathcal{F} \subseteq \langle K \circ \mathcal{M} \rangle_{\mathrm{h}}$, where*

$$\mathcal{M} = \{ f \mid f(\mathbf{x}) = 0 \textit{ unless } |\mathbf{x}| \leq 1 \},$$

then, for any finite subset $S \subseteq \mathcal{U}$, the problem HOLANT$(\mathcal{F}, S)$ *is polynomial-time computable. Otherwise, there exists a finite subset $S \subseteq \mathcal{U}$ such that* HOLANT$(\mathcal{F}, S)$ *is #P-hard. The dichotomy is still true even if the inputs are restricted to planar graphs.*

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We will refer to conditions 1–4 as the four conditions.

Universal quantum circuits as holant clones

The following is well-known in quantum computing:

Any $2^n \times 2^n$ *unitary matrix can be realised from* 2×2 *unitary matrices and*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

via Kronecker products and matrix products.

Lemma

Any algebraic complex-valued function is contained in $\langle \Upsilon_2, \mathrm{CNOT}, \delta_0 \rangle_{\mathrm{h}}$, *where*

$$\mathrm{CNOT}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \{0000, 0101, 1011, 1110\} \\ 0 & \text{otherwise.} \end{cases}$$

Universal conservative holant clones

Theorem

$\langle \mathcal{F}, \Upsilon_1 \rangle_{\mathrm{h}}$ *is universal unless* $\mathcal{F}$ *satisfies one of the four conditions.*

Proof (sketch).

- If $\mathcal{F} \subseteq \langle \Upsilon_1, \Upsilon_2 \rangle_{\mathrm{h}}$, it satisfies the first condition.
- So assume otherwise, i.e. there is $f \in \mathcal{F} \setminus \langle \Upsilon_1, \Upsilon_2 \rangle_{\mathrm{h}}$.
- Use this f to realise a ternary function f'.
- If $\mathcal{F}$ does not satisfy conditions 2–4, can use f' (and possibly some other functions in $\mathcal{F} \cup \Upsilon_1$) to realise EQ_3.
- Similarly, can show $\Upsilon_2 \subseteq \langle \mathcal{F}, \Upsilon_1 \rangle_{\mathrm{h}}$.
- Realise CNOT from EQ_3 and $h(x, y) = (-1)^{xy}$.
- Then apply the universality lemma. □

Approximating conservative holant

Theorem

Suppose that $\mathcal{F}$ is finite. Then there exists a finite subset $S \subseteq \Upsilon_1$ such that both HolantArg$(\mathcal{F} \cup S; \pi/3)$ *and* HolantNorm$(\mathcal{F} \cup S; 1.01)$ *are* #P*-hard, unless $\mathcal{F}$ satisfies one of the four conditions.*

Proof (sketch).

- If $\mathcal{F}$ does not satisfy any of the four conditions, $\langle F \cup \Upsilon_1 \rangle_{\mathrm{h}}$ is universal.
- Pick some computational problem that is known to be hard to approximate, e.g. approximating the independent set polynomial on graphs of maximum degree 3.
- Only need a finite number of functions from $\mathcal{F} \cup \Upsilon_1$ to build the gadgets necessary to reduce from that problem. □

Outline

Background: computational counting problems

Basic quantum information theory

Exact evaluation of HOLANT^c

Approximating conservative holant (joint work with Leslie Goldberg)

Conclusions

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Thank you!