Computational counting and quantum information theory

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Outline

Background: computational counting problems

Basic quantum information theory

Exact evaluation of $\text{HOLANT}^c$

Approximating conservative holant (joint work with Leslie Goldberg)

Conclusions
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Exact evaluation of $\text{HOLANT}^C$

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Conclusions
problems where the goal is to count number of solutions or compute total weight of all weighted solutions, e.g.

- number of satisfying assignments of some constraint satisfaction problem
- number of perfect matchings of some graph
- number of vertex covers
- strong classical simulation of quantum computations
Computational counting

problems where the goal is to count number of solutions or compute total weight of all weighted solutions, e.g.

▶ number of satisfying assignments of some constraint satisfaction problem
▶ number of perfect matchings of some graph
▶ number of vertex covers
▶ strong classical simulation of quantum computations

Complexity class $\#P$

▶ counting number of accepting paths of a polynomial-time Turing machine
▶ i.e. “counting complexity equivalent” of NP
Holant problems

signature grid $\Omega = (G, \mathcal{F}, \pi)$

- $G = (V, E)$ is a (pseudo-)graph
- $\mathcal{F}$ is a set of algebraic complex-valued functions of Boolean inputs, called signatures
- $\pi : V \to \mathcal{F}$
Holant problems

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$\text{HOLANT}(\mathcal{F})$

- Input: a signature grid $\Omega$ over $\mathcal{F}$
- Output:

$$\text{HOLANT}_\Omega := \sum_{\sigma : E \to \{0,1\}} \prod_{v \in V} f_v(\sigma|_{E(v)})$$
Example holant problem

Let $\mathcal{F} = \{\text{ONE}_k \mid 1 \leq k \leq n\}$ for some positive integer $n$, where

$$\text{ONE}_k : \{0, 1\}^k \to \{0, 1\} : x \mapsto \begin{cases} 1 & \text{if } |x| = 1 \\ 0 & \text{otherwise} \end{cases}$$

and consider the holant for the following signature grid $\Omega$:

$$\text{HOLANT}_\Omega = \sum_{x_1, \ldots, x_5 \in \{0, 1\}} \text{ONE}_3(x_1, x_2, x_3) \text{ONE}_2(x_1, x_4) \text{ONE}_3(x_4, x_2, x_5) \text{ONE}_2(x_5, x_3)$$
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\]

\[
= \sum_{x_2, \ldots, x_5 \in \{0, 1\}} \text{ONE}_3(0, x_2, x_3) \text{ONE}_2(0, x_4) \text{ONE}_3(x_4, x_2, x_5) \text{ONE}_2(x_5, x_3)
\]

\[
+ \sum_{x_4, x_5 \in \{0, 1\}} \text{ONE}_3(1, 0, 0) \text{ONE}_2(1, x_4) \text{ONE}_3(x_4, 0, x_5) \text{ONE}_2(x_5, 0)
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$$= 2$$
Holant variants and some existing complexity classifications

<table>
<thead>
<tr>
<th>Exact evaluation</th>
<th>HOLANT ((\mathcal{F}))</th>
<th>HOLANT(^c)((\mathcal{F})) pinning functions</th>
<th>HOLANT(^*)((\mathcal{F})) arbitrary unaries</th>
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Approximation

- partial result for \textsc{HOLANT}^* (\mathcal{F}) [Yamakami 2012]
- numerous results in \#CSP framework (i.e. holant with equality functions)
- full result for \textsc{HOLANT}^* (\mathcal{F}) here
Gadgets and holant clones

A signature grid with $k$ “dangling edges” and $m$ internal edges defines a function

$$g(y_1, \ldots, y_k) := \sum_{x_1, \ldots, x_m \in \{0,1\}} F(x_1, \ldots, x_m, y_1, \ldots, y_k)$$

If there exists a gadget over $\mathcal{F}$ which defines the function $g$, then

$$\text{HOLANT} (\mathcal{F} \cup \{g\}) \equiv_T \text{HOLANT} (\mathcal{F})$$
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The holant clone $\langle \mathcal{F} \rangle_h$ is the set of all functions that can be realised by gadgets using functions from $\mathcal{F}$. 
The vector perspective and holographic reductions

Let \( \{|x\rangle\}_{x \in \{0,1\}^n} \) be an orthonormal basis for \( \mathbb{C}^{2^n} \), then there exists a bijection:

\[
f : \{0, 1\}^n \rightarrow \mathbb{C} \quad \iff \quad |f\rangle = \sum_{x \in \{0,1\}^n} f(x) |x\rangle \in \mathbb{C}^{2^n}
\]

For any invertible \( 2 \times 2 \) matrix \( M \), define a holographic transformation:

\[
\begin{align*}
\text{if} & \quad M \odot f \\
\text{if} & \quad M \odot F = \{ M \odot f \mid f \in F \}
\end{align*}
\]

Theorem (Valiant's Holant Theorem)

Let \( F \) and \( G \) be two sets of functions, and \( M \) an invertible \( 2 \times 2 \) matrix. Then:

\[
\text{HOLANT}(F | G) \equiv \text{THOLANT}(M \odot F | M^{-1} \odot G)
\]
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For any invertible \( 2 \times 2 \) matrix \( M \), define a holographic transformation:

- \( M \circ f \) is the function corresponding to \( M^{\otimes \text{arity}(f)} |f\rangle \), and
- \( M \circ \mathcal{F} = \{M \circ f \mid f \in \mathcal{F}\} \).
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\[
\text{HOLANT} (\mathcal{F} \mid \mathcal{G}) \equiv_T \text{HOLANT} \left( M \circ \mathcal{F} \left| (M^{-1})^T \circ \mathcal{G} \right. \right).
\]
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Quantum states and their vector description

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$n$-ary signature $\leftrightarrow n$-qubit state
Quantum entanglement

A quantum state of multiple qubits is entangled if it cannot be written as a tensor product of single-qubit states.

\[
\left| \begin{array}{c}
00 \\
\end{array} \rightangle + \left| \begin{array}{c}
11 \\
\end{array} \rightangle \text{ is fully entangled}
\]

\[
\left| \begin{array}{c}
011 \\
\end{array} \rightangle - \left| \begin{array}{c}
101 \\
\end{array} \rightangle = \left( \left| \begin{array}{c}
01 \\
\end{array} \rightangle - \left| \begin{array}{c}
10 \\
\end{array} \rightangle \right) \otimes \left| \begin{array}{c}
1 \\
\end{array} \rightangle \text{ is entangled, but not fully so}
\]

Multipartite entanglement refers to states in which at least 3 qubits are fully entangled with each other, for example:

\[
\left( i \left| \begin{array}{c}
000 \\
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A quantum state of multiple qubits is entangled if it cannot be written as a tensor product of single-qubit states.

A quantum state is fully entangled if there is no way of writing it as a tensor product:

▶ $|00\rangle + |11\rangle$ is fully entangled

▶ $|011\rangle - |101\rangle = (|01\rangle - |10\rangle) \otimes |1\rangle$ is entangled, but not fully so
Quantum entanglement

A quantum state of multiple qubits is **entangled** if it cannot be written as a tensor product of single-qubit states.

A quantum state is **fully entangled** if there is no way of writing it as a tensor product:

- $|00⟩ + |11⟩$ is fully entangled
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$$(i |000⟩ − 3 |111⟩) ⊗ |101⟩$$
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<th>Quantum information theory</th>
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<tr>
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<td>non-degenerate signature</td>
<td>entangled state</td>
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<td>affine signature</td>
<td>stabiliser state</td>
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<tr>
<td>equality signature</td>
<td>generalised GHZ state</td>
</tr>
<tr>
<td>holographic transformation</td>
<td>stochastic local operation with classical communication (SLOCC)</td>
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<td>(exact) Holant problem</td>
<td>(strong) classical simulation of quantum circuits</td>
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Idea

\( \text{HOLANT}^c(\mathcal{F}) := \text{HOLANT}(\mathcal{F} \cup \{\delta_0, \delta_1\}) \)

\( \delta_0(x) = 1 - x \) and \( \delta_1(x) = x \)

know complexity classifications for

- \( \text{HOLANT}(\{f\} | \{g\}) \) for symmetric \( f, g \) with \( \text{arity}(f) = 3 \) and \( \text{arity}(g) = 2 \)  
  \[ \text{[Cai, Huang, Lu 2012]} \]

- \( \#\text{CSP}_2^c(\mathcal{F}) := \text{HOLANT}(\mathcal{F} \cup \{\delta_0, \delta_1\} \cup \{\text{EQ}_{2n} \mid n \in \mathbb{N}_{>0}\}) \)  
  \[ \text{[Cai, Lu, Xia 2017]} \]
**Idea**

\[
\text{HOLANT}^c(\mathcal{F}) := \text{HOLANT}(\mathcal{F} \cup \{\delta_0, \delta_1\})
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  \[\text{[Cai, Lu, Xia 2017]}\]

**Approach**

- if \(\text{HOLANT}(\mathcal{F} \cup \{\delta_0, \delta_1\})\) known to be polynomial-time computable: done

- in particular, this includes the case \(\mathcal{F} \subseteq \langle \Upsilon_1, \Upsilon_2 \rangle_h\)

- so assume there exists \(h \in \mathcal{F} \setminus \langle \Upsilon_1 \cup \Upsilon_2 \rangle_h\) with \(\text{arity}(h) \geq 3\), build gadgets of decreasing arity by pinning and self-loops

- either get function of arity 3 and reduce from \(\text{HOLANT}(\{f\} | \{g\})\)

- or get specific function or arity 4, interpolate \(\text{EQ}_4\), and reduce from \(\#\text{CSP}_2^c\)
Arity reduction

For any function $h$ with $n := \text{arity}(h) > 3$, consider the functions

\[ h_a(x_1, \ldots, x_{n-1}) = \sum_{y \in \{0,1\}} \delta_a(y) h(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_{n-1}) \quad \text{for } a \in \{0, 1\} \]

\[ h_{jk}(x_1, \ldots, x_{n-2}) = \sum_{y \in \{0,1\}} h(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_{k-1}, y, x_{k+1}, \ldots, x_{n-2}) \]

If one of these contains a non-decomposable tensor factor of arity $> 3$, replace $h$ by that factor and repeat. If the arity of the non-decomposable tensor factor is 3, stop and proceed to symmetrisation.
Arity reduction

For any function $h$ with $n := \text{arity}(h) > 3$, consider the functions

- $h^a_j(x_1, \ldots, x_{n-1}) = \sum_{y \in \{0,1\}} \delta_a(y) h(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_{n-1})$ for $a \in \{0, 1\}$
- $h_{jk}(x_1, \ldots, x_{n-2}) = \sum_{y \in \{0,1\}} h(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_{k-1}, y, x_{k+1}, \ldots, x_{n-2})$

If one of these contains a non-decomposable tensor factor of arity $> 3$, replace $h$ by that factor and repeat. If the arity of the non-decomposable tensor factor is 3, stop and proceed to symmetrisation.

The above process always works unless $n = 4$ and $h$ has the following property (up to permutations of the input variables)

$$h(x) = 0 \text{ unless } x \in \{0000, 0011, 1100, 1111\}$$

In that case, can realise or interpolate $EQ_4$, then realise $\{EQ_{2n} \mid n \in \mathbb{N}_{>0}\}$, thus problem is as hard as $\#\text{CSP}_2^\mathbb{C}$. 

A different perspective on ternary signatures

From quantum theory we know that, up to non-symmetric holographic transformations, there are only two distinct non-decomposable ternary signatures.

It can be convenient to think of signatures as gadgets even if they are not defined that way, e.g. for a non-decomposable ternary signature:

\[
\begin{array}{c}
\text{A} \\
\downarrow \\
\text{B} \\
\text{C}
\end{array}
\]

where \( \downarrow \) is assigned either

\[
\text{EQ}_3(x) = \begin{cases} 
1 & \text{if } x_1 = x_2 = x_3 \\
0 & \text{otherwise}
\end{cases}
\]

or

\[
\text{ONE}_3(x) = \begin{cases} 
1 & \text{if } x_1 + x_2 + x_3 = 1 \\
0 & \text{otherwise}
\end{cases}
\]

and \( A, B, C \) are non-decomposable binary signatures.
Symmetrising ternary signatures

To a holographic transformation, the following gadget is determined by 2 cases with 4 parameters each:

```
A
B
C
B
C
A
A
C
B
```
Symmetrising ternary signatures

Up to a holographic transformation, the following gadget is determined by 2 cases with 4 parameters each:
Gadgets for non-decomposable symmetric ternary signatures

Given a set $\mathcal{F}$ containing a non-decomposable ternary signature, can show:

- either it is possible to realise a symmetric non-decomposable ternary signature using the above gadget (among others),
- or $\mathcal{F} \cup \{\delta_0, \delta_1\}$ is one of the known polynomial-time computable families.

Can similarly realise non-decomposable symmetric binary $g$, and then get hardness by reduction from HOLANT ($\{f\} \mid \{g\}$).
The HOLANT\textsuperscript{c} dichotomy

**Theorem**

Let $\mathcal{F}$ be a set of algebraic complex-valued functions of Boolean inputs. Then $\text{HOLANT}^c(\mathcal{F})$ is polynomial time computable if:

- $\text{HOLANT}^*(\mathcal{F})$ can be solved in polynomial time, or
- $\mathcal{F}$ contains only affine functions under certain holographic transformations, or
- $\mathcal{F}$ contains only local affine functions.

In all other cases, $\text{HOLANT}^c(\mathcal{F})$ is $\#P$-hard.
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### Approximation vs exact evaluation

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<th>Function</th>
<th>Description</th>
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<tr>
<td>HOLANT $(\mathcal{F})$</td>
<td>determine holant value exactly</td>
</tr>
<tr>
<td>HolantNorm$(\mathcal{F}; 1.01)$</td>
<td>approximate norm to within multiplicative error of 1.01</td>
</tr>
<tr>
<td>HolantArg$(\mathcal{F}; \pi/3)$</td>
<td>approximate argument to within additive error of $\pi/3$</td>
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- Specific values of accuracy parameters not relevant
- Cannot use technique of polynomial interpolation when analysing approximation complexity, hence many existing results cannot be used
Exact dichotomy for conservative holant (aka HOLANT*)

Theorem ([Cai, Lu, Xia 2011])

Suppose $\mathcal{F}$ is a finite set of functions. If

1. $\mathcal{F} \subseteq \langle \Upsilon_1 \cup \Upsilon_2 \rangle_h$, or
2. there exists an orthogonal matrix $O$ such that $\mathcal{F} \subseteq O \circ \langle \Upsilon_1, \text{EQ}_3, \text{NEQ} \rangle_h$, or
3. $\mathcal{F} \subseteq \left( \begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix} \right) \circ \langle \Upsilon_1, \text{EQ}_3, \text{NEQ} \rangle_h$, or
4. there exists a matrix $K \in \left\{ \left( \begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix} \right), \left( \begin{smallmatrix} 1 & 1 \\ -i & i \end{smallmatrix} \right) \right\}$ such that $\mathcal{F} \subseteq \langle K \circ M \rangle_h$, where

$$M = \{ f \mid f(x) = 0 \text{ unless } |x| \leq 1 \},$$

then, for any finite subset $S \subseteq \mathcal{U}$, the problem $\text{HOLANT}(\mathcal{F}, S)$ is polynomial-time computable. Otherwise, there exists a finite subset $S \subseteq \mathcal{U}$ such that $\text{HOLANT}(\mathcal{F}, S)$ is $\#P$-hard. The dichotomy is still true even if the inputs are restricted to planar graphs.

We will refer to conditions 1–4 as the four conditions.
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We will refer to conditions 1–4 as the four conditions.
Universal quantum circuits as holant clones

The following is well-known in quantum computing:

*Any* $2^n \times 2^n$ *unitary matrix* can be realised from $2 \times 2$ *unitary matrices* and

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}$$

*via* Kronecker products and matrix products.

**Lemma**

*Any algebraic complex-valued function is contained in* $\langle \Upsilon_2, \text{CNOT}, \delta_0 \rangle_h$, *where*

$$\text{CNOT}(\mathbf{x}) = \begin{cases} 
1 & \text{if } \mathbf{x} \in \{0000, 0101, 1011, 1110\} \\
0 & \text{otherwise.}
\end{cases}$$
Universal conservative holant clones

Theorem
\( \langle F, \Upsilon_1 \rangle_h \) is universal unless \( F \) satisfies one of the four conditions.

Proof (sketch).

▷ If \( F \subseteq \langle \Upsilon_1, \Upsilon_2 \rangle_h \), it satisfies the first condition.
▷ So assume otherwise, i.e. there is \( f \in F \setminus \langle \Upsilon_1, \Upsilon_2 \rangle_h \).
▷ Use this \( f \) to realise a ternary function \( f' \).
▷ If \( F \) does not satisfy conditions 2–4, can use \( f' \) (and possibly some other functions in \( F \cup \Upsilon_1 \)) to realise \( \text{EQ}_3 \).
▷ Similarly, can show \( \Upsilon_2 \subseteq \langle F, \Upsilon_1 \rangle_h \).
▷ Realise CNOT from \( \text{EQ}_3 \) and \( h(x, y) = (-1)^{xy} \).
▷ Then apply the universality lemma.
Approximating conservative holant

Theorem
Suppose that \( \mathcal{F} \) is finite. Then there exists a finite subset \( S \subseteq \gamma_1 \) such that both \( \text{HolantArg}(\mathcal{F} \cup S; \pi/3) \) and \( \text{HolantNorm}(\mathcal{F} \cup S; 1.01) \) are \( \#P \)-hard, unless \( \mathcal{F} \) satisfies one of the four conditions.

Proof (sketch).

- If \( \mathcal{F} \) does not satisfy any of the four conditions, \( \langle F \cup \gamma_1 \rangle_h \) is universal.
- Pick some computational problem that is known to be hard to approximate, e.g. approximating the independent set polynomial on graphs of maximum degree 3.
- Only need a finite number of functions from \( \mathcal{F} \cup \gamma_1 \) to build the gadgets necessary to reduce from that problem.
Outline

Background: computational counting problems

Basic quantum information theory

Exact evaluation of HOLANT$^c$

Approximating conservative holant (joint work with Leslie Goldberg)

Conclusions
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- **Holant problems** are a framework for counting problems characterised by sets of signatures.

  - Holant problems have the same mathematical description as quantum states: vectors in $\mathbb{C}^{2^n}$.
  - Use knowledge from quantum theory, particularly about entanglement, to analyse Holant problems.
  - This has led to new complexity dichotomies.

  - Exact evaluation of $\text{Holant}(F)$ can be computed as $\text{Holant}(F \cup \{\delta_0, \delta_1\})$. [arXiv:1704.05798]
  - Approximation of conservative $\text{Holant}$ [with Leslie Goldberg, arXiv:1811.00817].

- Approach has also been taken up by other authors, leading to further exact holant dichotomies.
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