On the relation between Interpreted Systems and Kripke Models

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Abstract. We compare *Kripke models* and hypercube systems, a simplified notion of *Interpreted Systems*, as semantic structures for reasoning about knowledge. Our method is to define a map from the class of hypercube systems to the class of Kripke frames, another in the opposite direction, and study their properties and compositions. We show that it is possible to characterise semantically the frames that are images of the hypercube systems.

1 Introduction

The need for specifications of complex systems in Artificial Intelligence (AI), as in mainstream computer science, has brought forward the use of logic as formal tool for reasoning and proving properties about systems. In this respect, Multi-Agent Systems (MAS) constitute no exception and in the last thirty years many logics for modelling MAS have been proposed.

The design of a knowledge based agent is a central issue in agent theory, as knowledge is a key property of any intelligent system. Arguably the most successful approach is the modal logic $S5_n$, which was first proposed in Philosophical Logic by Hintikka ([Hin62]) and later used in Distributed Computing Theory by Halpern and Moses ([HF85]) and others.

The logic $S5_n$ models a community of *ideal* knowledge agents. Ideal knowledge agents have, among others, the properties of veridical knowledge (everything they know is true), positive introspection (they know what they know) and negative introspection (they know what they do not know). The modal logic $S5_n$ (see for example [HC96] and [Gol87]) can be axiomatised by taking all the propositional tautologies; the schemas of axioms

$K_i(\phi \Rightarrow \psi) \Rightarrow K_i\phi \Rightarrow K_i\psi$	Distribution of knowledge over implication
$K_i\phi \Rightarrow \phi$	Veridical knowledge
$K_i\phi \Rightarrow K_iK_i\phi$	Positive introspection
$\neg K_i\phi \Rightarrow K_i\neg K_i\phi$	Negative introspection

where $i \in A$ represents an agent in the set of agents $A = \{1, \ldots, n\}$; and the inference rules Modus Ponens and Necessitation.

The logic $S5_n$ has also been extended to deal with properties that arise when we investigate the state of knowledge of the group. Subtle concepts like common knowledge and distributed knowledge have been very well investigated ([FHMV95]). The logic $S5_n$ is a successful tool for the agent theorist also because, even in its extensions to common knowledge and distributed knowledge, it has important meta-properties like completeness and decidability (see for example [MvdH95]).

Two apparently different semantic treatments are available in symbolic AI to interpret the language of modal logic: *interpreted systems* and *Kripke models*.

Interpreted systems were first proposed by Fagin, Halpern, Moses and Vardi [HF85] to model distributed systems. The growing interest in complex MAS and in their specifications has brought forward the concept of interpreted system as useful formal tool to model key characteristics of the agents, such as the evolution of their knowledge, communication, etc. This work has culminated in the publication of [FHMV95] in which the authors use the notion of interpreted system to explore systematically fundamental classes of MAS (such as synchronous, asynchronous, with perfect recall ability, etc.) by the use of interpreted systems.

Kripke models [Kri59] were first proposed in Philosophical Logic and later used in Logic for AI as semantic structures for logics for belief, logics for knowledge, temporal logics, logics for actions, etc, all of which are modal logics. Over the last thirty years, many formal techniques have been developed for the study of modal logics grounded on Kripke semantics, such as completeness proofs via canonical models, decidability via the finite model property [HC96], and more recently, techniques for combining logics [KW91, Gab96].

The two approaches have different advantages and disadvantages. On the one hand, interpreted systems are more intuitive to model real MAS, on the other hand Kripke models come with a heritage of fundamental techniques (see for example [Gol87, HC96] that can allow the user to prove properties about his or her specification.

Given the common purpose of the two approaches, some questions arise naturally. Is one of the approaches more specialised than the other? What is the difference between the two generated logics? Is it possible to use the powerful techniques developed for Kripke models to MAS defined in terms of the more intuitive systems? Is it possible to identify in terms of frames key MAS usually defined in terms of interpreted systems? The rest of the paper answers only partially to some of these questions, but tries to bring us a step further in our understanding of the two notions.

In the article we isolate and study a special class of interpreted systems that we call hypercube systems or simply hypercubes, which are defined by taking not an arbitrary subset (as interpreted systems are defined) but the *full* Cartesian product of the local states for the agents. We show that hypercube systems are semantically equivalent to a special class of frames defined on equivalence relations commonly used to interpret an epistemic language.

Hypercube systems are a special case of interpreted systems but we hope that the methods we introduce to analyse them can be extended to analyse interpreted systems in the general settings.

The paper is organised as follows: In Section 2 we remind the reader of some basic mathematical notions that we will use throughout the paper. In Section 3 we define interpreted systems, Kripke models, and hypercube systems. In section 4 we define maps between hypercubes and Kripke models. In Section 5 we analyse the composition of these maps and we present results that relate the two semantics. In Section 6 we draw our conclusions and we suggest further work. Proofs of all theorems and lemmas are given in the Appendix.

2 Mathematical Preliminaries

We assume a modal propositional language, defined in the usual way from a set of propositional variables by the use of classical connectives, the operators K_i and D_B. The index i varies over a set $A = \{1, \dots, n\}$, representing the agents of the system and B varies over subsets of A. The modal operator K_i represents the knowledge of the agent i, while D_B represents the distributed knowledge among the group B (the reader is referred to [FHMV95] for an introduction to this terminology). We use the standard definitions for satisfaction for formulas on states, and validity for formulas on frames, on models, on class of frames, and on class of models - see [HC96] for details. If W is a set, id_W is the identity relation on W. If $\sim$ is an equivalence relation on W and $w \in W$, then $W/{\sim}$ is the set of equivalence classes, and $[w]_\sim$ is the equivalence class containing w.

3 Hypercube systems

We briefly remind the key definitions of Kripke frames and interpreted systems; then we define hypercube systems.

3.1 Kripke models

Kripke models are the fundamental semantic structures used in modal logic to reason about possibilities, necessities, knowledge, obligation, etc. In the case of epistemic logic the usual approach is to take Kripke models grounded on equivalence relations so that they constitute a complete semantics for the logic $S5_n$ described above. We report here the key definition.

Definition 1 (Equivalence frames). An *equivalence frame* $F = (W, \sim_1, \dots, \sim_n)$ is a tuple where W is a non-empty set and for every i in A, $\sim_i$ is an equivalence relation over $W \times W$. Elements of W are called worlds and are denoted as: $w_1, w_2, \dots$ $\mathcal{F}$ denotes the class of frames.

Intuitively points of W represent epistemic alternatives, i.e. possible configurations. Relations represent epistemic possibility between points; for example with

$w \sim_i w'$ we capture the fact that "w' is possible according to i's knowledge in the state w".

An equivalence Kripke model $M = (F, \pi)$ is a pair, where F is an equivalence frame and π is an interpretation for the atoms of the language.

For ease of reference, we state here the notion of validity on a class of frames.

Definition 2 (Validity on Kripke frames). A formula ϕ is valid on a class $\mathcal{F}$ of Kripke frames if for any frame $F \in \mathcal{F}$ for any valuation π, $(F, \pi) \models \phi$.

3.2 Interpreted systems

Interpreted systems can be defined as follows ([FHMV95]). Consider n sets of local states, one for every agent of the MAS, and a set of states for the environment.

Definition 3 (Global states of interpreted systems). *A set of global states for an interpreted system* is a subset S of the Cartesian product $L_e \times L_1 \times \cdots \times L_n$, where $L_e, L_1, \ldots, L_n$ are non-empty sets. The set L_i represents the local states possible for agent i and L_e represents the possible states of the environment.

A global state represents the configuration of all the agents and of the environment at a particular instant of time. The idea behind considering a subset is that some of the tuples that originate from the Cartesian product might not be possible because of explicit constraints present in the MAS. By considering functions (runs) $r : N \rightarrow S$ from the natural numbers to the set of global states, it is possible to represent the temporal evolution of the system. An interpreted system $IS = (R, \pi)$ is a set of functions $R = \{r : N \rightarrow S\}$ on the global states with a valuation π for the atoms of the language. Since here we carry out an analysis of the static properties of knowledge, we will not consider runs explicitly and we will consider interpreted systems to be pairs $IS = (S, \pi)$.

Interpreted systems can represent the knowledge of the MAS by considering two global states to be indistinguishable for an agent if its local state is the same in the two global states. Thus, a set of global states S denotes the Kripke frame $F = (W, \sim_1, \ldots, \sim_n)$, if $W = S, (l_1, \ldots, l_n) \sim_i (l'_1, \ldots, l'_n)$, if $l_i = l'_i, i \in A$.

3.3 Hypercube systems

Given n sets of local states for the agents of the MAS, the interpreted systems we analyse in this paper and that we call hypercube systems or hypercubes, result by considering the admissible state space of the MAS to be described by the full Cartesian product of its sets of local states. This means that every global state is in principle possible, i.e. there are no mutually exclusive configurations between such local states. Various scenarios comply with this specification, such as distributed systems that have just crashed, and more generally in MAS in which no information is available about their configuration[1]. In these cases the state space of the system is the whole full Cartesian product of the sets of locals states for the agents.

With hypercubes we are imposing a further simplification on the notion presented in Definition 3: in the tuples representing the configuration of the system we do not consider a slot for the environment. The presence of the environment in the notion of Fagin et al. is motivated in order to keep track of the changes in the system and in general to represent everything that cannot be captured by the local states of the single agents (most importantly messages in transit, etc.). By neglecting the dimension of the environment or, which comes to be the same thing, by treating it as a constant, we are projecting the notion of Fagin et al. of a time-dependent interpreted system to the product of its local states. Since we are focusing on a static case, in a way we can see this restriction as fixing the environment at the time in analysis, and investigate the possible configurations of the states of the agents. We formally define hypercube systems.

Definition 4 (Global states of hypercube systems). A *hypercube system*, or *hypercube*, is a Cartesian product $H = L_1 \times \cdots \times L_n$, where $L_1, \ldots, L_n$ are non-empty sets. The set L_i represents the local states possible for agent i. Elements of a local state L will be indicated with $l_1, l_2, \ldots$ The class of hypercube systems is denoted by $\mathcal{H}$.

Aim of the paper is to relate hypercube systems to Kripke models. More specifically we would like to identify the class of Kripke models that satisfy exactly the same formulas satisfied by the hypercubes. Given the notion of validity of formulas on interpreted systems and Kripke models, it is appropriate to compare the two underlying semantic structures: Kripke frames and global states of hypercube systems. This is what we do in the next two Sections, where, for brevity, we will use the terms "hypercube systems" and "hypercubes" also to refer to sets of global states of hypercube systems as in Definition 4.

4 Mappings between hypercubes and frames

Although hypercubes are intuitively a special class of Kripke frames, it is clear that they are not simply a subset. In order to clarify the relationship, we have to use the construction given implicitly in [FHMV95] for obtaining a frame from a system. Our framework will be the following (proofs are reported in the Appendix):

- We define the class of hypercubes $\mathcal{H}$, and the class of Kripke frames $\mathcal{F}$.
- We define two maps, $\mathcal{H} \xrightarrow{f} \mathcal{F}$ (based on [FHMV95]) and $\mathcal{F} \xrightarrow{g} \mathcal{H}$.
- We analyse the compositions of the maps f and g.

[1] It has also been suggested by Ron van der Meyden that there may be a connection between the full Cartesian product and the states of knowledge in certain classes of broadcast systems.

– We isolate the images of $\mathcal{H}$ in $\mathcal{F}$.

Hypercubes and frames are always defined over a set A of n agents, which we assume as given.

Every hypercube generates a frame ([FHMV95]):

Definition 5 (Hypercubes to frames). $f : \mathcal{H} \to \mathcal{F}$ is the function that maps the system H onto a Kripke frame in the following way:
If $H = L_1 \times \cdots \times L_n$, $f(H) = (L_1 \times \cdots \times L_n, \sim_1, \ldots, \sim_n)$, where $\sim_i$ is defined as: $(l_1, \ldots, l_n) \sim_i (l'_1, \ldots, l'_n)$ if and only if $l_i = l'_i$.

Lemma 6. *If H is a hypercube system, and $f(H) = (W, \sim_1, \ldots, \sim_n)$ is the frame defined from it by Definition 5, then*

1. $\bigcap_{i \in A} \sim_i = id_W$;
2. *For any $w_1, \ldots, w_n$ in W there exists a $\overline{w}$ such that $\overline{w} \sim_i w_i$, $i = 1, \ldots, n$.*

The proof of this (and other) results is given in the Appendix.

This shows that Kripke frames that we build from the hypercubes by means of the standard technique ([FHMV95]) constitute a subset of all the possible reflexive, symmetric and transitive Kripke frames. To relate the two semantic classes, we have to analyse the properties of Lemma 6.

The first one expresses the fact that in the images of the hypercubes there cannot be two states related by all the equivalence relations. This is a peculiarity of the construction f given in [FHMV95].

The second property reflects the fact that hypercubes are defined on full Cartesian products. The property expresses the circumstance that for every pair of points in the n dimensions space of the images of the hypercubes, there are $n!$ ways to connect them in two steps. In particular, we can change $n-1$ coordinates in n possible ways and change the last one in the last step.

Given these differences between the class of hypercubes and equivalence frames, it is likely that the two semantic structures satisfy different formulas. In fact we have the following.

Condition 1 of Lemma 6 imposes the following Lemma.

Lemma 7. *Consider a frame $F = (W, \sim_1, \ldots, \sim_n)$.*
$\bigcap_{i \in A} \sim_i = id_W$ *if and only if* $F \models \phi \Leftrightarrow D_A \phi$.

Corollary 8. *If H is a hypercube system, $f(H) \models \phi \Leftrightarrow D_A \phi$.*

This means that on hypercubes the notion of truth of a formula collapses to the one of distributed knowledge of the formula.

Condition 2 of Lemma 6 forces the frames generated from hypercubes to satisfy the following formula.

Lemma 9. *If H is a hypercube system, $f(H) \models \neg K_i \neg K_j \phi \Rightarrow K_j \neg K_i \neg \phi$, where $i \neq j$.*

The formula in Lemma 9 is an axiom that relates private knowledge between two arbitrary agents of the model.

It is easy to check that Formulas in Lemmas 7 and 9 are not generally valid on the class $\mathcal{F}$ of frames. In Figure 1 M_1 does not validate the Formula in Lemma 7 and M_2 does not validate the Formula in Lemma 9. In fact w_0 in M_1 does not satisfy $p \Leftrightarrow D_A p$, where $D_A p$ is as usual computed by taking the equivalence relation defined by the intersection of the equivalence relations $\sim_1, \sim_2$. In M_2, w_0 does not satisfy $\neg K_1 \neg K_2 p \Rightarrow K_2 \neg K_1 \neg p$.

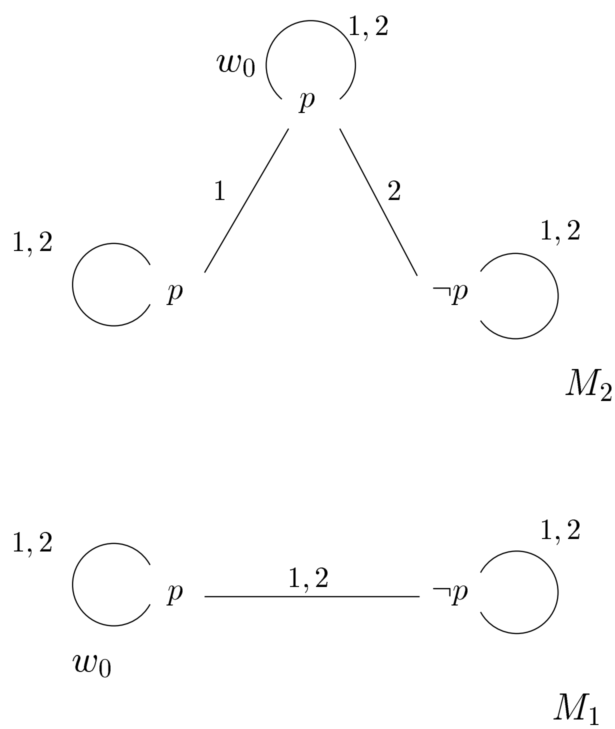

Fig. 1. Equivalence models not satisfying Formulas in Lemma 7 and Lemma 9

It is also possible to generate a system from a frame:

Definition 10 (Frames to hypercubes). $g : \mathcal{F} \rightarrow \mathcal{H}$ is the function that maps a frame $F = (W, \sim_1, \ldots, \sim_n)$ onto the hypercubes $g(F) = W/\!\sim_1 \times \cdots \times W/\!\sim_n$.

We now have defined maps between the two semantic structures. Our aim is to use them to identify the class of equivalence frames that are semantically equivalent, i.e. that satisfy the same formulas, to the hypercubes. In order to do

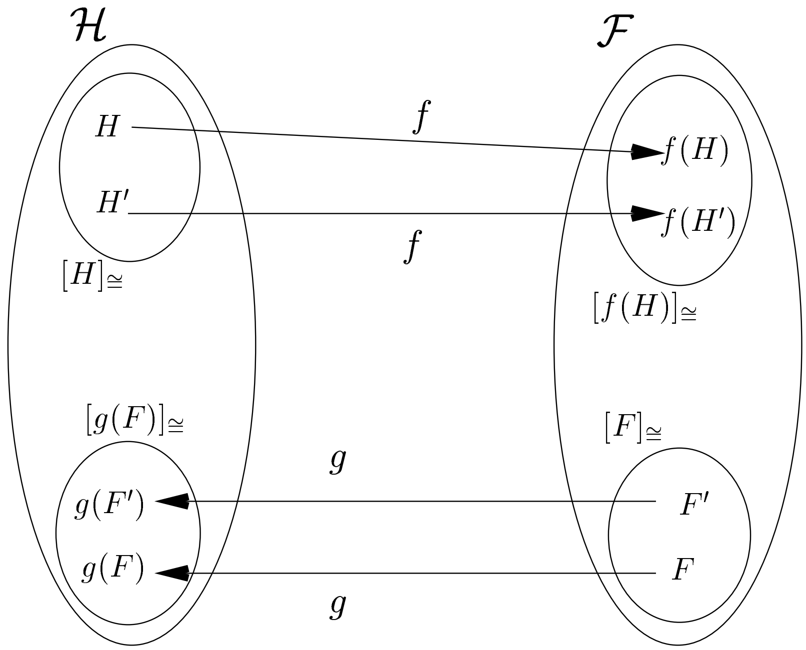

Fig. 2. Preservation of isomorphisms under the maps

so, we introduce a notion of isomorphism on $\mathcal{F}$ and $\mathcal{H}$. Many notions (such as p-morphisms or bisimulations for frames) may be appropriate for this task, but for our aims we need a strong equivalence between the structures.

Consider two MAS. If we can draw a bijection between the agents of the MAS such that the local states of the corresponding agents are themselves in a bijection, then in a way we can think that one MAS can simulate the other, and so the two MAS can be thought as being equivalent. We formalise this as follows:

Definition 11 (Isomorphism of hypercubes). Two hypercubes $H = L_1 \times \cdots \times L_n$, $H' = L'_1 \times \cdots \times L'_n$ are isomorphic ($H \cong_{\mathcal{H}} H'$) if $|L_i| = |L'_i|$ for $i = 1, \ldots, n$.

To reason about equivalent frames we take the standard notion of isomorphism.

Definition 12 (Isomorphism of frames). Two frames $F = (W, \sim_1, \ldots, \sim_n)$, $F' = (W', \sim'_1, \ldots, \sim'_n)$ are isomorphic ($F \cong_{\mathcal{F}} F'$) if and only if:

- There exists a bijection $b : W \to W'$,
- For all $s, t \in W$, and all $i \in A$, $s \sim_i t$ if and only if $b(s) \sim'_i b(t)$.

We can prove that the maps we defined preserve isomorphisms:

Lemma 13. *If $H \cong_{\mathcal{H}} H'$, then $f(H) \cong_{\mathcal{F}} f(H')$.*

Lemma 14. *If $F \cong_{\mathcal{F}} F'$, then $g(F) \cong_{\mathcal{H}} g(F')$.*

Figure 2 shows the preservation of isomorphisms under f and g between frames and hypercubes as proved Lemmas 13 and 14. Since we want to import and export results from one structure into the other, this is the result we need.

5 Characterisation of the class of hypercube systems

We now investigate the extent to which the composition of f with g (or g with f) results in a hypercube (frame) which is isomorphic to the one we started with. We do this for two reasons. First we want to check whether by going back and forth between the two class of structures we are going to lose information, i.e. the structure we obtain satisfies different formulas from the original one. Secondly, this will help us prove a result on the correspondence of the hypercubes into a subclass of frames. We operate as follows.

Given a hypercube $H = L_1 \times \cdots \times L_n$, consider the image under f of H, $f(H)$. Let $H' = (L_1 \times \cdots \times L_n)/{\sim_1} \times \cdots \times (L_1 \times \cdots \times L_n)/{\sim_n}$ be the image under g of $f(H)$. We want to investigate the relationship between H and H'.

Theorem 15. *For any system H in $\mathcal{H}$, $H \cong_{\mathcal{H}} g \circ f(H))$.*

In other words, if we start from a system H, build the corresponding Kripke frame $f(H)$, it is still possible to extract all the information from the frame by applying the function g that produces another system H', which is in a bijection with the original H.

We now investigate the other side of the relation. Consider a frame F and its image under g, $g(F)$. If we take the image under f of $g(F)$, that frame will satisfy the property stated by Lemma 6 and therefore will not in general be isomorphic to F. As we made clear in the previous Section, property one of Lemma 6 corresponds to the validity of a formula on such frames. Therefore, $f(g(F))$ is not only non-isomorphic to F, but it is not even even a p-morphic image of F.

What we can prove is the following:

Lemma 16. *If F is a frame such that there exists a system H, with $F \cong_{\mathcal{F}} f(H)$, then $F \cong_{\mathcal{F}} f \circ g(F)$.*

If we consider a frame $F = (W, \sim_1, \ldots, \sim_n)$ such that $\bigcap_{i \in A} \sim_i = id_W$, $f \circ g(F)$ will not in general be isomorphic to F. As an example, consider:

$$F = (\{w_1, w_2\}, \{(w_1, w_1), (w_2, w_2)\}, \{(w_1, w_1), (w_2, w_2)\}).$$

We need to restrict our attention to *both* the properties inherited from the mapping from hypercubes. Results of Lemmas 15 and 16 are shown in Figure 3.

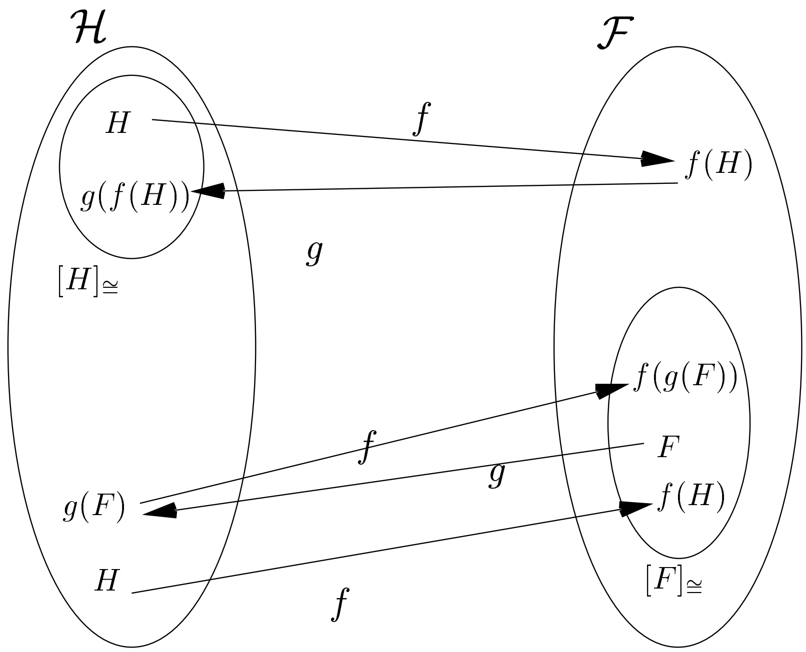

Fig. 3. Compositions of maps between frames and hypercubes as in Theorem 15 and Lemma 16

Theorem 17. *If $F = (W, \sim_1, \ldots, \sim_n)$ is a frame such that:*

- $\bigcap_i \sim_i = id_W$,
- $\forall w_1, \ldots, w_n, \exists \overline{w}$ *such that* $\overline{w} \sim_i w_i$, $i = 1, \ldots, n$;

then $F \cong_{\mathcal{F}} f \circ g(F)$.

Theorem 17 and Lemma 6 allows us to characterise the frames that are images of some system:

Theorem 18. *Let $F = (W, \sim_1, \ldots, \sim_n)$ be a frame. The following are equivalent:*

1. $\bigcap_i \sim_i = id_W$ *and* $\forall w_1, \ldots, w_n, \exists \overline{w}$ *such that* $\overline{w} \sim_i w_i$, $i = 1, \ldots, n$;
2. *there exists an H, such that $F \cong_{\mathcal{F}} f(H)$.*

Theorem 18 characterises the frames that we obtain by applying the map f to the class of hypercubes. Every member of this class of frames is isomorphic to a system and a frame not included in this class is not.

We can now identify a class of frames which is semantically equivalent to hypercube systems. To do this, we remind that satisfaction on a system H is defined by considering the image under f of H. In this context we need the notion of validity on a system:

Definition 19. A formula ϕ is valid on a system H, ($H \models \phi$), if $f(H) \models \phi$.

Validity of ϕ on the frame $f(H)$ in Definition 19 was defined in Definition 2.

We can no prove that:

Theorem 20. *Let $\mathcal{G}$ be the class of equivalence frames that satisfy property 1 and 2 of Lemma 6, then $\forall \phi (\mathcal{H} \models \phi$ if and only if $\mathcal{G} \models \phi)$.*
Proof: From right to left. If $\mathcal{G} \models \phi$, then, since $f(\mathcal{H}) \subseteq \mathcal{G}$, $f(\mathcal{H}) \models \phi$. So, by Definition 19 $\mathcal{H} \models \phi$.
From left to right. Assume $\mathcal{H} \models \phi$, i.e. $f(\mathcal{H}) \models \phi$, we want to show that for any $F \in \mathcal{G}$, $F \models \phi$. By Lemma 17 and Theorem 18, $F \cong_{\mathcal{F}} f(g(F))$. But then $F \models \phi$ if and only if $f(g(F)) \models \phi$. But $g(F) \in \mathcal{H}$, and so $f(g(F)) \models \phi$, and so $F \models \phi$. □

Theorems 18 and 20 completely characterise the hypercubes we focus in this note in terms of Kripke frames.

6 Conclusions and further work

Interpreted systems are a useful formalism for representing MAS knowledge. In this note we have analysed their relation with Kripke models in a simplified setting by looking at the case of hypercube systems.

We have defined mappings between hypercube systems and Kripke frames and we have completely characterised the Kripke structures which are semantically equivalent to hypercubes.

The methodology we presented here to map hypercubes into Kripke models suggests that further research could be undertaken to attempt to have a general methodology for translating interesting classes of interpreted systems into classes of Kripke models. This would help in the process of axiomatising key MAS defined in terms of interpreted systems as the analysis could be carried out in the class of Kripke models.

Should such a general methodology for inter-translating the two classes be achieved, this may also help in the attempt to apply combining logics techniques for modal logics (for example [KW91]) to the case of complex MAS defined in terms of systems. The idea is that complex MAS specifications would benefit from an approach focused on the identification of classes of interactions between basic and well-understood modal logics with respect to the transfer of important properties such as completeness (see [LR97a] for details).

Hypercubes seem to capture an interesting property concerning the relation between private knowledge of the agents of the group. Given the semantic equivalence expressed by Theorem 20 it is possible to axiomatise hypercubes by analysing the corresponding Kripke frames. This was presented in [LR97b] where a sound and complete axiomatisation for equivalence frames that satisfy properties 1 and 2 of Lemma 6 is shown.

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A Proofs of theorems

Lemma 6 *If H is a system, and $f(H) = (W, \sim_1, \ldots, \sim_n)$ is the frame defined from it by Definition 5, then*

1. $\bigcap_{i \in A} \sim_i = id_W$;
2. *For any $w_1, \ldots, w_n$ in W there exists a $\overline{w}$ such that $\overline{w} \sim_i w_i$, $i = 1, \ldots, n$.*

Proof: For 1, Consider any two elements $w = (l_1, \ldots, l_n), w' = (l'_1, \ldots, l'_n)$ in W such that $w(\bigcap_{i \in A} \sim_i)w'$. Then for all i in A, $(l_1, \ldots, l_n) \sim_i (l'_1, \ldots, l'_n)$. Therefore by definition, for all i in A, $l_i = l'_i$, that is $w = w'$.
For 2, consider any $w_1 = (l_1, \ldots, l_n), \ldots, w_n = (m_1, \ldots, m_n)$. Now let $\overline{w} = (l_1, \ldots, m_n)$. By definition 5, the element $\overline{w}$ is in W and for each i, $\overline{w} \sim_i w_i$. □

Lemma 7 *Consider a frame $F = (W, \sim_1, \ldots, \sim_n)$. $\bigcap_{i \in A} \sim_i = id_W$ if and only if $F \models \phi \Leftrightarrow D_A\phi$.*

Proof. Left to right. Let M be a model based on F such that $M \models_w \phi$. Since $\bigcap_{i \in A} \sim_i = id_W$, then $M \models_w D_A\phi$. Analogously, suppose $M \models_w D_A\phi$. Since $w(\bigcap_{i \in A} \sim_i)w'$ implies $w = w'$, then $M \models_w \phi$.

Right to left. Suppose $F \models \phi \Leftrightarrow D_A\phi$ and for all i $w_1 \sim_i w_2$. Take a valuation π such that $\pi(p) = \{w_1\}$. Since $F, \pi \models_{w_1} p \Leftrightarrow D_A p$ and $F, \pi \models_{w_1} p$, we have $F, \pi \models_{w_1} D_A p$ and so $F, \pi \models_{w_2} p$. But since $\pi(p) = \{w_1\}$, it must be that $w_1 = w_2$. □

Lemma 9 *If H is a system, $f(H) \models \neg K_i \neg K_j \phi \Rightarrow K_j \neg K_i \neg \phi$, where $i \neq j$.*

Proof: For a contradiction suppose that $f(H) \not\models \neg K_i \neg K_j \phi \Rightarrow K_j \neg K_i \neg \phi$. Then there exists a point w and a valuation π such that $(F, \pi) \models_w \neg K_i \neg K_j \phi \wedge \neg K_j \neg K_i \neg \phi$. Therefore there must exist two points w_1 and w_2 such that $w \sim_i w_1$ and $w \sim_j w_2$ and $(F, \pi) \models_{w_1} K_j \phi$ and $(F, \pi) \models_{w_2} K_i \neg \phi$. But by property 2. of Lemma 6 there exists a point $\overline{w}$ such that $\overline{w} \sim_j w_1$ and $\overline{w} \sim_i w_2$. Since $(F, \pi) \models_{w_1} K_j \phi$ and the relations are symmetric, we have $(F, \pi) \models_{\overline{w}} \phi$, but this contradicts $(F, \pi) \models_{w_2} K_i \neg \phi$ that requires $\overline{w}$ to satisfy $\neg \phi$. □

Lemma 13 *If $H \cong_{\mathcal{H}} H'$, then $f(H) \cong_{\mathcal{F}} f(H')$.*

Proof: Let $H = L_1 \times \cdots \times L_n$, and $H' = L'_1 \times \cdots \times L'_n$. Since $H \cong_{\mathcal{H}} H'$ there is a family of bijections $b_i : L_i \to L'_i$. Consider $b = b_1 \times \cdots \times b_n$. The function b is a bijection, and therefore the universes of the frames $f(H)$ and $f(H')$ are in a bijection.

Consider now $s = (l_1, \ldots, l_i, \ldots, l_n), s' = (l'_1, \ldots, l'_i, \ldots, l'_n)$ such that $s, s' \in H$, and $s \sim_i s'$ on $f(H)$. Consider $b(s) = (b_1(l_1), \ldots, b_i(l_i), \ldots, b_n(l_n))$ and $b(s') = (b_1(l'_1), \ldots, b_i(l'_i), \ldots, b_n(l'_n))$. Since, by definition, $l_i = l'_i$, then $b_i(l_i) = b_i(l'_i)$ and therefore $b(s) \sim'_i b(s')$.

Let now be $b(s) \sim'_i b(s')$. Then, by definition $b_i(l_i) = b_i(l'_i)$ and then $l_i = l'_i$, that implies $s \sim_i s'$. □

Lemma 14 *If $F \cong_{\mathcal{F}} F'$, then $g(F) \cong_{\mathcal{H}} g(F')$.*

Proof: Consider two isomorphic frames $F = (W, \sim_1, \ldots, \sim_n), F' = (W', \sim'_1, \ldots, \sim'_n)$ such that $b : W \to W'$ is a bijection. We want to prove that there is a family of bijections c_i between the components of $g(F) = W/\sim_1 \times \cdots \times W/\sim_n$ and $g(F') = W'/\sim'_1 \times \cdots \times W'/\sim'_n$. Let $c_i : W/\sim_i \to W'/\sim'_i$ such that $c_i([w]_{\sim_i}) = [b(w)]_{\sim'_i}$.

The function c_i is well defined. In fact, let $[w]_{\sim_i} = [w']_{\sim_i}$, with $w, w' \in W$. Then $c_i([w]_{\sim_i}) = [b(w)]_{\sim'_i} = [b(w')]_{\sim'_i} = c_i([w']_{\sim_i})$.

The function c_i is injective. $c_i([w]_{\sim_i}) = c_i([w']_{\sim_i})$, then $[b(w)]_{\sim'_i} = [b(w')]_{\sim'_i}$, that is $b(w) \sim_i b(w')$, $w \sim_i w'$ and then $[w]_{\sim_i} = [w]_{\sim'_i}$.

The function c_i is surjective. Consider $[w']_{\sim'_i}$, such that $w' \in W'$ and let $w \in W$ be such that $b(w) = w'$. Then $c_i([w]_{\sim_i}) = [w']_{\sim'_i}$. □

Theorem 15 *For any system H in $\mathcal{H}$, $H \cong_{\mathcal{H}} g \circ f(H))$.*

Proof: We prove that the function $b_i : L_i \to (L_1 \times \cdots \times L_n)/\sim_i$, defined as $b_i(l_i) = [(l_1, \ldots, l_i, \ldots, l_n)]_{\sim_i}$, where $l_j, i \neq j$, is any element in L_j, is a bijection.

The function b_i is well defined. In fact, let $l_i = l'_i$. So $b_i(l_i) = [(l_1, \ldots, l_i, \ldots, l_n)]_{\sim_i}$ and $b_i(l'_i) = [(l'_1, \ldots, l'_i, \ldots, l'_n)]_{\sim_i}$. But $(l_1, \ldots, l_i, \ldots, l_n) \sim_i (l'_1, \ldots, l'_i, \ldots, l'_n)$ and therefore $b_i(l_i) = b_i(l'_i)$.
The function b_i is an injection: let $b_i(l_i) = b_i(l'_i)$, so $[(l_1, \ldots, l_i, \ldots, l_n)]_{\sim_i} = [(l'_1, \ldots, l'_i, \ldots, l'_n)]_{\sim_i}$, that implies $l_i = l'_i$.
The function b_i is a surjection. In fact, consider any $[(l_1, \ldots, l_i, \ldots, l_n)]_{\sim_i} \in (L_1 \times \cdots \times L_n)/\sim_i$. $b_i(l_i) = [(l'_1, \ldots, l_i, \ldots, l'_n)]_{\sim_i} = [(l_1, \ldots, l_i, \ldots, l_n)]_{\sim_i}$. □

Theorem 17 *If $F = (W, \sim_1, \ldots, \sim_n)$ is a frame such that:*

- $\bigcap_i \sim_i = id_W$,
- $\forall w_1, \ldots, w_n, \exists \overline{w}$ *such that* $\overline{w} \sim_i w_i$, $i = 1, \ldots, n$;

then $F \cong_{\mathcal{F}} f \circ g(F)$.
Proof: Consider the frame $f \circ g(F) = (W/\sim_1 \times \cdots \times W/\sim_n, \sim'_1, \ldots, \sim'_n)$ built according to Definition 10 and Definition 5. Let now h be a mapping $h : W \to W/\sim_1 \times \cdots \times W/\sim_n$, defined by $h(w) = ([w]_{\sim_1}, \ldots, [w]_{\sim_n})$. We prove that h is a bijection.
Injective: suppose $h(w_1) = h(w_2)$, so $([w_1]_{\sim_1}, \ldots, [w_1]_{\sim_n}) = ([w_2]_{\sim_1}, \ldots, [w_2]_{\sim_n})$. Therefore, for all i, $[w_1] \sim_i [w_2]$, but since $\bigcap_i \sim_i = id_W$, it must be $w_1 = w_2$.
Surjective: consider any element $([w_1]_{\sim_1}, \ldots, [w_n]_{\sim_n})$ in $W/\sim_1 \times \cdots \times W/\sim_n$. By Hypothesis on F, there exists a world $\overline{w}$ in W, such that $[\overline{w}]_{\sim_i} = [w_i]_{\sim_i}$, for each $i = 1, \ldots, n$. Therefore $([w_1]_{\sim_1}, \ldots, [w_n]_{\sim_n}) = ([\overline{w}]_{\sim_1}, \ldots, [\overline{w}]_{\sim_n}) = h(\overline{w})$.
Now we prove that $w_1 \sim_i w_2$ in F if and only if $h(w_1) \sim'_i h(w_2)$ in $f \circ g(F)$. Suppose $w_1 \sim_i w_2$, that is $[w_1]_{\sim_i} = [w_2]_{\sim_i}$; by definition of $\sim_i$, this is equivalent to $([w_1]_{\sim_1}, \ldots, [w_1]_{\sim_n}) \sim'_i ([w_2]_{\sim_1}, \ldots, [w_2]_{\sim_n})$.
This proves that F and $f \circ g(F)$ are isomorphic. □

Theorem 18 *Let $F = (W, \sim_1, \ldots, \sim_n)$ be a frame. The following are equivalent:*

1. $\bigcap_i \sim_i = id_W$ *and* $\forall w_1, \ldots, w_n, \exists \overline{w}$ *such that* $\overline{w} \sim_i w_i$, $i = 1, \ldots, n$;
2. *there exists an H, such that* $F \cong_{\mathcal{F}} f(H)$

Proof: 1 implies 2: Under these conditions by Theorem 17, $F \cong_{\mathcal{F}} f \circ g(F)$. That is: $H = g(F)$.
2 implies 1: By Lemma 6 the frame $f(H)$ has the properties expressed by proposition 1. But F is isomorphic to $f(H)$ and therefore it has those properties as well. □

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