The geometry of constancy

(In HoTT and in `cubicaltt`)

Thierry Coquand
University of Gothenburg, Sweden

Martín Escardó
University of Birmingham, UK

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The papers [1, 2] consider the factorization of constant functions $f : X \to A$ through the propositional truncation $\|X\|$ of their domain, relating it to a number of phenomena, including a generalized version of Hedberg's Theorem that charaterizes those types which are sets. In particular, the above work considers general conditions on X that allow one to get $\|X\| \to X$, thus obtaining the explicit existence of an inhabitant of X from its anonymous existence.

For the purposes of this work, a function $f : X \to A$ is constant if any two of its values are equal:

$$\text{constant}\, f = \Pi(x, y : X).fx = fy.$$

This is not a proposition in general, and one may refer to a point of this type as a modulus of constancy of f. The above papers also give a number of sufficient conditions for the factorization to be possible, and conjecture that the factorization is not possible in general. In particular, if the type A is a set, then the factorization is always possible, and hence one needs to go beyond sets to settle the question.

Mike Shulman exhibited a family of constant functions for which a uniform factorization contradicts the univalence axiom, thus proving the conjecture [3]. Not all constant functions $f : X \to A$ factor through $\|X\|$.

Here we look at the problem from a more abstract, geometrical point of view. Moreover, we add a positive factorization result, originally conceived as an attempt to get a negative result, which here is offered as an illustration of the difficulty of the problem solved by Shulman.

Given any type X, we define a universal constant map $X \to S(X)$ by higher-induction. We have the constructors

$$\begin{aligned} \beta &: X \to S(X), \\ \ell &: \Pi(x, y : X).\beta(x) = \beta(y). \end{aligned}$$

When X is the terminal type 1, we have that $S(X)$ is the circle S^1:

$$S(1) = S^1.$$

The universal property of $S(-)$ is an equivalence

$$(S(X) \to A) \simeq \Sigma(f : X \to A).\,\text{constant}\, f,$$

which generalizes the universal property of the circle expressed as

$$(S^1 \to A) \simeq \Sigma(a : A).a = a.$$

This is particularly easy to construct in cubical type theory, compared to homotopy type theory.

To say that a function $f : X \to A$ is constant is equivalent to saying that it factors through $S(X)$. The type $S(X)$ is connected, the unit $\beta : X \to S(X)$ of the universal property of $S(X)$ is a constant surjection, and, because the universal map $X \to \|X\|$ into a proposition is constant, we always have a map $S(X) \to \|X\|$. We have a map $\|X\| \to S(X)$ for all X if and only if all constant functions $f : X \to A$ factor through $\|X\|$. Thus the general factorization problem is equivalent to the question of whether we have a function

$$\Pi(X : U).\|X\| \to S(X).$$

It seemed to us that perhaps the simplest potential counter-example could be

$$X = (s = \text{base}) \text{ for } s : S^1,$$

because then $\|X\| = 1$ as is well known and proved in the HoTT book, and so the question specialized to this particular case amounts to

$$\Pi(s : S^1).S(s = \text{base}).$$

It seemed preposterous to us to always be able to give an element of the type $S(s = \text{base})$ without being able to give an element of the type $(s = \text{base})$ in general. However, this is how things turn out be, and what we will present in the talk.

A consequence of this is that if we are given a point $s : S^1$ and a constant function $f : s = \text{base} \to A$ into a type A, then we can find a point of A which is the constant value of f, even in the absence of the knowledge of a point of the path space $(s = \text{base})$.

We will present `cubicaltt` code for the sake of illustration, but we emphasize that the definitions, theorems and proofs formulated here apply to the type theory of the HoTT book, and don't depend on the particularities of cubical type theory.

References

[1] Nicolai Kraus, Martín Escardó, Thierry Coquand, and Thorsten Altenkirch. Generalizations of hedberg's theorem. In *Typed Lambda Calculi and Applications*, volume 7941 of *Lecture Notes in Computer Science*, pages 173–188. Springer Berlin Heidelberg, 2013.

[2] Nicolai Kraus, Martín Escardó, Thierry Coquand, and Thorsten Altenkirch. Notions of anonymous existence in Martin-Löf Type Theory. Submitted for publication, 2014.

[3] Mike Shulman. Not every weakly constant function is conditionally constant. Blog post at the HoTT web site `http://homotopytypetheory.org`, June 2015.