The geometry of constancy
(in HoTT and in cubicaltt)

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Exiting propositional truncations

Often we have $\|X\| \to X$, even when we don’t know whether $X$ is empty or inhabited.

E.g. For any $f : \mathbb{N} \to \mathbb{N}$, we have $\| \sum_{n:\mathbb{N}} fn = 0 \| \to \sum_{n:\mathbb{N}} fn = 0$.

If there is a root of $f$, then we can find one.
Exiting propositional truncations

However, global choice

\[
\prod_{X : U} \Vert X \Vert \to X
\]

implies that all types have decidable equality.

(And even \( X + \neg X \) for all \( X : U \) if we have quotients.)

By Hedberg’s theorem, if every type has decidable equality then every type is a set and hence global choice negates univalence.

So global choice is both a constructive and a homotopy type theory taboo.
Exiting propositional truncations

**Theorem** (with Nicolai, Thierry and Thorsten):

A type $X$ has a choice function $\|X\| \to X$ iff it has a constant endomap $X \to X$.

**Question:**

*Can we eliminate $\|X\| \to A$ using a constant map $X \to A$?*

**Two answers:** Yes (Nicolai Kraus) and no (Mike Shulman).

Nicolai considers coherently constant functions.

Mike considers arbitrary constant functions.
Constancy

There are many notions of constancy.

We investigate the following:

1. A function \( f : X \rightarrow A \) is constant if any two of its values are equal.

\[
\text{constant } f \overset{\text{def}}{=} \prod_{x,y:X} fx = fy.
\]

2. This is data rather than property, unless \( A \) is a set.

Called a modulus of constancy of \( f \).

A function can have zero, one or more moduli of constancy.

3. E.g. the function \( f : 1 \rightarrow S^1 \) with definitional value base has \( \mathbb{Z} \)-many moduli of constancy \( \kappa_n : \text{constant } f \): 

\[
\kappa_n(x)(y) \overset{\text{def}}{=} \text{loop}^n.
\]
Set-valued constant functions

1. For any proposition $P$, by definition of truncation:

$$
\begin{array}{ccc}
X & \rightarrow & \|X\| \\
\downarrow f & & \downarrow f \\
\|X\| & \rightarrow & f \downarrow \downarrow P \\
\end{array}
$$

proposition (so $f$ constant)

2. Can replace $P$ by a set $A$:

$$
\begin{array}{ccc}
X & \rightarrow & \|X\| \\
\downarrow f & & \downarrow f \\
\|X\| & \rightarrow & \text{Im } f \\
\downarrow \downarrow \downarrow A \\
\end{array}
$$

constant by assumption
(a proposition because $A$ is a set and $f$ is constant

(but then in a unique way because $A$ is a set)
Propositional truncation as a set quotient

1. I.e. $\|X\|$ is the set-quotient of $X$ by the chaotic relation:

\[
\begin{array}{ccc}
X & \longrightarrow & \|X\| \\
\downarrow f & & \downarrow \bar{f} \\
A & \text{constant by assumption} & \text{set by assumption}
\end{array}
\]

2. Can we replace $A$ by an arbitrary type?

\[
\begin{array}{ccc}
X & \longrightarrow & \|X\| \\
\downarrow f & & \downarrow \bar{f} \\
A & \text{arbitrary} & ?
\end{array}
\]

No, not in general (Shulman, http://homotopytypetheory.org/2015/06/11/not-every-weakly-constant-function-is-conditionally-constant/)
When do we get a factorization of a constant function?

\[ X \xrightarrow{f} \|X\| \xrightarrow{\overline{f}} A \text{ arbitrary} \]

The factorization is possible if any of the following conditions holds:

1. \( X \) is empty.
2. \( X \) has a given point.
3. We have a function \( \|X\| \to X \).
4. We have a function \( A \to X \).
5. \( A \) is a set.

What other sufficient conditions?

And what about necessary conditions?

Also, given any factorization, we can construct another one for which the triangle commutes judgementally.
How to construct a counter example

cannot have a known point or be empty

X → ≅ X →

want: no f possible

constant by assumption

A cannot be a set
Natural attempt to get a counter-example

Let $s : S^1$ be an arbitrary point of the circle.
Let $A$ be an arbitrary type.
Let $f : s = \text{base} \to A$ be constant.

We can’t know a point of the path space $s = \text{base}$ in general.
But we know it is inhabited, that is, $\|s = \text{base}\|$
Hence $\|s = \text{base}\| = 1$ by propositional univalence/extensionality.
Attempt to get a counter-example

Can we expect to be able to get a point of an arbitrary type $A$, from any given constant function $f : s = \text{base} \rightarrow A$, even though we can’t expect to get a point of $s = \text{base}$ in general?

To our surprise, we can.

The attempt fails.
For any $s : S^1$ and any constant function $f : s = \text{base} \to A$ into an arbitrary type, we can find $a : A$ such that $fp = a$ for all $p : s = \text{base}$.
Proof outline

1. First show that for any given family of constant functions

\[ f : \prod_{s:S^1} s = \text{base} \to A(s), \]

each of them factors through 1. We get \( \bar{f} : \prod_{s:S^1} A(s) \).

This allows us to use induction on the circle and on paths.

2. For any type \( X \), consider the universal constant map on \( X \),
\( \beta_X : X \to S(X) \), constructed as a HIT.

3. By (1) applied to the family \( \beta_s : s = \text{base} \to S(s = \text{base}) \) given by (2), we get a function \( \bar{\beta} : \prod_{s:S^1} S(s = \text{base}) \).

4. Now, given a single constant function \( f : s = \text{base} \to A \), it factors through the universal constant map \( \beta_s : s = \text{base} \to S(s = \text{base}) \) as \( f' : S(s = \text{base}) \to A \) by (2), and hence we get the required point of \( A \) as using (3), as \( f'(\bar{\beta}(s)) \).
Step 1

For any \( f : \prod_{s:S^1} s = \text{base} \to A(s) \), with \( f \) \text{ base} constant, there is \( \bar{f} : \prod_{s:S^1} A(s) \) such that \( f \; s \; p = \bar{f} \; s \) for all \( p : s = \text{base} \).

1. **Lemma** Any transport of a value of \( f \) is a value of \( f \):

\[
\prod_{b,b':S^1} \prod_{r:b=b} \prod_{l:b=b'} \sum_{q:b'=b} \text{transport } l \left( f \; b \; r \right) = f \; b' \; q.
\]

This doesn’t depend on the fact that \( S^1 \) is the circle or on the constancy of \( f \) \text{ base}, and has a direct proof by based path induction.

2. We are interested in this particular case:

\[
\sum_{q:base=base} \text{transport loop } \left( f \; \text{base} \left( \text{refl base} \right) \right) = f \; \text{base} \; q.
\]

3. Then the constancy of \( f \) \text{ base} gives

\[
\text{transport loop } \left( f \; \text{base} \left( \text{refl base} \right) \right) = f \; \text{base} \left( \text{refl base} \right),
\]

which makes \( S^1 \)-induction work.
Step 2

For any type $X$, consider the universal constant map on $X$,

$$\beta : X \to S(X),$$
defined as a HIT with higher constructor

$$\ell : \prod_{x,y:X} \beta x = \beta y.$$ 

With modulus $k:
\prod_{x,y} fx = fy \to A$

such that $\text{ap} \bar{f}(\ell xy) = kxy$

When $X$ is the terminal type $1$, we get the circle $S^1$. 
Universal property of the constancy HIT

\[ \beta : X \to S(X), \]
\[ \ell : \prod_{x,y:X} \beta x = \beta y. \]

There is an equivalence

\[ SX \to A \cong \sum_{f:X \to A} \text{constant } f \]
\[ g \mapsto (g \circ \beta, \lambda xy. \text{ap } g (\ell xy)). \]

This generalizes the universal property of the circle

\[ S^1 \to A \cong \sum_{a:A} a = a \]
\[ \cong \sum_{f:1 \to A} \text{constant } f. \]
Side remark

(Not used in the proof, at least not explicitly.)

1. The universal constant map $\beta_X : X \to S(X)$ is a surjection.

2. The type $S(X)$ is conditionally connected, meaning that

$$\prod_{s,t:S(X)} \| s = t \|.$$ 

(“Conditionally” because it is empty if (and only if) $X$ is empty.)
Demonstrate and discuss some fragments of the \texttt{geometryOfConstancy.ctt} file (on my papers web page).
The constant factorization problem

Because the universal map $X \to \|X\|$ into a proposition is constant (in a unique way), the universal property of $S(X)$ gives a function

$$\prod_{X:U} S(X) \to \|X\|.$$  

The existence of a function in the other direction,

$$\prod_{X:U} \|X\| \to S(X),$$

is equivalent to the statement that all constant functions $f : X \to A$ factor through $X \to \|X\|$. 

But we know that this is not the case, by Shulman’s construction.

However, this does hold for $X = (s = \text{base})$ and all $A$. 
Step 3

By (1) applied to the family $\beta_s : s = \text{base} \rightarrow S(s = \text{base})$ of constant functions given by (2), we get a function

$$\bar{\beta} : \prod_{s:S^1} S(s = \text{base}).$$

This is perhaps surprising, because we don’t have, of course,

$$\prod_{s:S^1} s = \text{base},$$

as that would mean that that the circle is contractible.

How come we are able to pick a point of the generalized circle $S(s = \text{base})$, without being able to pick a point of the path space $s = \text{base}$, naturally in $s : S^1$?
Step 4

Now, given a single constant function $f : s = \text{base} \rightarrow A$, it factors through the universal constant map $\beta_s : s = \text{base} \rightarrow S(s = \text{base})$ as $f' : S(s = \text{base}) \rightarrow A$ by (2), and hence we get the required point of $A$ using (3), as

$$a \overset{\text{def}}{=} f'(\bar{\beta}(s)).$$

Theorem
Conjecture

In a type theory with $\| - \|$ and (hence) function extensionality.

All constant functions $f : X \to A$ of any two types factor through $X \to \|X\|$ if and only if all types are sets (zero-truncated).

And hence univalence fails if all constant functions factor through the truncations of their domains.

(Shulman’s construction exhibits a family of constant functions such that if all of them factor through the truncation of their domain, then univalence fails.)