Topologies on spaces of continuous functions*

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Abstract

It is well-known that a Hausdorff space is exponentiable if and only if it is locally compact, and that in this case the exponential topology is the compact-open topology. It is less well-known that among arbitrary topological spaces, the exponentiable spaces are precisely the core-compact spaces. The available approaches to the general characterization are based on either category theory or continuous-lattice theory, or even both. It is the main purpose of this paper to provide a self-contained, elementary and brief development of general function spaces. The only prerequisite to this development is a basic knowledge of general topology (continuous functions, product topology and compactness).

But another connection with the theory of continuous lattices lurks in this approach to function spaces, which is examined after the elementary exposition is completed. Continuity of the function-evaluation map is shown to coincide with a certain approximation property of a topology on the frame of open sets of the exponent space, and the existence of a smallest approximating topology is equivalent to exponentiability of the space. We show that the intersection of the approximating topologies of any preframe is the Scott topology. In particular, we conclude that a complete lattice is continuous if and only if it has a smallest approximating topology and finite meets distribute over directed joins.

1 Introduction

A topological space X is called *exponentiable* if for every space Y there is a topology on the set Y^X of continuous maps $X \to Y$ such that for any space A there is a natural bijection from the set of continuous maps $A \times X \to Y$ to the set of continuous maps $A \to Y^X$. This is elaborated in Section 2. It is known that a space is exponentiable if and only if it is *core-compact*, in the sense that any

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given open neighbourhood V of a point x contains an open neighbourhood U of x with the property that every open cover of V has a finite subcover of U. Moreover, if X is an exponentiable space and Y is any space, then the open sets of the function space Y^X are generated by the sets

$$\{f \in Y^X \mid \text{every open cover of } f^{-1}(V) \text{ has a finite subcover of } U\},$$

where U and V range over open sets of X and Y respectively.

In applications of function spaces Y^X to analysis, the spaces X and Y are usually Hausdorff, and this is the level of generality often considered in books on topology [3, 5, 14]. In this case, the exponential topology coincides with the more familiar compact-open topology. Moreover, for Hausdorff spaces, core-compactness is the same as local compactness.

However, in applications of topology to algebra via Stone duality [11] and to the theory of computation [1, 17, 19, 20, 21], non-Hausdorff spaces arise frequently. In the case of the theory of computation, where function spaces play a fundamental rôle, the exponential topologies have alternative descriptions [15].

Even if one is interested only in Hausdorff spaces, the non-Hausdorff ones may occasionally play an instrumental rôle in their study. For instance, spaces of continuous functions with values on a non-Hausdorff space have recently been used to obtain Hausdorff compactifications of spaces of continuous functions with values on a completely regular space [6].

The above characterization of the exponentiable spaces and the exponential topology has a long history, which is discussed in detail by Isbell [10] and goes back to at least 1945 with the work of Fox [7]. The first general solution is implicit in the work of Day and Kelly [2], who characterized the spaces X for which the function $q \times \mathrm{id}_X \colon Y \times X \to Z \times X$ is a quotient map for every quotient map $q\colon Y \to Z$. By virtue of the Adjoint Functor Theorem [16], such spaces coincide with the exponentiable spaces — see Isbell [10] for details. Day and Kelly's characterization amounts to the fact that the open sets of X form a continuous lattice in the sense of Scott [20] — but continuous lattices were introduced independently of the work of Day and Kelly. The above formulation of the characterization has been promoted by Isbell [10].

An alternative proof of the characterization is the following. For core-compact spaces, one shows directly that a certain topology known as the Isbell topology yields an exponential [10, 9]. Conversely, as observed by Johnstone and Joyal [12], if X is an exponentiable space and L is an injective space, then the exponential L^X is also an injective space; by considering the case in which L is the Sierpinski space, one sees that the open sets of X have to form a continuous lattice, because the injective spaces are characterized as the continuous lattices under the Scott topology [20].

It is the main purpose of this paper to provide a self-contained, elementary and brief development of the characterization of the exponentiable spaces as the core-compact spaces. In particular, we refrain from appealing to results from the theories of continuous lattices [8] and categories [16]. The only prerequisite to this development is a basic knowledge of general topology (continuous functions, product topology and compactness). Separation axioms are not needed. We hope that instructors and students of topology will find this development useful. Although there are one or two embellishments, the ingredients of the proofs are certainly not original.

When we wrote this elementary account in mid 1999, we were not aware of any such development in the literature. After we publicly advertised it, Fred Linton kindly let us know that Eilenberg had written a manuscript of the same kind [4], which apparently is expected to be published as part of collected works. The methods that we use are different, although naturally there are common ingredients. Eilenberg's method is to consider the largest topology on the set of continuous functions for which certain probe maps are continuous. In contrast, our method is to consider the smallest topology on the lattice of open sets of the exponent space which satisfies a certain approximation condition. In this sense, our method is closer to that used by Day and Kelly to characterize the quotient maps that are preserved by products [2].

But another connection with the theory of continuous lattices lurks in this approach to function spaces, which is examined after the elementary exposition is completed. Continuity of the function-evaluation map is shown to coincide with the approximation condition referred above, and existence of a smallest approximating topology is equivalent to exponentiability of the space. We show that the intersection of the approximating topologies of any preframe is the Scott topology. In particular, we conclude that a complete lattice is continuous if and only if it has a smallest approximating topology and finite meets distribute over directed joins. As far as we know, these results are new.

This paper is organized as follows. We formulate the exponentiability problem in Section 2. We then reduce it to a simpler problem in Section 3, which is solved in Section 4. To conclude the solution of the exponentiability problem in Section 5, we reformulate the solution obtained by the reduction process as the solution stated in the opening paragraph of this introduction. Up to this point, basic knowledge of general topology is the only prerequisite. We finish the paper by considering lattices more general than topologies in Section 6. Only in this last section, we assume some familiarity with the theory of continuous lattices. A small amount of repetition arises by postponing this material to the end of the paper. In a more logical development, this could be included at the place of Lemma 4.5, replacing it, but this would defeat our purpose of providing a self-contained development of function spaces, and, moreover, the result for the special case of the lattice of open sets of a topological space has a shorter proof. In order to also have a self-contained presentation of the continuous-lattice aspects, we have taken care of making the last section logically independent of the previous.

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Notation and terminology

The lattice of open sets of a topological space X is denoted by $\mathcal{O}X$. A topology T on a given set is *weaker* than another topology T' on the same set if $T \subseteq T'$. In this case we also say that the topology T' is *stronger* than the topology T.

2 Topologies on spaces of continuous functions

For topological spaces X and Y, we denote by

$$\mathrm{C}(X, Y)$$

the *set* of continuous maps from X to Y. The *transpose* $\overline{g} \colon A \to \mathrm{C}(X, Y)$ of a continuous map $g \colon A \times X \to Y$ is defined by

$$\overline{g}(a) = g_a$$

where $g_a \in \mathrm{C}(X, Y)$ is given by

$$g_a(x) = g(a, x).$$

More concisely, we write the definition of the transpose as $\overline{g}(a)(x) = g(a, x)$. A topology on the set $\mathrm{C}(X, Y)$ is called

1. *weak* if continuity of $g \colon A \times X \to Y$ implies that of $\overline{g} \colon A \to \mathrm{C}(X, Y)$,
2. *strong* if continuity of $\overline{g} \colon A \to \mathrm{C}(X, Y)$ implies that of $g \colon A \times X \to Y$,
3. *exponential* if it is both weak and strong.

Thus a topology on $\mathrm{C}(X, Y)$ is exponential if and only if it makes the transposition operation $g \mapsto \bar{g}$ into a well-defined bijection from the set $\mathrm{C}(A \times X, Y)$ to the set $\mathrm{C}(A, \mathrm{C}(X, Y))$. More standard terminologies for *weak* and *strong* are *splitting* and *conjoining* respectively. Our terminology is justified by Lemma 2.2 below.

Lemma 2.1 *A topology on* $\mathrm{C}(X, Y)$ *is strong if and only if it makes the* evaluation map

$$\begin{array}{rccc} \varepsilon_{X,Y} \colon & \mathrm{C}(X, Y) \times X & \to & Y \\ & (f, x) & \mapsto & f(x) \end{array}$$

into a continuous function.

Proof The transpose $\overline{\varepsilon} \colon \mathrm{C}(X, Y) \to \mathrm{C}(X, Y)$ of the evaluation map is continuous for any topology on $\mathrm{C}(X, Y)$ because $\overline{\varepsilon}(f)(x) = f(x)$ for all x and hence $\overline{\varepsilon}(f) = f$. This shows that evaluation is continuous if the topology on $\mathrm{C}(X, Y)$ is strong. Conversely, assume that evaluation is continuous for a given topology on $\mathrm{C}(X, Y)$ and let $g \colon A \times X \to Y$ be a map with a continuous transpose $\overline{g} \colon A \to \mathrm{C}(X, Y)$. Then g is also continuous because it is a composition $\varepsilon \circ (\overline{g} \times \mathrm{id}_X)$ of continuous maps as $g(a, x) = \overline{g}(a)(x) = \varepsilon(\overline{g}(a), x) = \varepsilon \circ (\overline{g} \times \mathrm{id}_X)(a, x)$, where $\mathrm{id}_X : X \to X$ is the identity. $\square$

Lemma 2.2 *1. Any weak topology is weaker than any strong topology.*

2. Any topology weaker than a weak topology is also weak.

3. Any topology stronger than a strong topology is also strong.

In particular, there is at most one exponential topology; when it exists, it is the weakest strong topology, or, equivalently, the strongest weak topology.

Proof Only (1) is not immediate. Endow $\mathrm{C}(X,Y)$ with a weak and a strong topology, obtaining spaces $\mathrm{W}(X,Y)$ and $\mathrm{S}(X,Y)$ respectively. By Lemma 2.1, the evaluation map $\varepsilon\colon\ \mathrm{S}(X,Y)\times X \to Y$ is continuous, and, by definition of weak topology, its transpose $\overline{\varepsilon}\colon\ \mathrm{S}(X,Y) \to \mathrm{W}(X,Y)$ is continuous. But we have seen that $\overline{\varepsilon}(f) = f$. Therefore $O = \overline{\varepsilon}^{-1}(O) \in \mathcal{O}\,\mathrm{S}(X,Y)$ for every $O \in \mathcal{O}\,\mathrm{W}(X,Y)$. $\square$

A space X is called *exponentiable* if the set $\mathrm{C}(X,Y)$ admits an exponential topology for every space Y. In this case, the set $\mathrm{C}(X,Y)$ endowed with the exponential topology is usually denoted by

$$Y^X$$

and referred to as an *exponential*. The problem tackled in Sections 3–5 is to develop a criterion for exponentiability and an explicit construction of exponential topologies.

3 Topologies on lattices of open sets

In this section we reduce the exponentiability problem to a simpler problem, which is solved in the next. It turns out that there is a *single* space $\mathbb{S}$ with the property that X is exponentiable if and only if $\mathrm{C}(X,\mathbb{S})$ has an exponential topology. Moreover, in this case, the exponential topology of $\mathrm{C}(X,Y)$ is uniquely determined by the exponential topology of $\mathrm{C}(X,\mathbb{S})$ and by the topology of Y in a simple fashion.

The *Sierpinski space* is the space $\mathbb{S}$ with two points 1 and 0 such that $\{1\}$ is open but $\{0\}$ is not. It is easy to see that the map $f \mapsto f^{-1}(1)$ is a bijection from $\mathrm{C}(X,\mathbb{S})$ to $\mathcal{O}\,X$. A topology on $\mathcal{O}\,X$ is *exponential* if it is induced by an exponential topology on $\mathrm{C}(X,\mathbb{S})$ via the bijection. Explicitly, this means that it is *strong* in the sense that the graph

$$\varepsilon_X = \{(U,x) \in \mathcal{O}\,X \times X \mid x \in U\}$$

of the membership relation is open, and *weak* in the sense that for each $W \in \mathcal{O}\,(A \times X)$, the function $\overline{w}\colon A \to \mathcal{O}\,X$ defined by

$$\overline{w}(a) = \{x \in X \mid (a,x) \in W\}$$

is continuous.

If T is a topology on $\mathcal{O}\,X$, the topology on $\mathrm{C}(X,Y)$ generated by the subbasic open sets

$$T(O,V) = \{f \in \mathrm{C}(X,Y) \mid f^{-1}(V) \in O\},$$

where O ranges over T and V ranges over $\mathcal{O}\,Y$, is referred to as the topology *induced* by T.

Lemma 3.1 *Let X be a topological space and T be a topology on $\mathcal{O}\,X$.*

1. *The topology T is weak if and only if it induces a weak topology on $\mathrm{C}(X,Y)$ for every space Y.*

2. *The topology T is strong if and only if induces a strong topology on $\mathrm{C}(X,Y)$ for every space Y.*

3. *The topology T is exponential if and only if it induces an exponential topology on $\mathrm{C}(X,Y)$ for every space Y.*

Proof Item (3) is an immediate consequence of (1) and (2), and the implications (1)($\Leftarrow$) and (2)($\Leftarrow$) follow from the fact that $\mathrm{C}(X,\mathbb{S})$ endowed with the topology induced by T is homeomorphic to $\mathcal{O}\,X$ endowed with T.

(1)($\Rightarrow$): To show that $\overline{g}\colon A \to \mathrm{C}(X,Y)$ is continuous for $\mathrm{C}(X,Y)$ endowed with the topology induced by T, it is enough to show that $\overline{g}^{-1}(T(O,V))$ is open for $O \in T$ and $V \in \mathcal{O}\,Y$. Let $W = g^{-1}(V)$. Since T is weak, $\overline{w}\colon A \to \mathcal{O}\,X$ is continuous for $\mathcal{O}\,X$ with T. Thus, in order to conclude the proof, it suffices to show that $\overline{g}^{-1}(T(O,V)) = \overline{w}^{-1}(O)$. This is equivalent to saying that $\overline{g}(a) \in T(O,V)$ if and only if $\overline{w}(a) \in O$. But we have that $\overline{g}(a) \in T(O,V)$ if and only if $\left(\overline{g}(a)\right)^{-1}(V) \in O$. Therefore, the chain of equivalences $x \in \left(\overline{g}(a)\right)^{-1}(V) \Leftrightarrow g(a,x) \in V \Leftrightarrow (a,x) \in g^{-1}(V) \Leftrightarrow (a,x) \in W \Leftrightarrow x \in \overline{w}(a)$ concludes the proof.

(2)($\Rightarrow$): In order to show that the evaluation map is continuous, let V be an open neighbourhood of $\varepsilon_{X,Y}(f,x) = f(x)$. Then $x \in f^{-1}(V)$, which shows that $(f^{-1}(V),x) \in \varepsilon_X$. Since ε_X is open in $\mathcal{O}\,X \times X$ for $\mathcal{O}\,X$ endowed with T, there are $O \in T$ and $U \in \mathcal{O}\,X$ such that $(f^{-1}(V),x) \in O \times U \subseteq \varepsilon_X$. Hence $(f,x) \in T(O,V) \times U$. But if $(g,u) \in T(O,V) \times U$ then $(g^{-1}(V),u) \in O \times U \subseteq \varepsilon_X$. Hence $u \in g^{-1}(V)$, i.e., $\varepsilon_{X,Y}(g,u) = g(u) \in V$, which shows that $\varepsilon_{X,Y}$ is continuous. □

We have thus obtained the promised reduction.

Corollary 3.2 *A space X is exponentiable if and only if $\mathcal{O}\,X$ has an exponential topology. In this case, the exponential topology of $\mathrm{C}(X,Y)$ is the topology induced by the exponential topology of $\mathcal{O}\,X$.*

4 Spaces with exponential topologies on the lattices of open sets

The discrete and indiscrete topologies of $\mathcal{O}\,X$ are strong and weak respectively. We begin by improving these bounds. A set $O \subseteq \mathcal{O}\,X$ is called *Alexandroff open* if the conditions $U \in O$ and $U \subseteq V \in \mathcal{O}\,X$ together imply that $V \in O$. It is immediate that the Alexandroff open sets form a topology.

Lemma 4.1 *The Alexandroff topology is strong.*

In particular, any open set in a weak topology is Alexandroff open.

Proof If $(U, x) \in \varepsilon_X$ then $(U, x) \in \{V \in \mathcal{O}X \mid U \subseteq V\} \times U$, which is a product of an Alexandroff open subset of $\mathcal{O}X$ with an open subset of X, and this product is clearly contained in ε_X. $\square$

An Alexandroff open set $O \subseteq \mathcal{O}X$ is called *Scott open* if every open cover of a member of O has a finite subcover of a member of O. For example, for any subset Q of X, the Alexandroff open set $\{V \in \mathcal{O}X \mid Q \subseteq V\}$ is Scott open if and only if Q is compact. Again, it is easy to check that the Scott open sets form a topology.

Lemma 4.2 *The Scott topology is weak.*

Proof Let $W \subseteq A \times X$ be open, let $a \in A$ and let $O \subseteq \mathcal{O}X$ be a Scott open neighbourhood of $\overline{w}(a)$. By openness of W in the product topology, for each $x \in \overline{w}(a)$ there are $U_x \in \mathcal{O}A$ and $V_x \in \mathcal{O}X$ with $(a, x) \in U_x \times V_x \subseteq W$. Since $\overline{w}(a)$ is the union of the sets V_x and since O is Scott open, the union V of finitely many such V_x belongs to O. Let U be the intersection of the corresponding open sets U_x. Clearly, U is a neighbourhood of a. To conclude the proof, we show that $\overline{w}(u) \in O$ for each $u \in U$. To this end, it is enough to show that $V \subseteq \overline{w}(u)$, because O is Alexandroff open and we know that $V \in O$. Let $v \in V$. Then $v \in V_x$ for some $x \in \overline{w}(a)$. Since $u \in U_x$, we have that $(u, v) \in U_x \times V_x \subseteq W$. Therefore $v \in \overline{w}(u)$. $\square$

Having improved the bounds, we now have a closer look at strong topologies. Let T be a topology on $\mathcal{O}X$. For opens $U, V \in \mathcal{O}X$, we write $U \prec_T V$ to mean that V belongs to the interior of the set $\{W \in \mathcal{O}X \mid U \subseteq W\}$ in the topology T. The following characterization, which is an immediate consequence of the definition of interior, is the criterion used in the proofs below.

Lemma 4.3 *The relation $U \prec_T V$ holds if and only if $V \in O$ for some $O \in T$ with $U \subseteq W$ for all $W \in O$.*

Notice that

1. $U \prec_T V$ implies $U \subseteq V$,
2. (a) $U' \subseteq U \prec_T V$ implies $U' \prec_T V$,
   (b) $U \prec_T V \subseteq V'$ implies $U \prec_T V'$, provided T is weaker than the Alexandroff topology,
3. $\emptyset \prec_T W$, and $U \prec_T W$ and $V \prec_T W$ together imply $U \cup V \prec_T W$.

Notice also that for a topology T weaker than the Alexandroff topology, the relation $U \prec_T U$ holds if and only if the set $\{V \in \mathcal{O}X \mid U \subseteq V\}$ is open. Hence, in this case, the relation $\prec_T$ is reflexive if and only if T *is* the Alexandroff topology, in which case $U \prec_T V$ if and only if $U \subseteq V$. Therefore the following generalizes the fact that the Alexandroff topology is strong.

Lemma 4.4 *A topology T on $\mathcal{O}X$ is strong if and only if it is* approximating, *in the sense that for every open neighbourhood V of a point x of X, there is an open neighbourhood $U \prec_T V$ of x.*

Notice that this is equivalent to saying that every open set V is the union of the opens $U \prec_T V$.

Proof Assume that T is strong and let V be an open neighbourhood of a point x of X. Since these two conditions mean that $\varepsilon_X \subseteq \mathcal{O}\,X \times X$ is open with respect to T and that $(V,x) \in \varepsilon_X$, there are $O \in T$ and $U \in \mathcal{O}\,X$ such that $(V,x) \in O \times U \subseteq \varepsilon_X$. Hence, if $(W,u) \in O \times U$ then $u \in W$. Therefore $U \subseteq W$ for every $W \in O$, which shows that $x \in U \prec_T V$. Conversely, assume that T is approximating and that $(V,x) \in \varepsilon_X$. Then $x \in V$ and there is $U \prec_T V$ with $x \in U$. Let $O \in T$ with $V \in O$ and $U \subseteq W$ for all $W \in O$. Then $(V,x) \in O \times U \subseteq \varepsilon_X$, which shows that ε_X is open and hence that T is strong. □

This is used in order to prove the following.

Lemma 4.5 *The Scott topology is the intersection of the strong topologies.*

Therefore it is the strongest weak topology.

Proof Being weak, it is contained in the intersection. Conversely, for each $\mathcal{C} \subseteq \mathcal{O}\,X$, let $T_\mathcal{C}$ be the set of all Alexandroff open subsets O of $\mathcal{O}\,X$ with the property that if $\mathcal{C}$ covers a member of O then $\mathcal{C}$ has a finite subcover of a member of O. This is easily seen to be a topology on $\mathcal{O}\,X$, and, by construction, the Scott topology is the intersection of all such topologies. To conclude the proof, it suffices to show that they are strong. Assume that $x \in U \in \mathcal{O}\,X$. If $x \notin \bigcup \mathcal{C}$, then $\{V \in \mathcal{O}\,X \mid U \subseteq V\} \in T_\mathcal{C}$, and so $x \in U \prec_{T_\mathcal{C}} U$. If, on the other hand, $x \in U'$ for some $U' \in \mathcal{C}$, then $\{V \in \mathcal{O}\,X \mid U' \cap U \subseteq V\} \in T_\mathcal{C}$, whence $x \in (U' \cap U) \prec_{T_\mathcal{C}} U$. Therefore $T_\mathcal{C}$ is strong by Lemma 4.4. □

We have thus obtained a characterization of the spaces with exponential topologies on their lattices of open sets.

Corollary 4.6 *A space X has an exponential topology on $\mathcal{O}\,X$ if and only if the Scott topology of $\mathcal{O}\,X$ is approximating, in which case the exponential topology is the Scott topology.*

The topology on $\mathrm{C}(X,Y)$ induced by the Scott topology of $\mathcal{O}\,X$ is known as the *Isbell topology*. Combining Corollaries 3.2 and 4.6, the following characterization of exponentiable spaces is obtained.

Theorem 4.7 *A space is exponentiable if and only if the Scott topology of its lattice of open sets is approximating. Moreover, the topology of an exponential is the Isbell topology.*

The definition of exponentiability quantifies over all topological spaces. The criterion provided by this theorem reduces exponentiability of a space X to an intrinsic property of X. A related criterion, which avoids considering a topology on the topology $\mathcal{O}\,X$ of X, is developed in the next section.

In the remainder of this section we digress slightly from our main goal. For X an exponentiable space, we regard $\mathcal{O}\,X$ as a topological space under the exponential topology. (Then $\mathcal{O}\,X$ is homeomorphic to the function space $\mathbb{S}^X$.)

By the above corollary, $\mathcal{O}\,\mathcal{O}\,X$ is the Scott topology of $\mathcal{O}\,X$. The following is an application of Theorem 4.7.

Proposition 4.8 *If X is an exponentiable space then so is $\mathcal{O}\,X$.*

Proof It is enough to show that the Scott topology of $\mathcal{O}\,\mathcal{O}\,X$ is approximating. Assume that $V \in \mathcal{V} \in \mathcal{O}\,\mathcal{O}\,X$. It suffices to conclude that $V \in \mathcal{U} \prec \mathcal{V}$ for some $\mathcal{U} \in \mathcal{O}\,\mathcal{O}\,X$. By exponentiability of X, we know that $\mathcal{O}\,\mathcal{O}\,X$ is an approximating topology on $\mathcal{O}\,X$. Hence, by Scott openness of $\mathcal{V}$, there is some $V' \prec V$ in $\mathcal{V}$. The open set V' induces a set $\mathbb{W} = \{\mathcal{W} \in \mathcal{O}\,\mathcal{O}\,X \mid V' \in \mathcal{W}\}$, which is easily seen to be Scott open. By definition of $V' \prec V$, there is some $\mathcal{U} \in \mathcal{O}\,\mathcal{O}\,X$ with $V \in \mathcal{U}$ and $V' \subseteq U$ for all $U \in \mathcal{U}$. Hence $U \in \mathcal{U}$ implies $U \in \mathcal{W}$ for any $\mathcal{W} \in \mathbb{W}$ because $\mathcal{W}$ is Alexandroff open. This shows that $\mathcal{U} \subseteq \mathcal{W}$ for all $\mathcal{W} \in \mathbb{W}$, and, because $\mathcal{V} \in \mathbb{W}$, we conclude that $\mathcal{U} \prec \mathcal{V}$. □

The following, which generalizes the fact that finite intersections of open sets are open, is an entertaining application of Corollary 4.6 and Proposition 4.8.

Proposition 4.9 *If X is an exponentiable space and $\mathcal{Q} \subseteq \mathcal{O}\,X$ is compact, then $\bigcap \mathcal{Q}$ is open.*

Proof Because $\mathcal{O}\,\mathcal{O}\,X$ is a strong topology on $\mathcal{O}\,X$ as X is exponentiable, the set $W = \{(x, U) \in X \times \mathcal{O}\,X \mid x \in U\}$ is open, and, because $\mathcal{O}\,\mathcal{O}\,\mathcal{O}\,X$ is a weak topology on $\mathcal{O}\,\mathcal{O}\,X$ as the space $\mathcal{O}\,X$ is also exponentiable, its transpose $\overline{w} : X \to \mathcal{O}\,\mathcal{O}\,X$ is continuous. It is clear that $\overline{w}(x) = \{U \in \mathcal{O}\,X \mid x \in U\}$. By compactness of $\mathcal{Q}$, the set $\mathbb{U} = \{\mathcal{U} \in \mathcal{O}\,\mathcal{O}\,X \mid \mathcal{Q} \subseteq \mathcal{U}\}$ is open, and hence so is $\overline{w}^{-1}(\mathbb{U})$. But $x \in \overline{w}^{-1}(\mathbb{U})$ if and only if $\overline{w}(x) \in \mathbb{U}$ if and only if $x \in U$ for all $U \in \mathcal{Q}$ if and only if $x \in \bigcap \mathcal{Q}$, which shows that $\bigcap \mathcal{Q} = \overline{w}^{-1}(\mathbb{U})$. □

Nachbin considers a similar result with different hypotheses [18]. Keimel and Gierz show that the pointwise meet of a compact set of extended real-valued lower semicontinuous functions defined on a locally compact space is itself lower semicontinuous [13]. As it is discussed in the next section, locally compact spaces are exponentiable. It follows that, for X locally compact, their result generalizes the above proposition, because a set is open if and only if its characteristic function is lower semicontinuous.

5 Core-compact spaces

Our next goal is to avoid explicit references to the Scott topology in the criterion for exponentiability given in Theorem 4.7. The tools used for that purpose are Lemmas 5.1 and 5.2 below, which are really pieces of continuous-lattice theory specialized to the lattice of open sets of a topological space. Readers who are familiar with continuous lattices can jump directly to Theorem 5.3.

For open sets U and V of a topological space X, one writes $U \ll V$, and says that U is *way below* V, to mean that every open cover of V has a finite subcover of U. For example, this is the case if there is a compact set $Q \subseteq X$ with $U \subseteq Q \subseteq V$. If X is locally compact, in the sense that any neighbourhood of a point contains a compact (not necessarily open) neighbourhood of the point, it is easy to show that the converse also holds. Notice that this relative notion of compactness enjoys the following properties.

1. $U \ll V$ implies $U \subseteq V$,
2. $U' \subseteq U \ll V \subseteq V'$ implies $U' \ll V'$,
3. $\emptyset \ll W$, and $U \ll W$ and $V \ll W$ together imply $U \cup V \ll W$.

Although the relations $U \prec_{\text{Scott}} V$ and $U \ll V$ don't coincide in general, they do for exponentiable spaces.

Lemma 5.1 *For open sets U and V of a space X,*

1. *the relation $U \prec_{\text{Scott}} V$ implies $U \ll V$,*
2. *if the Scott topology of $\mathcal{O}X$ is approximating then the relation $U \ll V$ implies $U \prec_{\text{Scott}} V$.*

Proof Assume that $U \prec_{\text{Scott}} V$. Then there is a Scott open neighbourhood O of V such that $U \subseteq W$ for all $W \in O$. By Scott openness, any open cover of V has a finite subcover of a member of O and hence of U. Therefore $U \ll V$. Conversely, assume that the Scott topology of $\mathcal{O}X$ is approximating and that $U \ll V$. Since V is the join of the opens $V' \prec_{\text{Scott}} V$, we have that $U \subseteq W$ where W is a union of finitely many $V' \prec_{\text{Scott}} V$. Since $W \prec_{\text{Scott}} V$, we conclude that $U \prec_{\text{Scott}} V$. □

A space X is called *core-compact* if every open neighbourhood V of a point x of X contains an open neighbourhood $U \ll V$ of x. Again, this is equivalent to saying that every open V is the union of the opens $U \ll V$. By the above observations, every locally compact space is core-compact.

Lemma 5.2 *Let X be a core-compact space.*

1. *If $U \ll W$ in $\mathcal{O}X$ then $U \ll V \ll W$ for some $V \in \mathcal{O}X$.*
2. *The set $\Uparrow U \stackrel{\text{def}}{=} \{V \in \mathcal{O}X \mid U \ll V\}$ is Scott open.*
3. *If $O \subseteq \mathcal{O}X$ is Scott open and $V \in O$ then $U \ll V$ for some $U \in O$.*
4. *The sets $\Uparrow U$ for $U \in \mathcal{O}X$ form a base of the Scott topology of $\mathcal{O}X$.*
5. *If $U \ll V$ then $U \prec_{\text{Scott}} V$.*

Proof (1): The open set W is the union of the open sets $V \ll W$, and, in turn, each open set $V \ll W$ is the union of the open sets $V' \ll V$. Hence W is the union of the collection $\mathcal{C}$ of open sets V' for which there exists an open set V with $V' \ll V \ll W$. Since $\mathcal{C}$ is closed under the formation of finite unions, we have that $U \subseteq V'$ for some $V' \in \mathcal{C}$. By definition of $\mathcal{C}$, there is an open V with $V' \ll V \ll W$ and hence with $U \ll V \ll W$.

(2): The set $\Uparrow U$ is clearly Alexandroff open. If $W \in \Uparrow U$ then $V \in \Uparrow U$ for some $V \ll W$ by (1), which shows that every open cover of a member of $\Uparrow U$ has a finite subcover of a member of $\Uparrow U$.

(3): The open set V is the union of the open sets $U \ll V$, and such open sets are closed under the formation of finite unions.

(4): This is an immediate consequence of (2) and (3).

(5): $\Uparrow U$ is a Scott open set with $V \in \Uparrow U$ and $U \subseteq W$ for all $W \in \Uparrow U$. □

We have thus obtained an alternative characterization of the exponentiable spaces and an alternative construction of the exponential topology which don't rely on the Scott topology.

Theorem 5.3 *A space is exponentiable if and only if it is core-compact. Moreover, if X is a core-compact space and Y is any space then the topology of the exponential Y^X is generated by the sets*

$$\{f \in Y^X \mid U \ll f^{-1}(V)\},$$

where U and V range over $\mathcal{O}X$ and $\mathcal{O}Y$ respectively.

Proof If X is exponentiable then $\mathcal{O}X$ has an exponential topology and hence it is core-compact by Theorem 4.7 and by Lemma 5.1. Conversely, if X is core-compact then $\mathcal{O}X$ has an exponential topology by Theorem 4.7 and by Lemma 5.2(5), and hence it is exponentiable. For the second part, it is easy to see that if T is a topology on $\mathcal{O}X$ with a base B then the topology on $\mathrm{C}(X, Y)$ induced by T has as a subbase the sets $T(O, V)$ for O in B. The result then follows from the fact that the sets $\Uparrow U$ for $U \in \mathcal{O}X$ form a base of the Scott topology of $\mathcal{O}X$ if X is core-compact, and from the fact that the Isbell topology is induced by the Scott topology. □

We finish this section by briefly considering exponentiable Hausdorff spaces. For open subsets U and V of a Hausdorff space, if $U \ll V$ then $U \subseteq Q \subseteq V$ for some compact set Q — see [8]. It follows that a Hausdorff space is core-compact if and only if it is locally compact. Moreover, if X is a locally compact Hausdorff space, then the exponential topology of Y^X is generated by the sets

$$\{f \in Y^X \mid f(Q) \subseteq V\},$$

where Q and V range over compact subsets of X and open subsets of Y respectively. That is, the exponential topology is the compact-open topology [3, 7, 14]. The reason is that in this case the Scott topology of $\mathcal{O}X$ has the sets $\{U \in \mathcal{O}X \mid Q \subseteq U\}$ as a base and that the condition $Q \subseteq f^{-1}(V)$ is equivalent to $f(Q) \subseteq V$.

6 Locating the Scott topology

The proof of Lemma 4.5 identifies the Scott topology of the lattice of open sets of a topological space as the intersection of the approximating topologies. It is the purpose of this last section to extend this from lattices of open sets to more general posets.

Scott defined a complete lattice to be *continuous* if its Scott topology is approximating [20]. Many extensions of this notion to posets more general than complete lattices have been considered in the literature. In all such extensions, directed sets play a major rôle. Recall that a subset Δ of a poset is called *directed* if it is non-empty and any two members of Δ have an upper bound in Δ. The results of this section apply to posets with binary meets and directed joins in which the former distribute over the latter. Such a poset is known as a *preframe*. (Strictly speaking, a preframe is required to have all finite meets, not just binary meets — but the meet of the empty set, a top element, plays no rôle in our development.) For example, the topology of any topological space is a preframe (and, in fact, a frame — a complete lattice in which finite meets distribute over arbitrary joins).

Let P be a poset. A subset U of P is called *Alexandroff open* if it is an upper set (that is, $u \in U$ and $u \leq x$ together imply $x \in U$) and it is called *Scott open* if it is Alexandroff open and every directed set with join in U actually intersects U. For example, the Scott topology of the natural order of the real line is the topology of lower semicontinuity (the topology for which the non-trivial open sets are the intervals (a, ∞)). The Scott relation of P is defined by $x \prec y$ if and only if y belongs to the interior of the principal filter

$$\uparrow x = \{u \in P \mid x \leq u\}$$

in the Scott topology. The poset P is called *continuous* if its Scott relation is *approximating*, in the sense that, for any $x \in P$, the set $\{u \in P \mid u \prec x\}$ of approximants of x is directed and has x as its join. This is the original definition given by Scott for the case in which P is a complete lattice. An alternative formulation that occurs more often in the literature involves a certain way-below relation $\ll$, which has already occurred in the above development for the lattice of open sets of a topological space — but this is not needed for the purposes of this section.

More generally, following the pattern of Section 4, for any topology T on a poset P, we write $x \prec_T y$ to mean that y belongs to the interior of $\uparrow x$ in the topology T, and in this case we say that x is an *approximant* of y with respect to T. This is equivalent to saying that there is a neighbourhood $U \in T$ of y such that $x \leq u$ for each $u \in U$. The following properties are easily verified.

1. $x \prec_T y$ implies $x \leq y$.
2. (a) $x' \leq x \prec_T y$ implies $x' \prec_T y$.
   (b) $x \prec_T y \leq y'$ implies $x \prec_T y'$ if T is weaker than the Alexandroff topology.
3. (a) If $\perp$ is a least element then $\perp \prec_T x$.
   (b) If u and v have a least upper bound then $u \prec_T x$ and $v \prec_T x$ together imply $u \vee v \prec_T x$.

In particular, if the poset has joins of upper bounded finite sets, then the set of approximants of any element is directed. But, in the absence of such a condition, there is no reason why this should be the case.

Example 6.1 Add a top element $\top$ and two minimal elements a and b to a discretely ordered infinite set. Then a and b are approximants of $\top$ with respect to the cofinite topology, but there is no approximant above a and b. □

We thus explicitly require directedness in the following definition. We say that a topology on a poset is *approximating* if every element of the poset is the directed join of its approximants. Notice that the stronger the topology, the more approximants any element has, and hence, in a poset with bounded finite joins, any topology stronger than an approximating topology is itself approximating.

The Alexandroff topology is always approximating, but in a rather uninteresting way, as the approximation relation coincides with the (reflexive) partial ordering. Here we consider *order-consistent* topologies, that this, those whose specialization ordering coincides with the ordering of the poset. Recall that, by definition, the relation $x \leq y$ holds in the *specialization order* of a topology T if and only if $x \in U \in T$ implies $y \in U$. It follows that the Alexandroff topology is the strongest order-consistent topology. In particular, the Alexandroff topology is the strongest approximating topology amongst those which are order-consistent. The weakest order-consistent topology is generated by the sets $\{u \mid u \not\leq x\}$, where x ranges over arbitrary elements of the poset. Since such sets are Scott open, it follows that the Scott topology is between this and the Alexandroff topologies. In particular, we conclude that any topology between the Scott and the Alexandroff topologies is order-consistent.

The proof of the following generalizes that of Lemma 4.5. There are two main differences. The first is that the absolute notion of weakness is not available in this general setting, and hence a different argument is needed to show that any approximating topology contains the Scott topology. The second is that, by virtue of the above observations, sets of approximants have to be explicitly shown to be directed.

Theorem 6.2

1. *The Scott topology of any poset is weaker than any approximating topology.*
2. *The Scott topology of a preframe is the intersection of the order-consistent approximating topologies.*

(It follows that the Scott topology of a preframe is the intersection of *all* approximating topologies, but it is the stronger result involving fewer topologies that is relevant to the theory of continuously ordered sets.)

Proof (1): Let T be an approximating topology and x be a member of a Scott open set U. By definition of approximation, the set of approximants of x is directed and hence, by definition of Scott openness, there is some $u \prec_T x$ in U. By definition of $\prec_T$, there is a neighbourhood $V \in T$ of x with $u \leq v$ for all $v \in V$. Since U, being Scott open, is Alexandroff open, we conclude that V is contained in U. And since x is arbitrary, we conclude that U is a union of members of T and hence that it is itself a member of T.

(2): By (1), it suffices to construct a family of order-consistent approximating topologies that has the Scott topology as its intersection. For each directed lower set I let T_I be the collection of all Alexandroff open sets U, except those such that the join of I belongs to U but I doesn't intersect U. Such collections T_I are easily seen to form topologies, and, by construction, their intersection is the Scott topology. Since they are between the Alexandorff and the Scott topologies, they are order-consistent. To see that T_I is approximating, we consider two cases for an arbitrary element x. (i) $x \not\leq \bigvee I$: Then $\uparrow x \in T_I$ and hence $x \prec_{T_I} x$, which shows that x is trivially the directed join of its approximants. (ii) $x \leq \bigvee I$: For $i \in I$ we have that $\uparrow i \in T_I$ and hence, for any u, the relation $i \prec_{T_I} u$ holds if and only if $i \leq u$. In particular, $x \wedge i \prec_{T_I} x$ for any $i \in I$ because $x \wedge i \in I$ as I is a lower set. Conversely, if $v \prec_{T_I} x$, then x belongs to the interior of $\uparrow v$, which, being Alexandroff open by definition of T_I, has to then have $\bigvee I$ as member because $x \leq \bigvee I$, and hence also some $i \in I$ as a member, again by definition of T_I. Thus v, being below a member of I, is itself in I. Since $v = x \wedge v$, we conclude that the approximants of x are those elements of the form $x \wedge i$ for $i \in I$. In particular, they form a directed set because I is directed. By the preframe distributivity law and the fact that $x \leq \bigvee I$, we conclude that $\bigvee\{x \wedge i \mid i \in I\} = x \wedge \bigvee I = x$. □

It is well-known that continuity of a poset with finite meets and directed joins implies the preframe distributivity law. Hence the following is an immediate consequence of the theorem.

Corollary 6.3 *A poset with binary meets and directed joins is continuous if and only if it satisfies the preframe distributivity law and has a smallest approximating topology.*

In the *Compendium of continuous lattices* [8, pages 43–45], it is shown that the way-below relation of a preframe is the intersection of the approximating auxiliary relations. However, this and the theorem just proved are not corollaries of each other. In fact, the translation from topologies to auxiliary relations doesn't preserve or reflect intersections as, in general, neither the Scott topology induces the way-below relation nor is the Scott topology recoverable from knowledge of the way-below relation alone.

An assumption such as distributivity in the above theorem and corollary cannot be removed, as the following example shows.

Example 6.4 Consider two disjoint copies $\mathbb{N}$ and $\mathbb{N}'$ of the set of natural numbers under their natural order, with elements denoted by $0, 1, 2, \ldots$ and $0', 1', 2', \ldots$ respectively. To make this into a complete lattice, add bottom and top elements $\bot$ and $\top$. Let T be an order-consistent approximating topology. For all $x \neq \top$, the condition $x \prec_T x$ holds, and thus $\uparrow x$ is open because T, being order-consistent, is weaker than the Alexandroff topology. Then $\{\top\} = \uparrow 0 \cap \uparrow 0'$ is open as well, and so all upper sets are open, and hence T is the Alexandroff topology. Therefore the Alexandroff topology is the only order-consistent approximating topology. But $\{\top\}$ is not Scott open, and hence the Scott topology is strictly weaker. It follows from the theorem that the preframe distributivity law cannot hold. An explicit instance of the failure is $\bigvee\{0' \wedge n \mid n \in \mathbb{N}\} = \bot \neq 0' = 0' \wedge \bigvee \mathbb{N}$. □

We finish by considering an alternative formulation of approximation, first considered by Scott for the Scott topology of a complete lattice. For this, we need to assume joins of directed sets, as before, and meets of all non-empty sets, not just the finite ones. Since such posets automatically have joins of finite upper bounded sets, we don't need to worry about checking directedness of sets of approximants.

Lemma 6.5 *A topology T on a poset with non-empty meets and directed joins is approximating if and only if every element x of the poset is the* limit inferior *of its neighbourhoods, in the sense that*

$$x = \bigvee \{ \bigwedge U \mid x \in U \in T \}.$$

Proof It is clear that the meets are of non-empty sets and that the join is of a directed set. Assume that T is approximating. It is enough to conclude that $x \leq \bigvee \{ \bigwedge U \mid x \in U \in T \}$, because the other inequality always holds. To establish the inequality it is enough to show that $y \leq \bigvee \{ \bigwedge U \mid x \in U \in T \}$ for all $y \prec_T x$, because x is the join of its approximants. By definition of $y \prec_T x$, there is a $U \in T$ with $x \in U$ and $y \leq u$ for $u \in U$. Hence $y \leq \bigwedge U$ by definition of meet, and we are done. Conversely, $x \in U$ implies $\bigwedge U \prec_T x$ because $\bigwedge U \leq u$ for all $u \in U$ by definition of meet, and, if the equation holds, x is join of some, and hence all, of its approximants. □

This gives rise to a characterization of continuity that doesn't refer to the Scott topology or the way-below relation.

Theorem 6.6 *A poset with non-empty meets and directed joins is continuous if and only if it satisfies the preframe distributivity law and has a smallest topology for which every element is the limit inferior of its neighbourhoods.*

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