# Universes in sheaf models

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We extend Coquand's construction of universe in presheaf models [1] to the one in sheaf models. When attempting to verify the sheaf condition for this "universe", we face two problems which are discussed in the end of the note.

Recall that a *coverage* $\mathcal{J}$ on a category $\mathbf{C}$ assigns to each object $X \in \mathbf{C}$ a collection $\mathcal{J}(X)$ of families of morphisms $\{\varphi_i : X_i \to X\}_{i \in I}$, called *covering families*, such that for any covering family $\{\varphi_i : X_i \to X\}_{i \in I} \in \mathcal{J}(X)$ and any morphism $f : Y \to X$, there exists a covering family $\{\psi_j : Y_j \to Y\}_{j \in J} \in \mathcal{J}(Y)$ such that each composite $f \circ \psi_j$ factors through some $\varphi_i$.

If $\Gamma$ is a *sheaf* on a site $(\mathbf{C}, \mathcal{J})$, then for each object $X \in \mathbf{C}$ we have a set $\Gamma(X)$ and a map

$$\_ \cdot f : \Gamma(X) \to \Gamma(Y)$$

for each $f : Y \to X$ such that

(s1) $\rho \cdot 1_X = \rho$ for all $\rho : \Gamma(X)$.

(s2) $(\rho \cdot f) \cdot g = \rho \cdot (f \circ g)$ for all $\rho : \Gamma(X)$, $f : Y \to X$ and $g : Z \to Y$.

(s3) For any $\{\varphi_i : X_i \to X\}_{i \in I} \in \mathcal{J}(X)$ and any compatible family of elements $\{\rho_i : \Gamma(X_i)\}_{i \in I}$, there is a unique element $\rho : \Gamma(X)$ such that $\rho \cdot \varphi_i = \rho_i$ for each $i \in I$.

In this note, we write $\Gamma, \Delta$ to denote sheaves, $X, Y, Z$ to denote $\mathbf{C}$-objects, $f, g, h$ to denote $\mathbf{C}$-morphisms, and $\varphi, \psi, \phi$ to denote maps in covering families in $\mathcal{J}$.

Now we attempt to construct the (first) universe in the sheaf model. For $X : \mathbf{C}$, we define $\mathrm{U}(X)$ as a collection of families of sets. Specifically, an element $A : \mathrm{U}(X)$ is a family of sets indexed by the $\mathbf{C}$-morphisms into $X$, together with a map

$$\_ \bullet g : A_f \to A_{(f \circ g)}$$

for each $f : Y \to X$ and $g : Z \to Y$, satisfying the following conditions:

(u1) $a \bullet 1_Y = a$, for all $f : Y \to X$ and $a : A_f$.

(u2) $(a \bullet g) \bullet h = a \bullet (g \circ h)$, for all $f : Y \to X$, $a : A_f$, $g : Z \to Y$ and $h : W \to Z$.

(u3) For any morphism $f : Y \to X$, any covering family $\{\varphi_i : Y_i \to Y\}_{i \in I}$, and any compatible family $\{a_i : A_{(f \circ \varphi_i)}\}_{i \in I}$, there is a unique $a : A_f$ such that $a \bullet \varphi_i = a_i$ for all $i \in I$.

Then (the underlying family of) the universe $\Gamma \vdash \mathrm{U}$ is defined by $\mathrm{U}_\rho :\equiv \mathrm{U}(X)$ for all $\rho : \Gamma(X)$. The restriction map of U is defined by, for $A : \mathrm{U}(X)$, $f : Y \to X$ and $g : Z \to Y$,

$$(A \cdot f)_g \;:\equiv\; A_{(f \circ g)}.$$

To be a type U needs to satisfy the sheaf condition. However, we only manage to prove the following:

**Proposition 1.** *For any covering family $\{\varphi_i : X_i \to X\}_{i \in I}$, for any compatible family of elements $\{A^i : \mathrm{U}(X_i)\}_{i \in I}$, there is an element $A : \mathrm{U}(X)$ such that $A \cdot \varphi_i = A^i$ for all $i \in I$, which is unique up to (pointwise) isomorphism.*

We prove this by additionally requiring $\{1_Y : Y \to Y\} \in \mathcal{J}(Y)$ of the coverage $\mathcal{J}$. In the following proof, we explicitly point out where this additional property is needed.

*Proof.* Given a covering family $\{\varphi_i\colon X_i \to X\}_{i\in I}$ and a compatible family $\{A^i\colon \mathrm{U}(X_i)\}_{i\in I}$, we define $A\colon \mathrm{U}(X)$ as follows: Given $f\colon Y \to X$, by the coverage axiom, we can find a covering family $\{\psi_j\colon Y_j \to Y\}_{j\in J}$ such that

$$\forall j{\in}J.\ \exists i_j{\in}I.\ \exists g_j\colon Y_j{\to}X_{i_j}.\ f\circ\psi_j = \varphi_{i_j}\circ g_j. \tag{†}$$

We define

$$A_f \ :\equiv\ \prod_{j\in J} A^{i_j}_{g_j}.$$

Notice that the above definition uses the axiom of choice, and that different choices of $i$ and $g$ in (†) give different results. However, if $\{1_Y\colon Y\to Y\}\in\mathcal{J}(Y)$ then all the resulting products are isomorphic. The proof is essentially the same as the one below of the uniqueness of $A$.

For $w\colon A_f$ and $g\colon Z\to Y$, we define $w\bullet g\colon A_{(f\circ g)}$ as follows: Using the coverage axiom for $\{\varphi_i\}$ and $f$, we get a covering family $\{\psi_j\colon Y_j\to Y\}_{j\in J}$ satisfying (†). Using the coverage axiom again for $\{\psi_j\}$ and $g$, we get another covering family $\{\phi_k\colon Z_k\to Z\}_{k\in K}$ such that

$$\forall k{\in}K.\ \exists j_k{\in}J.\ \exists h_k\colon Z_k{\to}Y_{j_k}.\ g\circ\phi_k = \psi_{j_k}\circ h_k. \tag{‡}$$

If we combine (†) and (‡), then we have

$$\forall k{\in}K.\exists j_k{\in}J.\exists h_k{:}Z_k{\to}Y_{j_k}.\exists i_k{\in}I.\exists g_k{:}Y_{j_k}{\to}X_{i_k}.\ (f\circ g)\circ\phi_k = \varphi_{i_k}\circ(g_k\circ h_k).$$

Thus, for $k\in K$, we define

$$(w\bullet g)(k)\ :\equiv\ w(i_k)\bullet h_k : A^{i_k}_{(g_k\circ h_k)}.$$

Notice that $A_{f\circ g}$ (obtained using the coverage axiom for $\{\varphi_i\}$ and $f\circ g$) and $\prod_{k\in K} A^{i_k}_{(g_k\circ h_k)}$ may not be the same, but they are isomorphic as discussed above. Hence we may need to apply the isomorphism to make $w\bullet g$ well-defined.

We skip the (complicated) proof that $A$ satisfies (u1) to (u3).

Then we prove $A\cdot\varphi_i = A^i$ for each $i\in I$: Given $f\colon Y\to X_i$, we want to show $A_{(\varphi_i\circ f)} = A^i_f$. If we have $\{1_Y\}\in\mathcal{J}(Y)$, then this singleton family satisfies the equation given by the coverage axiom for $\{\varphi_i\}$ and $\varphi_i\circ f$, *i.e.* $(\varphi_i\circ f)\circ \mathrm{id}_Y = \varphi_i\circ f$. Hence we have $(A\cdot\varphi_i)_f = A_{(\varphi_i\circ f)} = A^i_f$ according to the construction of $A$.

The above $A$ is unique up to isomorphism: Suppose that $B\colon \mathrm{U}(X)$ satisfies $B\cdot\varphi_i = A^i$ for all $i\in I$. We want to show that $A_f$ and $B_f$ are isomorphic for all $f\colon Y\to X$. Let $\{\psi_j\colon Y_j\to Y\}_{j\in J}$ be a covering family obtained using the coverage axiom satisfying (†).

(⇒) Given $w\colon A_f$, we have $w(j)\colon A^{i_j}_{g_j}$ for each $j\in J$. Because

$$\begin{aligned} A^{i_j}_{g_j} &= (B\cdot\varphi_{i_j})_{g_j} && \text{(by the assumption } B\cdot\varphi_i = A^i\text{)}\\ &= B_{(\varphi_{i_j}\circ g_j)} && \text{(by the definition of restriction maps of U)}\\ &= B_{(f\circ\psi_j)} && \text{(by (†))}\end{aligned}$$

we have a family of elements $\{w(j)\colon B_{(f\circ\psi_j)}\}_{j\in J}$. Using condition (u3) of $B$, we get a unique element of $B_f$.

(⇐) Given $b\colon B_f$, we have $b\bullet\psi_j\colon B_{(f\cdot\psi_j)}$. We have shown $A^{i_j}_{g_j} = B_{(f\circ\psi_j)}$, thus, the element $b\bullet\psi_j$ is in $A^{i_j}_{g_j}$. Then the map $\lambda j.(b\bullet\psi_j)$ is in $A$.

The composite of the above operations are identity due to the uniqueness property in condition (u3) of $B$. □

In summary, we have two problems when trying to prove the sheaf condition for U: (1) Our construction of amalgamation additionally requires the additional property $\{1_Y\}\in\mathcal{J}(Y)$ of the coverage. (2) Amalgamations (if exist) are unique only up to isomorphism.

# References


[1] T. Coquand. Sheaf model of type theory. Unpulished note, 2013.