Spatiality of the patch frame

Martín H. Escardó

School of Mathematical and Computational Sciences, St Andrews University,
North Haugh, St Andrews, Fife KY16 9SS, Scotland
`mhe@dcs.st-and.ac.uk` `http://www.dcs.st-and.ac.uk/ mhe/`

Abstract

The Scott continuous nuclei form a subframe of the frame of all nuclei. This subframe is referred to as the *patch frame*. In this short note we show that the patch of a topology is isomorphic to a finer topology on the same set. We also give a point-set construction of the patch topology of a sober space.

Keywords: Frame of nuclei, Scott continuous nucleus, patch topology, perfect map.

Mathematics Subject Classification: 06E15, 54A10.

The Scott continuous nuclei form a subframe of the frame of all nuclei. This subframe is referred to as the *patch frame*. It is shown in [2] that the patch construction gives rise to the right adjoint to the (full) inclusion of the category of continuous maps of compact regular locales into the category of perfect maps of stably compact locales. This implies that the patch frame of the frame of open sets of a stably compact space is isomorphic to the patch topology. In particular, the frame of Scott continuous nuclei on the Scott topology of a continuous lattice is isomorphic to the Lawson topology.

In this short note we discuss the general topological manifestation of this localic patch construction. We assume the terminology and notation of [2].

We begin by showing that locales with enough points are closed under the patch construction. It is enough to show that if X is a locale with enough (Scott) open filters, so is $\operatorname{Patch} X$. Assume that $j \not\leq k$ holds in $\mathcal{O} \operatorname{Patch} X$. Then $j(U) \not\leq k(U)$ for some $U \in \mathcal{O} X$. Since X has enough open filters, there is an open filter $\phi \subseteq \mathcal{O} X$ with $j(U) \in \phi$ but $k(U) \notin \phi$. Let Φ be the set $\{l \in \mathcal{O} \operatorname{Patch} X \mid l(U) \in \phi\}$. Then, by construction, $j \in \Phi$ but $k \notin \Phi$. To conclude the proof, it suffices to show that Φ is an open filter. To see that Φ is a filter, assume that $l \in \Phi$ and that $l \leq l'$. Then $l(U) \in \phi$ by definition of Φ, and $l'(U) \in \phi$ because ϕ is a filter and $l(U) \leq l'(U)$. Therefore $l' \in \Phi$. Now assume that $l, l' \in \Phi$. Then, by definition, $l(U), l'(U) \in \phi$, and, because ϕ is a filter

18 May 2000

and because meets of nuclei are computed pointwise, $(l \wedge l')(U) \in \phi$. Therefore, by definition of Φ, we have that $l \wedge l' \in \Phi$. To see that Φ is open, assume that $\mathcal{L} \subseteq \mathcal{O}\,\mathrm{Patch}\,X$ is a directed set with $\bigvee \mathcal{L} \in \Phi$. Then, because directed joins of Scott continuous nuclei are computed pointwise, $\bigvee\{l(U) \mid l \in \mathcal{L}\} \in \phi$. Since this set is directed and ϕ is open, $l(U) \in \phi$ for some $l \in \mathcal{L}$. Therefore $l \in \Phi$ for some $l \in \mathcal{L}$, and the proof is concluded.

In [1, Theorem 3.1.4] it is shown that a perfectly embedded sublocale of a spatial locale is again spatial. Therefore, the frame of nuclei on the topology of a sober space is dually isomorphic to the collection of perfectly embedded sober subspaces ordered by inclusion. Thus, such subspaces form the closed sets of a topology, finer than the given topology, which we refer to as the *patch topology*, as it generalizes the patch topology from stably compact spaces to sober spaces. By [3, Proposition 3.3], the embedding of a sober subspace F of a sober space X is a perfect map if and only if the following two conditions hold:

(1) $F \cap Q$ is compact for every compact saturated set $Q \subseteq X$,
(2) the lower set of C in X is closed in X for every relatively closed set $C \subseteq F$.

Here lower sets are taken with respect to the specialization order. Thus, the patch construction is idempotent for precisely those spaces X for which the sober subspaces F satisfying the above two conditions coincide with the closed sets. There are some immediate questions. Is the patch construction functorial for perfect maps? Does the patch construction produce a coreflection in this generality?

Acknowledgement The above development partially answers questions made to me by Harold Simmons.

References

[1] M.H. Escardó. Properly injective spaces and function spaces. *Topology and its Applications*, 89(1–2):75–120, 1998.

[2] M.H. Escardó. The regular-locally-compact coreflection of stably locally compact locale. *Journal of Pure and Applied Algebra*, to appear. Available at `http://www.elsevier.nl/locate/jpaa`.

[3] K.H. Hofmann and J.D. Lawson. On the order theoretical foundation of a theory of quasicompactly generated spaces without separation axiom. In R.-E. Hoffmann, editor, *Continuous Lattices and Related Topics*, volume 27 of *Mathematik-Arbeitspapiere*, pages 143–160. University of Bremen, 1982.