Typed λ-calculus: Denotational Semantics of Call-By-Value

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Preliminary note: substitution in CBV

For the pure calculus, we gave a substitution lemma expressing $[\![M[N/\mathtt{x}]]\!]$ in terms of $[\![M]\!]$ and $[\![N]\!]$. But that will not be possible in CBV, as the following example demonstrates. We define terms $\mathtt{x} : \mathtt{bool} \vdash M, M' : \mathtt{bool}$ and $\vdash N : \mathtt{bool}$ by

$$\begin{aligned} M &\stackrel{\text{def}}{=} \mathtt{true} \\ M' &\stackrel{\text{def}}{=} \mathtt{case}\ \mathtt{x}\ \mathtt{of}\ \{\mathtt{true}.\ \mathtt{true},\ \mathtt{false}.\ \mathtt{true}\} \\ N &\stackrel{\text{def}}{=} \mathtt{error}\ \mathtt{CRASH} \end{aligned}$$

But in any CBV semantics we will have

$$\begin{aligned} [\![M]\!] &= [\![M']\!] \qquad \text{because } M =_{\eta\,\mathtt{bool}} M' \\ [\![M[N/\mathtt{x}]]\!] &\neq [\![M'[N/\mathtt{x}]]\!] \end{aligned}$$

However, what we *will* be able to describe semantically is the substitution of a restricted class of terms, called *values.*

$$V ::= \ \mathtt{x} \mid \underline{n} \mid \mathtt{true} \mid \mathtt{false} \mid \mathtt{inl}\ V \mid \mathtt{inr}\ V \mid \lambda\mathtt{x}.M$$

A value, in any syntactic environment, is terminal. And a closed term is a value iff it is terminal. In the study of call-by-value, we define a *substitution* $\Gamma \xrightarrow{k} \Delta$ to be a function mapping each identifier $\mathtt{x} : A$ in Γ to a *value* $\Delta \vdash V : A$. If W is a value, then k^*W is a value, for any substitution k.

1 Denotational Semantics for CBV

Let us think about how to give a denotational semantics for call-by-value λ-calculus with errors. Let E be the set of errors.

1.1 First Attempt

Let's propose that for a type A, its denotation $[\![A]\!]$ will be a set that's a *universe for terms*: by this I mean that a closed term of type A will denote an element of $[\![A]\!]$. Then we should have

$$\begin{aligned}[\![\mathtt{bool}]\!] &= \mathbb{B} + E \\ [\![\mathtt{int}]\!] &= \mathbb{Z} + E \\ [\![\mathtt{bool} \times \mathtt{int}]\!] &= (\mathbb{B} \times \mathbb{Z}) + E \\ [\![A \times B]\!] &= [\![A]\!] * [\![B]\!]\end{aligned}$$

where $*$ is an operation on sets that would have to satisfy

$$(\mathbb{B} + E) * (\mathbb{Z} + E) = (\mathbb{B} \times \mathbb{Z}) + E$$

I can't see any such operation, so we give up on this proposal.

1.2 Second Attempt

Let's make $[\![A]\!]$ a set that's a *universe for values*, meaning that a closed value of type A will denote an element of type $[\![A]\!]$. In particular we want

$$\begin{aligned}[\![\mathtt{bool}]\!] &= \mathbb{B} \\ [\![\mathtt{int}]\!] &= \mathbb{Z} \\ [\![A + B]\!] &= [\![A]\!] + [\![B]\!] \\ [\![A \times B]\!] &= [\![A]\!] \times [\![B]\!]\end{aligned}$$

and we postpone the semantic equation for $\rightarrow$.

A *semantic environment* for Γ maps each identifier $\mathtt{x} : A$ in Γ to an element of $[\![A]\!]$. We write $[\![\Gamma]\!]$ for the set of semantic environments.

A closed term of type B either returns a closed value or raises an error. So it should denote an element of $[\![B]\!] + E$. More generally, a term $\Gamma \vdash M : B$ should denote, for each semantic environment $\rho \in [\![\Gamma]\!]$, an element of $[\![B]\!] + E$. Hence

$$[\![\Gamma]\!] \xrightarrow{[\![M]\!]} [\![B]\!] + E$$

Now let's think about $[\![A \to B]\!]$. A closed value of type $A \to B$ is a λ-abstraction $\lambda \mathtt{x}_A.M$. This can be applied to a closed *value* V of type A, and gives a closed term $M[V/\mathtt{x}]$ of type B. So we define

$$[\![A \to B]\!] = [\![A]\!] \to ([\![B]\!] + E)$$

We can easily write out the semantics of terms now.

1.3 Substitution Lemma

According to what we have said, a value $\Gamma \vdash V : A$ denotes a function

$$[\![\Gamma]\!] \xrightarrow{[\![V]\!]} [\![A]\!] + E$$

To formulate a substitution lemma, we *also* want V to denote a function

$$[\![\Gamma]\!] \xrightarrow{[\![V]\!]^{\mathsf{val}}} [\![A]\!]$$

and $[\![V]\!]^{\mathsf{val}}$ should be related to $[\![V]\!]$ by

$$[\![V]\!]\rho = \mathsf{inl}\ [\![V]\!]^{\mathsf{val}}\rho \qquad (1)$$

or as a diagram:

$$\begin{array}{ccc} [\![\Gamma]\!] & \xrightarrow{[\![V]\!]^{\mathsf{val}}} & [\![A]\!] \\ & \searrow^{[\![V]\!]} & \downarrow \mathsf{inl} \\ & & [\![A]\!] + E \end{array}$$

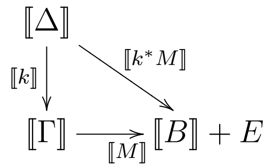
We define $[\![V]\!]^{\mathsf{val}}$ and verify (1) by induction on V.

Given a substitution $\Gamma \xrightarrow{k} \Delta$, we obtain a function $[\![\Delta]\!] \xrightarrow{[\![k]\!]} [\![\Gamma]\!]$. It maps $\rho \in [\![\Delta]\!]$ to the semantic environment for Γ that takes each identifier $\mathtt{x} : A$ in $\Gamma +$ to $[\![k(x)]\!]^{\mathsf{val}}\rho$.

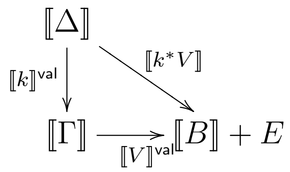
Now we can formulate two substitution lemmas: one for substitution into terms, and one for substitution into values.

Proposition 1. *Let* $\Gamma \xrightarrow{k} \Delta$ *be a substitution, and let* ρ *be a semantic environment for* Δ.

1. *For any term* $\Gamma \vdash M : B$*, we have* $\llbracket k^*M \rrbracket \rho = \llbracket M \rrbracket (\llbracket k \rrbracket \rho)$*, or as a diagram:*



2. *For any value* $\Gamma \vdash V : B$*, we have* $\llbracket k^*V \rrbracket^{\mathsf{val}} \rho = \llbracket V \rrbracket^{\mathsf{val}} (\llbracket k \rrbracket \rho)$*, or as a diagram:*



As usual we first prove this for renamings (or at least weakening).

1.4 Computational Adequacy

It is all very well to define a denotational semantics, but it's no good if it doesn't agree with the way the language was defined (the operational semantics).

Proposition 2. *Let* M *be a closed term.*

1. *If* $M \Downarrow V$*, then* $\llbracket M \rrbracket = \mathsf{inl}\ \llbracket V \rrbracket^{\mathsf{val}}$*.*
2. *If* $M \lightning e$*, then* $\llbracket M \rrbracket = \mathsf{inr}\ e$*.*

We prove this by induction on $\Downarrow$ and $\lightning$.