λ-calculus, effects and call-by-push-value

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Outline

Types

We're going to look at simply typed λ-calculus with arithmetic, including not just function types, but also sum and product types.

Here is the syntax of types:

$$A \quad ::= \quad \texttt{bool} \mid \texttt{nat} \mid A \to A \mid 1 \mid A \times A \mid 0 \mid A + A$$
$$\mid \textstyle\sum_{i \in \mathbb{N}} A_i \mid \prod_{i \in \mathbb{N}} A_i \quad \text{(optional extra)}$$

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Why no brackets?

- You might expect $A ::= \quad \cdots \mid (A)$.
- But our definition is abstract syntax.
- This means a type—or a term—is a tree of symbols, not a string of symbols.

Typing Judgement

Example

$$\mathtt{x} : \mathtt{nat},\ \mathtt{y} : \mathtt{nat} \vdash \lambda \mathtt{z}_{\mathtt{nat} \to \mathtt{nat}}.\ \mathtt{z}\,(\mathtt{x} + \mathtt{x}) : (\mathtt{nat} \to \mathtt{nat}) \to \mathtt{nat}$$

In English:

Given declarations of $\mathtt{x} : \mathtt{nat}$ and $\mathtt{y} : \mathtt{nat}$,

$\lambda \mathtt{z}_{\mathtt{nat} \to \mathtt{nat}}.\ \mathtt{z}\,(\mathtt{x} + \mathtt{x})$ is a term of type $(\mathtt{nat} \to \mathtt{nat}) \to \mathtt{nat}$.

The typing judgement takes the form $\Gamma \vdash M : A$.

- Γ is a typing context, a finite set of typed distinct identifiers.
- M is a term.
- A is a type.

Identifiers

The most basic typing rules, not associated with any particular type.

Free identifier

$$\frac{}{\Gamma \vdash \mathtt{x} : A}\ (\mathtt{x} : A) \in \Gamma$$

Multiple local declaration, e.g. of two identifiers

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash M' : B \quad \Gamma, \mathtt{x} : A, \mathtt{y} : B \vdash N : C}{\Gamma \vdash \mathtt{let}\ (\mathtt{x}\ \mathtt{be}\ M,\ \mathtt{y}\ \mathtt{be}\ M').\ N : C}$$

Typing rules for $A \to B$

Introduction rule

$$\frac{\Gamma, \mathtt{x} : A \vdash M : B}{\Gamma \vdash \lambda \mathtt{x}_A . M : A \to B}$$

Elimination rule

$$\frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B}$$

Type annotations in terms

- For Γ and M, there's at most one A such that $\Gamma \vdash M : A$
- and at most one derivation of $\Gamma \vdash M : A$.
- This is because of our type annotations.
- Some formulations omit some or all of these.

Typing rules for bool

Two introduction rules:

$$\frac{}{\Gamma \vdash \texttt{true} : \texttt{bool}} \qquad \frac{}{\Gamma \vdash \texttt{false} : \texttt{bool}}$$

Elimination rule

$$\frac{\Gamma \vdash M : \texttt{bool} \quad \Gamma \vdash N : B \quad \Gamma \vdash N' : B}{\Gamma \vdash \texttt{match}\ M\ \texttt{as}\ \{\texttt{true}.\ N,\ \texttt{false}.\ N'\} : B}$$

It's a pretentious notation for $\texttt{if}\ M\ \texttt{then}\ N\ \texttt{else}\ N'$.

Typing rules for arithmetic

These are *ad hoc* rules.

$$\frac{}{\Gamma \vdash 17 : \mathtt{nat}} \qquad \frac{\Gamma \vdash M : \mathtt{nat} \quad \Gamma \vdash M' : \mathtt{nat}}{\Gamma \vdash M + M' : \mathtt{nat}}$$

Typing rules for $A + B$

Two introduction rules

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathtt{inl}^{A,B}\ M : A + B} \qquad \frac{\Gamma \vdash M : B}{\Gamma \vdash \mathtt{inr}^{A,B}\ M : A + B}$$

Elimination rule

$$\frac{\Gamma \vdash M : A + B \quad \Gamma, \mathtt{x} : A \vdash N : C \quad \Gamma, \mathtt{y} : B \vdash N' : C}{\Gamma \vdash \mathtt{match}\ M\ \mathtt{as}\ \{\mathtt{inl}\ \mathtt{x}.\ N,\ \mathtt{inr}\ \mathtt{y}.\ N'\} : C}$$

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Likewise for $\sum_{i \in \mathbb{N}} A_i$.

Typing rules for 0

Zero introduction rules

Elimination rule

$$\frac{\Gamma \vdash M \ : \ 0}{\Gamma \vdash \texttt{match}\ M\ \texttt{as}\ \{\}^{A} \ : \ A}$$

Typing rules for $A \times B$

Introduction rule

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash N : B}{\Gamma \vdash \langle M, N \rangle : A \times B}$$

Two options for elimination

- Pattern-matching product. Elimination rule

$$\frac{\Gamma \vdash M : A \times B \quad \Gamma, \mathtt{x} : A, \mathtt{y} : B \vdash N : C}{\Gamma \vdash \mathtt{match}\ M\ \mathtt{as}\ \langle \mathtt{x}, \mathtt{y} \rangle.\ N : C}$$

- Projection product. Two elimination rules

$$\frac{\Gamma \vdash M : A \times B}{\Gamma \vdash M^{\mathtt{l}} : A} \qquad \frac{\Gamma \vdash M : A \times B}{\Gamma \vdash M^{\mathtt{r}} : B}$$

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$$\frac{\Gamma \vdash M : A \times B}{\Gamma \vdash M^{\mathtt{l}} : A} \qquad \frac{\Gamma \vdash M : A \times B}{\Gamma \vdash M^{\mathtt{r}} : B}$$

$\prod_{i \in \mathbb{N}} A_i$ is a projection product.

Typing rules for 1

Introduction rule

$$\frac{}{\Gamma \vdash \langle\rangle : 1}$$

Two options for elimination

- Pattern-match unit. Elimination rule

$$\frac{\Gamma \vdash M : 1 \quad \Gamma \vdash N : C}{\Gamma \vdash \mathtt{match}\ M\ \mathtt{as}\ \langle\rangle.\ N : C}$$

- Projection unit. Zero elimination rules

Weakening is admissible

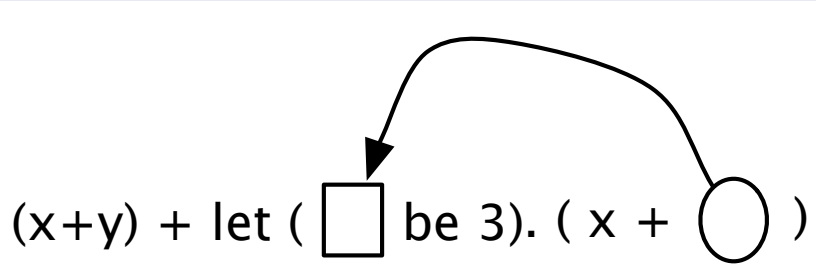
Theorem

If $\Gamma \vdash M : A$ and $\Gamma \subseteq \Gamma'$ then $\Gamma' \vdash M : A$.

Binding diagrams

Example

The term $(\mathtt{x} + \mathtt{y}) + \mathtt{let}\ (\mathtt{y}\ \mathtt{be}\ 3).\ (\mathtt{x} + \mathtt{y})$ has binding diagram



- Terms are α-equivalent when they have the same binding diagram.

$$M \equiv_\alpha N \quad \overset{\text{def}}{\Leftrightarrow} \quad \mathrm{BD}(M) = \mathrm{BD}(N)$$

- The collection of binding diagrams forms an initial algebra [FPT; AR].
- We'll skate over this issue. It's not specific to λ-calculus.

Substitution

Subsitution is an operation on binding diagrams, not on terms.

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Multiple substitution, e.g. for two identifiers

If $\Gamma \vdash M : A$ and $\Gamma \vdash M' : B$ and $\Gamma, \mathtt{x} : A, \mathtt{y} : B \vdash N : C$,

we define $\Gamma \vdash N[M/\mathtt{x}, M'/\mathtt{y}] : C$.

Example

$$\begin{aligned} M &= \lambda \mathtt{y}_{\mathtt{nat}}.\ \mathtt{y} + 3 \\ M' &= 7 \\ N &= \mathtt{x}\,(5 + \mathtt{y}) \\ N[M/\mathtt{x}, M'/\mathtt{y}] &= (\lambda \mathtt{z}_{\mathtt{nat}}.\ \mathtt{z} + 3)\,(5 + 7) \end{aligned}$$

Types denote sets

- Every type A denotes a set $[\![A]\!]$.
- For example, $[\![\mathtt{nat} \to \mathtt{nat}]\!]$ is the set of functions $\mathbb{N} \to \mathbb{N}$.

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  denotes an element of $[\![A]\!]$.

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  denotes an element of $[\![A]\!]$.
- For example, $\lambda \mathtt{x}_{\mathtt{nat}}.\, \mathtt{x} + 3$ denotes $\lambda a \in \mathbb{N}.\, a + 3$.

Semantics of types

Notation

For sets X and Y,

- $X \to Y$ is the set of functions from X to Y.
- $X \times Y$ is $\{\langle x, y\rangle \mid x \in X, y \in Y\}$.
- $X + Y$ is $\{\mathsf{inl}\ x \mid x \in X\} \cup \{\mathsf{inr}\ y \mid y \in Y\}$.

$$
\begin{aligned}
[\![\mathtt{bool}]\!] &= \mathbb{B} = \{\mathsf{true}, \mathsf{false}\} \\
[\![\mathtt{nat}]\!] &= \mathbb{N} \\
[\![A \to B]\!] &= [\![A]\!] \to [\![B]\!] \\
[\![1]\!] &= 1 = \{\langle\rangle\} \\
[\![A + B]\!] &= [\![A]\!] + [\![B]\!] \\
[\![A \times B]\!] &= [\![A]\!] \times [\![B]\!] \\
[\![0]\!] &= \emptyset
\end{aligned}
$$

Semantic environments

Let Γ be a typing context.

- A semantic environment ρ for Γ provides an element $\rho_{\mathtt{x}} \in [\![A]\!]$ for each $(\mathtt{x} : A) \in \Gamma$.
- $[\![\Gamma]\!]$ is the set of semantic environments for Γ.

$$[\![\Gamma]\!] \stackrel{\text{def}}{=} \prod_{(\mathtt{x}:A)\in\Gamma} [\![A]\!]$$

Semantics of typing judgement

Given a typing judgement $\Gamma \vdash M : A$,

we shall define $[\![M]\!]$, or more precisely $[\![\Gamma \vdash M : A]\!]$.

It's a function from $[\![\Gamma]\!]$ to $[\![A]\!]$.

Example

$$\mathtt{x} : \mathtt{nat}, \mathtt{y} : \mathtt{nat} \vdash \lambda \mathtt{z}_{\mathtt{nat} \to \mathtt{nat}}.\, \mathtt{z}(\mathtt{x} + \mathtt{y}) : (\mathtt{nat} \to \mathtt{nat}) \to \mathtt{nat}$$

denotes the function

$$\begin{array}{rcl} [\![\mathtt{x} : \mathtt{nat}, \mathtt{y} : \mathtt{nat}]\!] & \longrightarrow & (\mathbb{N} \to \mathbb{N}) \to \mathbb{N} \\ \rho & \longmapsto & \lambda z \in \mathbb{N} \to \mathbb{N}.\, z(\rho_{\mathtt{x}} + \rho_{\mathtt{y}}) \end{array}$$

Semantics of terms

$$\frac{}{\Gamma \vdash 17 : \mathtt{nat}}$$

$$[\![17]\!] : \rho \longmapsto 17$$

$$\frac{\Gamma \vdash M : \mathtt{nat} \quad \Gamma \vdash M' : \mathtt{nat}}{\Gamma \vdash M + M' : \mathtt{nat}}$$

$$[\![M + M']\!] : \rho \longmapsto [\![M]\!]\rho + [\![M']\!]\rho$$

More semantic equations

$$\frac{}{\Gamma \vdash \mathtt{x} : A}\ (\mathtt{x} : A) \in \Gamma$$

$$[\![\mathtt{x}]\!] \ : \ \rho \longmapsto \rho_{\mathtt{x}}$$

$$\frac{\Gamma, \mathtt{x} : A \vdash M : B}{\Gamma \vdash \lambda \mathtt{x}_A.\, M : A \to B}$$

$$[\![\lambda \mathtt{x}_A.\, M]\!] \ : \ \rho \longmapsto \lambda a \in [\![A]\!].\, [\![M]\!](\rho, \mathtt{x} \mapsto a)$$

More semantic equations

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathtt{inl}^{A,B}\ M : A + B}$$

$$[\![\mathtt{inl}^{A,B}\ M]\!] : \rho \longmapsto \mathsf{inl}\ [\![M]\!]\rho$$

$$\frac{\Gamma \vdash M : A + B \quad \Gamma, \mathtt{x} : A \vdash N : C \quad \Gamma, \mathtt{y} : B \vdash N' : C}{\Gamma \vdash \mathtt{match}\ M\ \mathtt{as}\ \{\mathtt{inl}\ \mathtt{x}.\ N, \mathtt{inr}\ \mathtt{y}.\ N'\} : C}$$

$$[\![\mathtt{match}\ M\ \mathtt{as}\ \{\mathtt{inl}\ \mathtt{x}.\ N, \mathtt{inr}\ \mathtt{y}.\ N'\}]\!] : \rho \longmapsto$$
$$\mathsf{match}\ [\![M]\!]\rho\ \mathsf{as}\ \{\mathsf{inl}\ a.\ [\![N]\!](\rho, \mathtt{x} \mapsto a), \mathsf{inr}\ b.\ [\![N']\!](\rho, \mathtt{x} \mapsto b)\}$$

Basic properties

Semantic Coherence

If type annotations are omitted,

then $\Gamma \vdash M : A$ can have more than one derivation.

We must prove that $[\![\Gamma \vdash M : A]\!]$ doesn't depend on the derivation.

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Weakening Lemma

If $\Gamma \vdash M : A$ and $\Gamma \subseteq \Gamma'$ then

$$[\![\Gamma' \vdash M : A]\!]\rho = [\![\Gamma \vdash M]\!](\rho \restriction_\Gamma)$$

Substitution

Binding Diagrams

- We can give denotational semantics of binding diagrams.
- $[\![\Gamma \vdash M : A]\!] = [\![\Gamma \vdash \mathsf{BD}(M) : A]\!]$
- So α-equivalent terms have the same denotation.

Substitution

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Substitution Lemma

For binding diagrams $\Gamma \vdash M : A$ and $\Gamma \vdash M' : B$ and $\Gamma, \mathtt{x} : A \vdash N : C$, we can recover $[\![N[M/\mathtt{x}, M'/\mathtt{y}]]\!]$ from $[\![M]\!]$ and $[\![N]\!]$.

$$[\![N[M/\mathtt{x}, M'/\mathtt{y}]]\!] : \rho \longmapsto [\![N]\!](\rho, \mathtt{x} \mapsto [\![M]\!]\rho, \mathtt{y} \mapsto [\![M']\!]\rho)$$

β-laws

The β-law for $A \to B$

$$\frac{\Gamma \vdash M : A \quad \Gamma, \mathtt{x} : A \vdash N : B}{\Gamma \vdash (\lambda \mathtt{x}_A . N)\, M = N[M/\mathtt{x}] : B}$$

Introduction inside an elimination may be removed.

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Introduction inside an elimination may be removed.

Two β-laws for projection product $A \times B$

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash N : A'}{\Gamma \vdash \langle M, N \rangle^{\mathtt{1}} = M : A}$$

Zero β-laws for projection unit 1

More β-laws

Two β-laws for `bool`

$$\frac{\Gamma \vdash N : C \quad \Gamma \vdash N' : C}{\Gamma \vdash \texttt{match true as } \{\texttt{true}.\, N,\ \texttt{false}.\, N'\} = N : C}$$

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$$\frac{\Gamma \vdash M : A \quad \Gamma, \mathtt{x} : A \vdash N : C \quad \Gamma, \mathtt{y} : B \vdash N' : C}{\Gamma \vdash \mathtt{match}\ \mathtt{inl}^{A,B}\ M\ \mathtt{as}\ \{\mathtt{inl}\ \mathtt{x}.\, N,\ \mathtt{inr}\ \mathtt{y}.\, N'\} = N[M/\mathtt{x}] : C}$$

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Zero β-laws for 0

β-law for local declaration

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash M' : B \quad \Gamma, \mathtt{x} : A, \mathtt{y} : B \vdash N : C}{\Gamma \vdash \mathtt{let}\ (\mathtt{x}\ \mathtt{be}\ M,\ \mathtt{y}\ \mathtt{be}\ M').\ N = N[M/\mathtt{x}, M'/\mathtt{y}] : C}$$

η-laws

η-law for $A \to B$, everything is λ

$$\frac{\Gamma \vdash M : A \to B}{\Gamma \vdash M = \lambda \mathtt{x}_A .\, M\, \mathtt{x} : A \to B}\ \mathtt{x} \notin \Gamma$$

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η-law for projection product $A \times B$, everything is λ

$$\frac{\Gamma \vdash M : A \times B}{\Gamma \vdash M = \langle M^{\mathtt{l}}, M^{\mathtt{r}} \rangle : A \times B}$$

η-law for projection unit 1, everything is λ

$$\frac{\Gamma \vdash M : 1}{\Gamma \vdash M = \langle\rangle : 1}$$

More η-laws

η-law for `bool`, everything is `true` or `false`

$$\frac{\Gamma \vdash M : \mathtt{bool} \quad \Gamma, \mathtt{z} : \mathtt{bool} \vdash N : C}{\Gamma \vdash N[M/\mathtt{z}] = \mathtt{match}\ M\ \mathtt{as}\ \{N[\mathtt{true}/\mathtt{z}],\ N[\mathtt{false}/\mathtt{z}]\} : C}\ \mathtt{z} \notin \Gamma$$

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η-law for $A + B$, everything is `inl` or `inr`

$$\frac{\Gamma \vdash M : A + B \quad \Gamma, \mathtt{z} : \mathtt{bool} \vdash N : C}{\Gamma \vdash N[M/\mathtt{z}] = \mathtt{match}\ M\ \mathtt{as}\ \{\mathtt{inl}\ \mathtt{x}.\ N[\mathtt{inl}\ \mathtt{x}/\mathtt{z}],\ \mathtt{inr}\ \mathtt{y}.\ N[\mathtt{inr}\ \mathtt{y}/\mathtt{z}]\} : C}\ \mathtt{z} \notin \Gamma$$

More η-laws

η-law for $\mathtt{bool}$, everything is $\mathtt{true}$ or $\mathtt{false}$

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η-law for 0, nothing exists

$$\frac{\Gamma \vdash M : 0 \quad \Gamma, \mathtt{z} : 0 \vdash N : C}{\Gamma \vdash N[M/\mathtt{z}] = \mathtt{match}\ M\ \mathtt{as}\ \{\}_C : C}\ \mathtt{z} \notin \Gamma$$

The $\beta\eta$-theory

We define $\Gamma \vdash M =_{\beta\eta} M' : A$ inductively as follows.

All the β- and η-laws are taken as axioms,

and it is a congruence i.e. an equivalence relation preserved by each term constructor. For example:

$$\frac{\Gamma, \mathtt{x} : A \vdash M = M' : B}{\Gamma \vdash \lambda \mathtt{x}_A . M = \lambda \mathtt{x}_A . M' : A \to B}$$

Properties of $=_{\beta\eta}$

Closure Theorems

- $=_{\beta\eta}$ is closed under weakening. But not conversely, e.g.

$$\begin{aligned} \mathtt{z} : 0 \quad &\vdash \quad \mathtt{true} =_{\beta\eta} \mathtt{false} : \mathtt{bool} \\ &\vdash \quad \mathtt{true} \neq_{\beta\eta} \mathtt{false} : \mathtt{bool} \end{aligned}$$

- $=_{\beta\eta}$ is closed under substitution.

Soundness theorem

If $\Gamma \vdash M =_{\beta\eta} M' : A$ then $[\![M]\!] = [\![M']\!]$.

Follows from the weakening and substitution lemmas.

Reversible rule for $A \to B$

The connective $\to$ is rightist: it has a reversible rule

$$\frac{\Gamma, \mathtt{x} : A \vdash B}{\Gamma \vdash A \to B}$$

natural in Γ—we'll skate over naturality.

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- Downwards, a term $\Gamma, \mathtt{x} : A \vdash M : B$ is sent to $\lambda \mathtt{x}_A . M$.
- Upwards, a term $\Gamma \vdash N : A \to B$ is sent to $N\, \mathtt{x}$.
- These are inverse up to $=_{\beta\eta}$.

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$A \to B$ appears on the right of $\vdash$ in the conclusion.

Reversible rule for bool

The (nullary) connective bool is leftist.
That means: it has a reversible rule

$$\frac{\Gamma \vdash C \quad \Gamma \vdash C}{\Gamma, \mathtt{z} : \mathtt{bool} \vdash C}$$

natural in Γ and C—we'll skate over naturality.

- Downwards, a pair $\Gamma \vdash M : C$ and $\Gamma \vdash M' : C$ is sent to $\mathtt{match\ z\ as}\ \{\mathtt{true}.\, M,\ \mathtt{false}.\, M'\}$.
- Upwards, a term $\Gamma, \mathtt{z} : \mathtt{bool} \vdash N : C$ is sent to $N[\mathtt{true}/\mathtt{z}]$ and $N[\mathtt{false}/\mathtt{z}]$.
- These are inverse up to $=_{\beta\eta}$.

bool appears on the left of $\vdash$ in the conclusion.

Reversible rule for $A + B$

The connective $+$ is leftist, having a reversible rule

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The (nullary) connective 0 is leftist, having a reversible rule

$$\frac{}{\Gamma, \mathtt{z} : 0 \vdash C}$$

natural in Γ and C.

Bipartisan connectives

The connective $\times$ has a reversible rule

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \times B}$$

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natural in Γ, so it's rightist.

It also has a reversible rule

$$\frac{\Gamma, \mathtt{x} : A, \mathtt{y} : B \vdash C}{\Gamma, \mathtt{z} : A \times B \vdash C}$$

natural in Γ and C, so it's leftist.

In summary, the connective $\times$ is bipartisan.
Likewise the (nullary) connective 1.

Most general leftist connective

The variant tuple type $\boxed{\sum}\{^0 A, A';\ ^1 B, B', B''\}$ denotes a sum of products

$$([\![A]\!] \times [\![A']\!]) + ([\![B]\!] \times [\![B']\!] \times [\![B'']\!])$$

This gives a leftist connective.

$$\frac{\Gamma, A, A' \vdash C \quad \Gamma, B, B', B'' \vdash C}{\Gamma, \boxed{\sum}\{^0 A, A';\ ^1 B, B', B''\} \vdash C}$$

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Here is its term syntax:

$$\mathtt{in}_0(M, M')$$
$$\mathtt{in}_1(M, M', M'')$$
$$\mathtt{match}\ M\ \mathtt{as}\ \{\mathtt{in}_0(\mathtt{x}, \mathtt{x}').\, N, \mathtt{in}_1(\mathtt{y}, \mathtt{y}', \mathtt{y}'').\, N'\}$$

Most general rightist connective

The variant function type $\boxed{\prod}\{^0\, A, A' \vdash B;\ ^1\, C, C', C' \vdash D\}$ denotes a product of multi-ary function types

$$((\llbracket A \rrbracket \times \llbracket A' \rrbracket) \to \llbracket B \rrbracket) \times ((\llbracket C \rrbracket \times \llbracket C' \rrbracket \times \llbracket C'' \rrbracket) \to \llbracket D \rrbracket)$$

This gives a rightist connective.

$$\frac{\Gamma, A, A' \vdash B \quad \Gamma, C, C', C'' \vdash D}{\Gamma \vdash \boxed{\prod}\{^0\, A, A' \vdash B;\ ^1\, C, C', C' \vdash D\}}$$

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Here is its term syntax:

$$\lambda\{^0(\mathtt{x}, \mathtt{x}').M, ^1(\mathtt{y}, \mathtt{y}', \mathtt{y}'').M'\}$$
$$M\ ^0(N, N')$$
$$M\ ^1(N, N', N'')$$

Jumbo λ-calculus

Type syntax

$$A \quad ::= \quad \boxed{\Sigma}\,\{\overrightarrow{A_i}\}_{i<n} \quad \mid \quad \boxed{\Pi}\,\{\overrightarrow{A_i} \vdash B_i\}_{i<n} \qquad (n \in \mathbb{N} \text{ or } n = \infty)$$

Term syntax, with type annotations omitted

$$\begin{aligned} M \quad ::= \quad & \mathtt{x} \mid \mathtt{let}\ (\overrightarrow{\mathtt{x}\ \mathtt{be}\ M}).\ M \\ & \mid \mathtt{in}_i(\overrightarrow{M}) \\ & \mid \mathtt{match}\ M\ \mathtt{as}\ \{\mathtt{in}_i(\overrightarrow{\mathtt{x}}).\ M_i\}_{i<n} \\ & \mid \lambda\{^i(\overrightarrow{\mathtt{x}}).\ M_i\}_{i<n} \\ & \mid M\,^i(\overrightarrow{M}) \end{aligned}$$

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Includes both pattern-match product $A \times B$
and projection product $A \amalg B$.

Jumbo vs non-jumbo

Jumbo λ-calculus is the most expressive form of simply typed λ-calculus: it contains all leftist and rightist connectives as primitives.

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Modulo $=_{\beta\eta}$ it is no more expressive than the non-jumbo version.

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Jumbo λ-calculus is the most expressive form of simply typed λ-calculus:

it contains all leftist and rightist connectives as primitives.

Modulo $=_{\beta\eta}$ it is no more expressive than the non-jumbo version.

But the β- and η-laws are not going to survive.

Evaluating terms

We want to evaluate every closed term $\vdash M : A$ to a terminal term.

We want $\lambda \mathrm{x}_A.\, M$ to be terminal, since M is not closed.

But there are many options.

Three decisions we must make

1. To evaluate $\texttt{let}\ (\texttt{x}\ \texttt{be}\ M,\ \texttt{y}\ \texttt{be}\ M').\ N$, do we
   - evaluate M to T and M' to T', then evaluate $N[T/\texttt{x}, T'/\texttt{y}]$?
   - just evaluate $N[M/\texttt{x}, M'/\texttt{y}]$?

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2. To evaluate $M\,N$, we must evaluate M to $\lambda \mathtt{x}_A.\, P$. Do we
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3. Any terminal term of type $A + B$ must be $\texttt{inl}\ M$ or $\texttt{inr}\ M$. Do we
   - deem $\texttt{inl}\ T$ and $\texttt{inr}\ T$ terminal only if T is terminal?
   - always deem $\texttt{inl}\ M$ and $\texttt{inr}\ M$ terminal?

One fundamental decision

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Substituting terminal terms gives call-by-value or eager evaluation.

Substituting unevaluated terms gives call-by-name.

Terminology: lazy and call-by-name

- "Lazy" evaluation usually means call-by-need, except in Abramsky's "lazy λ-calculus".
- In the untyped literature, "call-by-name" evaluation means reduction to head normal form.

Evaluation order for `let`

To evaluate $\texttt{let}\ (\texttt{x}\ \texttt{be}\ M,\ \texttt{y}\ \texttt{be}\ M').\ N$, do we

- evaluate M to T and M' to T', then evaluate $N[T/\texttt{x}, T'/\texttt{y}]$? Call-by-value
- just evaluate $N[M/\texttt{x}, M'/\texttt{y}]$? Call-by-name

Evaluation order for application

To evaluate $M\,N$, we must evaluate M to $\lambda \mathtt{x}_A.\,P$. Do we

- evaluate N to T (before or after evaluating M), then evaluate $P[T/\mathtt{x}]$? Call-by-value
- just evaluate $P[N/\mathtt{x}]$? Call-by-name

Terminal terms of type $A + B$

Any terminal term of type $A + B$ must be $\texttt{inl}\ M$ or $\texttt{inr}\ M$. Do we

- deem $\texttt{inl}\ T$ and $\texttt{inr}\ T$ terminal only if T is terminal? Call-by-value
- always deem $\texttt{inl}\ M$ and $\texttt{inr}\ M$ terminal? Call-by-name

Consider evaluation of $\texttt{match}\ P\ \texttt{as}\ \{\texttt{inl}\ \texttt{x}.\, N,\ \texttt{inr}\ \texttt{y}.\, N'\}$ to see this.

Definitional interpreter for call-by-value

CBV terminals $T ::= \texttt{true} \mid \texttt{false} \mid \texttt{inl}\ T \mid \texttt{inr}\ T \mid \lambda \texttt{x}.M$

To evaluate

- $\texttt{true}$: return $\texttt{true}$.
- $M + N$: evaluate M. If this returns m, evaluate N. If this returns n, return $m + n$.
- $\lambda \texttt{x}.M$: return $\lambda \texttt{x}.M$.
- $\texttt{inl}\ M$: evaluate M. If this returns T, return $\texttt{inl}\ T$.
- $\texttt{let}\ (\texttt{x be}\ M,\ \texttt{y be}\ M').\ N$: evaluate M. If this returns T, evaluate M'. If this returns T', evaluate $N[T/\texttt{x}, T'/\texttt{y}]$.
- $\texttt{match}\ M\ \texttt{as}\ \{\texttt{true}.\ N,\ \texttt{false}.\ N'\}$: evaluate M. If this returns $\texttt{true}$, evaluate N, but if it returns $\texttt{false}$, evaluate N'.
- $\texttt{match}\ M\ \texttt{as}\ \{\texttt{inl x}.\ N,\ \texttt{inr x}.\ N'\}$: evaluate M. If this returns $\texttt{inl}\ T$, evaluate $N[T/\texttt{x}]$, but if it returns $\texttt{inr}\ T$, evaluate $N'[T/\texttt{x}]$.
- MN: evaluate M. If this returns $\lambda \texttt{x}.P$, evaluate N. If this returns T, evaluate $P[T/\texttt{x}]$.

Definitional interpreter for call-by-name

In CBN the terminals are $\mathtt{true}$, $\mathtt{false}$, $\mathtt{inl}\ M$, $\mathtt{inr}\ M$, $\lambda \mathtt{x}.M$
To evaluate

- $\mathtt{true}$: return $\mathtt{true}$.
- $M + N$: evaluate M. If this returns m, evaluate N. If this returns n, return $m + n$.
- $\lambda \mathtt{x}.M$: return $\lambda \mathtt{x}.M$.
- $\mathtt{inl}\ M$: return $\mathtt{inl}\ M$.
- $\mathtt{let}\ (\mathtt{x}\ \mathtt{be}\ M,\ \mathtt{y}\ \mathtt{be}\ M').\ N$: evaluate $N[M/\mathtt{x}, M'/\mathtt{y}]$.
- $\mathtt{match}\ M\ \mathtt{as}\ \{\mathtt{true}.\ N,\ \mathtt{false}.\ N'\}$: evaluate M. If this returns $\mathtt{true}$, evaluate N, but if it returns $\mathtt{false}$, evaluate N'.
- $\mathtt{match}\ M\ \mathtt{as}\ \{\mathtt{inl}\ \mathtt{x}.\ N,\ \mathtt{inr}\ \mathtt{x}.\ N'\}$: evaluate M. If this returns $\mathtt{inl}\ P$, evaluate $N[P/\mathtt{x}]$, but if it returns $\mathtt{inr}\ P$, evaluate $N'[P/\mathtt{x}]$.
- MN: evaluate M. If this returns $\lambda \mathtt{x}.P$, evaluate $P[N/\mathtt{x}]$.

Big-step semantics for call-by-value

We write $M \Downarrow T$ to mean that M evaluates to T.

This is defined inductively, for example

$$\frac{M \Downarrow \lambda \mathtt{x}_A.\, P \quad N \Downarrow T \quad P[T/\mathtt{x}] \Downarrow T'}{M\ N \Downarrow T'}$$

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If $\vdash M : A$ then $M \Downarrow T$ for unique T.

Moreover $\vdash T : A$ and $[\![M]\!] = [\![T]\!]$.

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Long story

The experiment

- Add effects to (jumbo) λ-calculus, with CBV or CBN evaluation.
- See what equations and isomorphisms survive.
- Seek a denotational semantics for each language.

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Analyzing CBV with a microscope

- Look closely at the CBV models: there's a pattern.
- CBV contains particles of meaning, constituting fine-grain call-by-value.

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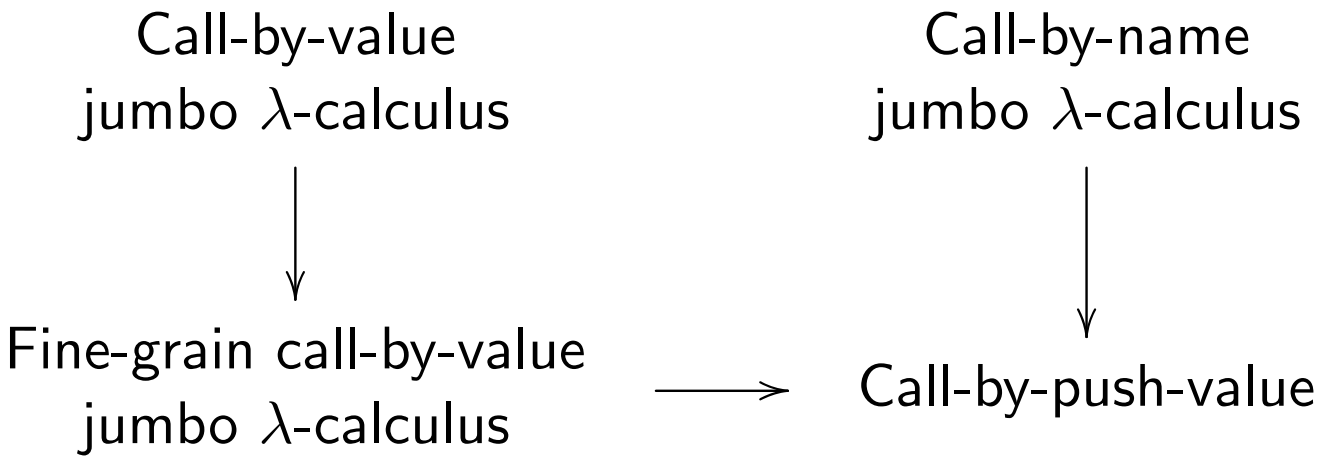
Analyzing CBV with a microscope

- Look closely at the CBV models: there's a pattern.
- CBV contains particles of meaning, constituting fine-grain call-by-value.

Increasing the magnification

- Look very closely at the CBN and fine-grain CBV models: there's a pattern.
- Both contain tiny particles of meaning, constituting call-by-push-value.

The big picture



Both fine-grain call-by-value and call-by-push-value are obtained empirically, by observing particles of meaning within a range of denotational models.

Where this story comes from

- Plotkin: semantics of recursion for call-by-name (PCF) and call-by-value (FPC)
- Moggi: list of monads for denotational semantics
- Moggi: monadic metalanguage
- Power and Robinson: Freyd categories
- Plotkin and Felleisen: call-by-value continuation semantics
- Reynolds' Idealized Algol, a call-by-name language with state
- O'Hearn: semantics of type identifiers in such a language
- Streicher and Reus: call-by-name continuation semantics
- Filinski: Effect-PCF

Adding computational effects

Errors

Let $E = \{\mathrm{CRASH}, \mathrm{BANG}\}$ be a set of "errors". We add

$$\frac{}{\Gamma \vdash \mathtt{error}^B\ e : B}\ e \in E$$

To evaluate $\mathtt{error}^B\ e$: halt with error message e.

Printing

Let $\mathcal{A} = \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}\}$ be a set of "characters". We add

$$\frac{\Gamma \vdash M : B}{\Gamma \vdash \mathtt{print}\ c.\ M : B}\ c \in \mathcal{A}$$

To evaluate $\mathtt{print}\ c.\ M$: print c and then evaluate M.

Exercises

1. Evaluate

$$\texttt{let (x be error CRASH). 5}$$

   in CBV and CBN.

2. Evaluate

$$(\lambda \texttt{x}.(\texttt{x} + \texttt{x}))(\texttt{print "hello". } 4)$$

   in CBV and CBN.

3. Evaluate

$$\begin{array}{l}\texttt{match (print "hello". inr error CRASH) as} \\ \quad \{\texttt{inl x. x} + 1,\ \texttt{inr y. 5}\}\end{array}$$

   in CBV and CBN.

Big-step semantics for errors

For call-by-value, we inductively define two big-step relations:

- $M \Downarrow T$ means M evaluates to T.
- $M \lightning e$ means M raises error e.

Here are the rules for application:

$$\frac{M \lightning e}{M\ N \lightning e} \qquad \frac{M \Downarrow \lambda \mathtt{x}.\, P \quad N \lightning e}{M\ N \lightning e}$$

$$\frac{M \Downarrow \lambda \mathtt{x}.\, P \quad N \Downarrow T \quad P[T/\mathtt{x}] \lightning e}{M\ N \lightning e}$$

$$\frac{M \Downarrow \lambda \mathtt{x}.\, P \quad N \Downarrow T \quad P[T/\mathtt{x}] \Downarrow T'}{M\ N \Downarrow T'}$$

Likewise for call-by-name.

Observational equivalence

A program is a closed term of type `nat` or `bool`.

Two terms $\Gamma \vdash M, M' : B$ are observationally equivalent

when $\mathcal{C}[M]$ and $\mathcal{C}[M']$ have the same behaviour

for every program with a hole $\mathcal{C}[\cdot]$.

Same behaviour means: print the same string, raise the same error, return the same boolean.

We write $M \simeq_{\mathbf{CBV}} M'$ and $M \simeq_{\mathbf{CBN}} M'$.

The η-law for boolean type: has it survived?

η-law for bool

Any term $\Gamma, \mathtt{z} : \mathtt{bool} \vdash M : B$ can be expanded as

$$\mathtt{match\ z\ as}\ \{\mathtt{true}.\ M[\mathtt{true}/\mathtt{z}],\ \mathtt{false}.\ M[\mathtt{false}/\mathtt{z}]\}$$

Anything of boolean type is a boolean.

This holds in CBV, because z can only be replaced by true or false.

But it's broken in CBN, because z might raise an error. For example,

$$\mathtt{true} \not\simeq_{\mathbf{CBN}} \mathtt{match\ z\ as}\ \{\mathtt{true}.\ \mathtt{true},\ \mathtt{false}.\ \mathtt{true}\}$$

because we can apply the context

$$\mathtt{let}\ (\mathtt{z\ be\ error}\ \mathrm{CRASH}).\ [\cdot]$$

Similarly the η-law for sum types is valid in CBV but not in CBN.

The η-law for functions: has it survived?

η-law for $A \to B$ and $A \mathbin{\pi} B$

Any term $\Gamma \vdash M : A \to B$ can be expanded as $\lambda \mathtt{x}.M\,\mathtt{x}$.
Any term $\Gamma \vdash M : A \mathbin{\pi} B$ can be expanded as $\lambda\{^{\mathtt{l}}.\,M^{\mathtt{l}},\ ^{\mathtt{r}}.\,M^{\mathtt{r}}\}$.

Although these fail in CBV, they hold in CBN. Consequences:

$$\begin{aligned}
\mathtt{error}\ e &\simeq_{\mathbf{CBN}} \lambda \mathtt{x}.\ \mathtt{error}\ e \\
\mathtt{error}\ e &\simeq_{\mathbf{CBN}} \lambda\{^{\mathtt{l}}.\,\mathtt{error}\ e,\ ^{\mathtt{r}}.\,\mathtt{error}\ e\} \\
\mathtt{print}\ c.\ \lambda \mathtt{x}.\ M &\simeq_{\mathbf{CBN}} \lambda \mathtt{x}.\ \mathtt{print}\ c.\ M \\
\mathtt{print}\ c.\ \lambda\{^{\mathtt{l}}.\,M,\ ^{\mathtt{r}}.\,N\} &\simeq_{\mathbf{CBN}} \lambda\{^{\mathtt{l}}.\,\mathtt{print}\ c.\ M,\ ^{\mathtt{r}}.\,\mathtt{print}\ c.\ N\}
\end{aligned}$$

Yet the two sides have different operational behaviour! What's going on?

In CBN, a function gets evaluated only by being applied.

Summary

The pure λ-calculus satisfies all the β- and η-laws.

With computational effects,

- CBV satisfies η for leftist connectives (tuple types), but not rightist ones (function types)
- CBN satisfies η for rightist connectives (function types), but not leftist ones (tuple types).

Summary

The pure λ-calculus satisfies all the β- and η-laws.

With computational effects,

- CBV satisfies η for leftist connectives (tuple types), but not rightist ones (function types)
- CBN satisfies η for rightist connectives (function types), but not leftist ones (tuple types).

Similarly for isomorphisms:

- $(A + B) + C \cong A + (B + C)$ survives in CBV but not CBN.
- $A \times B \cong A \amalg B$ survives in neither CBV nor CBN.
- $A \to (B \to C) \cong (A \amalg B) \to C$ survives in CBN but not CBV.

Naive CBV semantics

Our first attempt.

Each type A denotes a set, a semantic domain for terms.

$$\begin{aligned} [\![\mathtt{bool}]\!]_* &= \mathbb{B} + E \\ [\![\mathtt{bool} + \mathtt{bool}]\!]_* &= (\mathbb{B} + \mathbb{B}) + E \\ [\![\mathtt{bool} \times \mathtt{bool}]\!]_* &= (\mathbb{B} \times \mathbb{B}) + E \end{aligned}$$

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Not easy to make this compositional, so we abandon it.

CBV denotational semantics

Each type denotes a set, a semantic domain for terminals.

$$
\begin{aligned}
[\![\mathtt{bool}]\!] &= \mathbb{B} \\
[\![A + B]\!] &= [\![A]\!] + [\![B]\!] \\
[\![A \to B]\!] &= [\![A]\!] \to ([\![B]\!] + E) \\
[\![() \to B]\!] &= [\![B]\!] + E \\
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\end{aligned}$$

Each term $\Gamma \vdash M : B$ denotes a function $[\![M]\!] : [\![\Gamma]\!] \longrightarrow ([\![B]\!] + E)$.

Semantics of term constructors

$$\frac{\Gamma, \mathtt{x} : A \vdash M : B}{\Gamma \vdash \lambda \mathtt{x} \in A.\, M : A \to B}$$

$$[\![\lambda \mathtt{x}_A.\, M]\!] \;:\; \rho \longmapsto \mathsf{inl}\ \lambda a \in [\![A]\!].\, [\![M]\!](\rho, \mathtt{x} \mapsto a)$$

$$\frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B}$$

$$[\![M\ N]\!] \;:\; \rho \longmapsto \mathsf{match}\ [\![M]\!]\rho\ \mathsf{as} \begin{cases} \mathsf{inl}\ f. & \mathsf{match}\ [\![N]\!]\rho\ \mathsf{as} \begin{cases} \mathsf{inl}\ x. & f(x) \\ \mathsf{inr}\ e. & \mathsf{inr}\ e \end{cases} \\ \mathsf{inr}\ e. & \mathsf{inr}\ e \end{cases}$$

More term constructors

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathtt{inl}^{A,B}\ M : A + B}$$

$$[\![\mathtt{inl}^{A,B}\ M]\!] : \rho \longmapsto \begin{cases} \mathsf{inl}\ a. & \mathsf{inl}\ \mathsf{inl}\ a \\ \mathsf{inr}\ e. & \mathsf{inr}\ e \end{cases}$$

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To prove the soundness of the denotational semantics, we need a substitution lemma.

CBV Substitution Lemma: What Doesn't Work

Can we obtain $[\![N[M/\mathrm{x}]]\!]$ from $[\![M]\!]$ and $[\![N]\!]$?

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Example that rules out a general substitution lemma

Define $\vdash M : \mathtt{bool}$ and $\mathtt{x} : \mathtt{bool} \vdash N, N' : \mathtt{bool}$.

$$\begin{array}{rcl} M & \stackrel{\text{def}}{=} & \mathtt{error}\ \mathrm{CRASH} \\ N & \stackrel{\text{def}}{=} & \mathtt{true} \\ N' & \stackrel{\text{def}}{=} & \mathtt{match}\ \mathtt{x}\ \mathtt{as}\ \{\mathtt{true}.\mathtt{true},\ \mathtt{false}.\mathtt{true}\} \\ [\![N]\!] & = & [\![N']\!] \qquad \text{because } N =_{\eta\,\mathtt{bool}} N' \\ [\![N[M/\mathtt{x}]]\!] & \neq & [\![N'[M/\mathtt{x}]]\!] \end{array}$$

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But we can give a lemma for the substitution of values.

Values

The following terms are called values.

$$V \quad ::= \quad \texttt{true} \mid \texttt{false} \mid \texttt{inl}\ V \mid \texttt{inr}\ V \mid \lambda \mathrm{x}.M \mid \mathrm{x}$$

The closed values are just the terminals:
we don't allow "complex values" such as

$$\texttt{match true as } \{\texttt{true.false}, \texttt{ false.true}\}$$

Denotational semantics of values

Each value $\Gamma \vdash V : A$ denotes a function $[\![V]\!]^{\mathsf{val}} : [\![\Gamma]\!] \longrightarrow [\![A]\!]$.

$$
\begin{array}{lcll}
[\![\mathtt{x}]\!]^{\mathsf{val}} & : & \rho & \longmapsto \quad \rho_{\mathtt{x}} \\
[\![\mathtt{true}]\!]^{\mathsf{val}} & : & \rho & \longmapsto \quad \mathsf{true} \\
[\![\mathtt{inl}\ V]\!]^{\mathsf{val}} & : & \rho & \longmapsto \quad \mathsf{inl}\ [\![V]\!]^{\mathsf{val}}\rho \\
[\![\lambda \mathtt{x}_A.\, M]\!]^{\mathsf{val}} & : & \rho & \longmapsto \quad \lambda a \in [\![A]\!].\, [\![M]\!](\rho, \mathtt{x} \mapsto [\![a]\!])
\end{array}
$$

We can recover $[\![V]\!]$ from $[\![V]\!]^{\mathsf{val}}$.

$$[\![V]\!] : \rho \longmapsto \mathsf{inl}\ [\![V]\!]^{\mathsf{val}}\rho$$

Substitution Lemma For Values

Given values $\Gamma \vdash V : A$ and $\Gamma \vdash^{\mathsf{v}} W : B$ and a term $\Gamma, \mathtt{x} : A, \mathtt{y} : B \vdash M : C$

we can obtain $[\![M[V/\mathtt{x}, W/\mathtt{y}]]\!]$ from $[\![V]\!]^{\mathsf{val}}$ and $[\![W]\!]^{\mathsf{val}}$ and $[\![M]\!]$.

$$[\![M[V/\mathtt{x}, W/\mathtt{y}]]\!] : \rho \longmapsto [\![M]\!](\rho, \mathtt{x} \mapsto [\![V]\!]^{\mathsf{val}}\rho, \mathtt{y} \mapsto [\![W]\!]^{\mathsf{val}}\rho)$$

Likewise for substitution of values into values.

Soundness of CBV Denotational Semantics

- If $M \Downarrow V$ then $[\![M]\!]\varepsilon = \mathsf{inl}\ ([\![V]\!]^{\mathsf{val}}\varepsilon)$.
- If $M \not\downarrow e$ then $[\![M]\!]\varepsilon = \mathsf{inr}\ e$.

Proof by induction, using the substitution lemma.

Fine-Grain Call-By-Value

Fine-grain call-by-value has two judgements:

- A value $\Gamma \vdash^{\mathsf{v}} V : A$ denotes a function $[\![V]\!] : [\![\Gamma]\!] \longrightarrow [\![A]\!]$.
- A computation $\Gamma \vdash^{\mathsf{c}} M : A$ denotes a function $[\![M]\!] : [\![\Gamma]\!] \longrightarrow [\![A]\!] + E$.

Key typing rules

$$\frac{\Gamma \vdash^{\mathsf{v}} V : A}{\Gamma \vdash^{\mathsf{c}} \mathtt{return}\ V : A} \qquad \frac{\Gamma \vdash^{\mathsf{v}} M : A \quad \Gamma, \mathtt{x} : A \vdash^{\mathsf{c}} N : B}{\Gamma \vdash^{\mathsf{c}} M\ \mathtt{to}\ \mathtt{x}.\ N : B}$$

Corresponds to Power and Robinson's notion of a Freyd category.

Semantics of returning and sequencing

$$\frac{\Gamma \vdash^{\mathsf{v}} V : A}{\Gamma \vdash^{\mathsf{c}} \mathtt{return}\ V : A}$$

$$[\![\mathtt{return}\ V]\!] : \rho \longmapsto \mathsf{inl}\ [\![V]\!]\rho$$

$$\frac{\Gamma \vdash^{\mathsf{c}} M : A \quad \Gamma, \mathtt{x} : A \vdash^{\mathsf{c}} N : B}{\Gamma \vdash^{\mathsf{c}} M\ \mathtt{to}\ \mathtt{x}.\ N : B}$$

$$[\![M\ \mathtt{to}\ \mathtt{x}.\ N]\!] : \rho \longmapsto \mathsf{match}\ [\![M]\!]\rho\ \mathsf{as} \begin{cases} \mathsf{inl}\ a. & [\![N]\!](\rho, \mathtt{x} \mapsto a) \\ \mathsf{inr}\ e. & \mathsf{inr}\ e \end{cases}$$

Syntax

For connectives $\mathtt{bool}, +, \to$ the syntax is as follows.

$$
\begin{array}{rcl}
V & ::= & \mathtt{x} \mid \mathtt{true} \mid \mathtt{false} \\
& & \mid \mathtt{inl}\ V \mid \mathtt{inr}\ V \mid \lambda \mathtt{x}.\, M \\
M & ::= & M\ \mathtt{to}\ \mathtt{x}.\ M \mid \mathtt{return}\ V \\
& & \mid \mathtt{let}\ (\overrightarrow{\mathtt{x}\ \mathtt{be}\ V}).\ M \mid V\ V \\
& & \mid \mathtt{match}\ V\ \mathtt{as}\ \{\mathtt{true}.\, M,\ \mathtt{false}.\, M\} \\
& & \mid \mathtt{match}\ V\ \mathtt{as}\ \{\mathtt{inl}\ \mathtt{x}.\, M,\ \mathtt{inr}\ \mathtt{x}.\, M\} \\
& & \mid \mathtt{error}\ e
\end{array}
$$

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V & ::= & \mathtt{x} \mid \mathtt{true} \mid \mathtt{false} \\
& & \mid \mathtt{inl}\ V \mid \mathtt{inr}\ V \mid \lambda \mathtt{x}.\, M \\
M & ::= & M\ \mathtt{to}\ \mathtt{x}.\ M \mid \mathtt{return}\ V \\
& & \mid \mathtt{let}\ (\overrightarrow{\mathtt{x}\ \mathtt{be}\ V}).\ M \mid V\, V \\
& & \mid \mathtt{match}\ V\ \mathtt{as}\ \{\mathtt{true}.\, M,\ \mathtt{false}.\, M\} \\
& & \mid \mathtt{match}\ V\ \mathtt{as}\ \{\mathtt{inl}\ \mathtt{x}.\, M,\ \mathtt{inr}\ \mathtt{x}.\, M\} \\
& & \mid \mathtt{error}\ e
\end{array}
$$

We don't allow "complex values" such as

$$\mathtt{match}\ \mathtt{true}\ \mathtt{as}\ \{\mathtt{true}.\, \mathtt{false},\ \mathtt{false}.\, \mathtt{true}\}$$

These would complicate the operational semantics.

Definitional interpreter for fine-grain CBV

We evaluate a closed computation $\vdash^{\mathsf{c}} M : A$ to a closed value $\vdash^{\mathsf{v}} V : A$.
To evaluate

- $\mathtt{return}\ V$: return V.
- $M\ \mathtt{to}\ \mathtt{x}.\ N$, evaluate M. If this returns V, evaluate $N[V/\mathtt{x}]$.
- $\mathtt{let}\ (\mathtt{x}\ \mathtt{be}\ V,\ \mathtt{y}\ \mathtt{be}\ W).\ M$, evaluate $M[V/\mathtt{x}, W/\mathtt{y}]$.
- $(\lambda \mathtt{x}.\, M)\, V$, evaluate $M[V/\mathtt{x}]$.
- $\mathtt{match}\ \mathtt{inl}\ V\ \mathtt{as}\ \{\mathtt{inl}\ \mathtt{x}.\ N,\ \mathtt{inr}\ \mathtt{x}.\ N'\}$: evaluate $N[V/\mathtt{x}]$.

Equational theory

β-laws

$$\begin{aligned} \texttt{match}\ (\texttt{inl}\ V)\ \texttt{as}\ \{\texttt{true}.\, M, \texttt{false}.\, M'\} &= M[V/\texttt{x}] \\ (\lambda \texttt{x}.\, M)\, V &= M[V/\texttt{x}] \\ \texttt{let}\ (\texttt{x}\ \texttt{be}\ V,\ \texttt{y}\ \texttt{be}\ W).\ M &= M[V/\texttt{x}, W/\texttt{y}] \end{aligned}$$

η-laws

$$\begin{aligned} M[V/\texttt{z}] &= \texttt{match}\ V\ \texttt{as}\ \{\texttt{inl}\ \texttt{x}.\, M[\texttt{inl}\ \texttt{x}/\texttt{z}],\ \texttt{inr}\ \texttt{y}.\, M[\texttt{inr}\ \texttt{x}/\texttt{z}]\} \\ V &= \lambda \texttt{x}.\, V\texttt{x} \end{aligned}$$

Sequencing laws

$$\begin{aligned} (\texttt{return}\ V)\ \texttt{to}\ \texttt{x}.\ M &= M[V/\texttt{x}] \\ M &= M\ \texttt{to}\ \texttt{x}.\ \texttt{return}\ \texttt{x} \\ (M\ \texttt{to}\ \texttt{x}.\ N)\ \texttt{to}\ \texttt{y}.\ P &= M\ \texttt{to}\ \texttt{x}.\ (N\ \texttt{to}\ \texttt{y}.\ P) \end{aligned}$$

CBV to fine-grain call-by-value

Term $\Gamma \vdash M : A$ to computation $\Gamma \vdash^{\mathsf{c}} \hat{M} : A$.

$$\begin{aligned}
\mathtt{x} &\longmapsto \mathtt{return}\ \mathtt{x} \\
\lambda \mathtt{x}.\, M &\longmapsto \mathtt{return}\ \lambda \mathtt{x}.\, \hat{M} \\
\mathtt{inl}\ M &\longmapsto \hat{M}\ \mathtt{to}\ \mathtt{x}.\ \mathtt{return}\ \mathtt{inl}\ \mathtt{x} \\
M\, N &\longmapsto \hat{M}\ \mathtt{to}\ \mathtt{x}.\ \hat{N}\ \mathtt{to}\ \mathtt{y}.\ \mathtt{x}\,\mathtt{y} \\
\mathtt{let}\ (\mathtt{x}\ \mathtt{be}\ M,\ \mathtt{y}\ \mathtt{be}\ M').\ N &\longmapsto \hat{M}\ \mathtt{to}\ \mathtt{x}.\ \hat{M'}\ \mathtt{to}\ \mathtt{y}.\ \hat{N}
\end{aligned}$$

Value $\Gamma \vdash V : A$ to value $\Gamma \vdash^{\mathsf{v}} \check{V} : A$.

$$\begin{aligned}
\mathtt{x} &\longmapsto \mathtt{x} \\
\lambda \mathtt{x}.\, M &\longmapsto \lambda \mathtt{x}.\, \hat{M} \\
\mathtt{inl}\ V &\longmapsto \mathtt{inl}\ \check{V}
\end{aligned}$$

Nullary functions

Call-by-value programmers use nullary functions to delay evaluation, and call them thunks.

$$\begin{array}{rcl@{\qquad\qquad}rcl} TA & \stackrel{\text{def}}{=} & () \to A & [\![TA]\!] & = & [\![A]\!] + E \\ \texttt{thunk}\ M & \stackrel{\text{def}}{=} & \lambda().\, M & [\![\texttt{thunk}\ M]\!] & = & [\![M]\!] \\ \texttt{force}\ V & \stackrel{\text{def}}{=} & V\,() & [\![\texttt{force}\ V]\!] & = & [\![V]\!] \end{array}$$

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The type TA has a reversible rule $\dfrac{\Gamma \vdash^{\mathsf{c}} A}{\Gamma \vdash^{\mathsf{v}} TA}$

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The type TA has a reversible rule $\dfrac{\Gamma \vdash^{\mathsf{c}} A}{\Gamma \vdash^{\mathsf{v}} TA}$

Fine-grain CBV (unlike the monadic metalanguage) distinguishes computations from thunks.

Naive CBN semantics of errors

Each type denotes a set, a semantic domain for terms. For example:

$$\begin{aligned}
[\![\mathtt{bool} \to (\mathtt{bool} \to \mathtt{bool})]\!]_* &= (\mathbb{B} + E) \to ((\mathbb{B} + E) \to (\mathbb{B} + E)) \\
[\![\mathtt{bool} + \mathtt{bool}]\!]_* &= ((\mathbb{B} + E) + (\mathbb{B} + E)) + E \\
[\![\mathtt{bool} \,\pi\, \mathtt{bool}]\!]_* &= (\mathbb{B} + E) \times (\mathbb{B} + E)
\end{aligned}$$

Thus we define

$$\begin{aligned}
[\![\mathtt{bool}]\!]_* &= \mathbb{B} + E \\
[\![A + B]\!]_* &= ([\![A]\!]_* + [\![B]\!]_*) + E \\
[\![A \to B]\!]_* &= [\![A]\!]_* \to [\![B]\!]_* \\
[\![A \,\pi\, B]\!]_* &= [\![A]\!]_* \times [\![B]\!]_* \\
[\![\Gamma]\!] &= \prod_{(\mathtt{x}:A)\in\Gamma} [\![A]\!]_*
\end{aligned}$$

Each term $\Gamma \vdash M : B$ should denote a function $[\![M]\!] : [\![\Gamma]\!] \longrightarrow [\![B]\!]_*$.

Naive semantics: what goes wrong

$$\frac{}{\Gamma \vdash \texttt{error}\ \mathrm{CRASH} : B}$$

denotes $\rho \longmapsto$?

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Example:

- suppose $B = \texttt{bool} \to (\texttt{bool} \to \texttt{bool})$
- then B denotes $(\mathbb{B} + E) \to ((\mathbb{B} + E) \to (\mathbb{B} + E))$
- and $\texttt{error}\ \mathrm{CRASH} \simeq_{\mathbf{CBN}} \lambda \mathtt{x}.\ \lambda \mathtt{y}.\ \texttt{error}\ \mathrm{CRASH}$
- so the answer should be $\lambda x.\ \lambda y.$ inr CRASH.

Intuition: go down through the function types until we hit a tuple type.

Naive semantics: what goes wrong

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Example:

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- then B denotes $(\mathbb{B} + E) \to ((\mathbb{B} + E) \to (\mathbb{B} + E))$
- and $\texttt{error}\ \mathrm{CRASH} \simeq_{\mathbf{CBN}} \lambda \mathtt{x}.\ \lambda \mathtt{y}.\ \texttt{error}\ \mathrm{CRASH}$
- so the answer should be $\lambda x.\ \lambda y.\ \mathrm{inr}\ \mathrm{CRASH}$.

Intuition: go down through the function types until we hit a tuple type.
A similar problem arises with `match`.

Solution: E-pointed sets

Definition

An E-pointed set is a set X with two distinguished elements $c, b \in X$.

A type should denote an E-pointed set, a semantic domain for terms.

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Definition

An E-pointed set is a set X with two distinguished elements $c, b \in X$.

A type should denote an E-pointed set, a semantic domain for terms.
Examples:

$$\begin{aligned}
[\![\mathtt{bool} \to (\mathtt{bool} \to \mathtt{bool})]\!] &= ((\mathbb{B} + E) \to ((\mathbb{B} + E) \to (\mathbb{B} + E)),\\
&\quad \lambda \mathrm{x}.\lambda \mathrm{y}.\mathrm{inr\ CRASH},\\
&\quad \lambda \mathrm{x}.\lambda \mathrm{y}.\mathrm{inr\ BANG})\\
[\![\mathtt{bool} + \mathtt{bool}]\!] &= (((\mathbb{B} + E) + (\mathbb{B} + E)) + E,\\
&\quad \mathrm{inr\ CRASH},\\
&\quad \mathrm{inr\ BANG})\\
[\![\mathtt{bool} \,\Pi\, \mathtt{bool}]\!] &= ((\mathbb{B} + E) \times (\mathbb{B} + E),\\
&\quad (\mathrm{inr\ CRASH}, \mathrm{inr\ CRASH}),\\
&\quad (\mathrm{inr\ BANG}, \mathrm{inr\ BANG}))
\end{aligned}$$

CBN semantics of errors

$$[\![\texttt{bool}]\!] \quad = \quad (\mathbb{B} + E, \text{inr CRASH}, \text{inr BANG})$$

If $[\![A]\!] = (X, c, b)$ and $[\![B]\!] = (Y, c', b')$

then $[\![A + B]\!] = ((X + Y) + E, \text{inr CRASH}, \text{inr BANG})$

and $[\![A \to B]\!] = (X \to Y, \lambda x.\, c', \lambda x.\, b')$

and $[\![A \,\pi\, B]\!] = (X \times Y, (c, c'), (b, b'))$

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and $[\![A \to B]\!] = (X \to Y, \lambda x.\, c', \lambda x.\, b')$

and $[\![A \,\pi\, B]\!] = (X \times Y, (c, c'), (b, b'))$

$$[\![\Gamma]\!] \quad = \quad \prod_{\substack{(\mathtt{x}:A)\in\Gamma \\ [\![A]\!]=(X,c,b)}} X$$

A term $\Gamma \vdash M : B$ denotes a function $[\![M]\!] : [\![\Gamma]\!] \longrightarrow [\![B]\!]$.

Semantics of term constructors

$$\frac{}{\Gamma \vdash \texttt{true} : \texttt{bool}}$$

$$[\![\texttt{true}]\!] : \rho \longmapsto \mathsf{inl}\ \mathsf{true}$$

$$\frac{\Gamma \vdash M : \texttt{bool} \quad \Gamma \vdash N : B \quad \Gamma \vdash N' : B}{\Gamma \vdash \texttt{match}\ M\ \texttt{as}\ \{\texttt{true}.\ N,\ \texttt{false}.\ N'\} : B}$$

$$[\![\texttt{match}\ M\ \texttt{as}\ \{\texttt{true}.\ N,\ \texttt{false}.\ N'\}]\!] : \rho \quad \longmapsto$$

$$\mathsf{match}\ [\![M]\!]\rho\ \mathsf{as} \begin{cases} \mathsf{inl}\ \mathsf{true}. & [\![N]\!]\rho \\ \mathsf{inl}\ \mathsf{false}. & [\![N']\!]\rho \\ \mathsf{inr}\ \mathrm{CRASH}. & c \\ \mathsf{inr}\ \mathrm{BANG}. & b \end{cases} \qquad \text{where } [\![B]\!] = (Y, c, b)$$

More term constructors

$$
\begin{array}{lcccl}
[\![\lambda \mathtt{x}.M]\!] & : & \rho & \longmapsto & \lambda a.\, [\![M]\!](\rho, \mathtt{x} \mapsto a) \\
[\![M\ N]\!] & : & \rho & \longmapsto & [\![M]\!]\,[\![N]\!] \\
[\![\mathtt{x}]\!] & : & \rho & \longmapsto & \rho_{\mathtt{x}} \\
\mathtt{error}\ \mathrm{CRASH} & : & \rho & \longmapsto & c
\end{array}
$$

Soundness/adequacy

- If $M \Downarrow T$ then $[\![M]\!]\varepsilon = [\![T]\!]\varepsilon$.
- If $M \not\downarrow \mathrm{CRASH}$ then $[\![M]\!]\varepsilon = c$.
- If $M \not\downarrow \mathrm{BANG}$ then $[\![M]\!]\varepsilon = b$.

Proved by induction, using the substitution lemma.

Notation for E-pointed sets

- Free E-pointed set on a set X.

$$F^E X \stackrel{\text{def}}{=} (X + E, \text{inr CRASH}, \text{inr BANG})$$

- Product of two E-pointed sets.

$$(X, c, b) \amalg (Y, c', b') \stackrel{\text{def}}{=} (X \times Y, (c, c'), (b, b'))$$

- Unit E-pointed set. $1_\amalg \stackrel{\text{def}}{=} (1, (), ())$
- Product of a family of E-pointed sets.

$$\prod_{i \in I} (X_i, c_i, b_i) \stackrel{\text{def}}{=} (\prod_{i \in I} X_i, \lambda i.\, c_i, \lambda i.\, b_i)$$

- Exponential E-pointed set.

$$\begin{aligned} X \to (Y, c, b) &\stackrel{\text{def}}{=} \prod_{x \in X} (Y, c, b) \\ &= (X \to Y, \lambda x.\, c, \lambda x.\, b) \end{aligned}$$

- Carrier of an E-pointed set. $U^E(X, c, b) \stackrel{\text{def}}{=} X$

Summary of call-by-name semantics

A type denotes an E-pointed set.

$$\begin{aligned}
[\![\mathtt{bool}]\!] &= F^E(1+1) \\
[\![A+B]\!] &= F^E(U^E[\![A]\!] + U^E[\![B]\!]) \\
[\![A \to B]\!] &= U^E[\![A]\!] \to [\![B]\!] \\
[\![A \mathbin{\pi} B]\!] &= [\![A]\!] \mathbin{\pi} [\![B]\!]
\end{aligned}$$

A typing context denotes a set.

$$[\![\Gamma]\!] = \prod_{(\mathtt{x}:A)\in\Gamma} U^E[\![A]\!]$$

A term $\Gamma \vdash^{\mathsf{c}} M : \underline{B}$ denotes a function $[\![\Gamma]\!] \longrightarrow [\![\underline{B}]\!]$.

Summary of call-by-value semantics

A type denotes a set.

$$\begin{aligned}
[\![\mathtt{bool}]\!] &= 1+1 \\
[\![A+B]\!] &= [\![A]\!]+[\![B]\!] \\
[\![A\to B]\!] &= U^E([\![A]\!]\to F^E[\![B]\!]) \\
[\![TB]\!] &= U^E F^E[\![B]\!]
\end{aligned}$$

A typing context denotes a set.

$$[\![\Gamma]\!] = \prod_{(\mathtt{x}:A)\in\Gamma} [\![A]\!]$$

A computation $\Gamma \vdash^{\mathsf{c}} M : B$ denotes a function $[\![\Gamma]\!] \longrightarrow F^E[\![B]\!]$.

Call-By-Push-Value Types

Two kinds of type:

- A value type denotes a set.
- A computation type denotes an E-pointed set.

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value type $\quad A ::= \quad U\underline{B} \mid 1 \mid A \times A \mid 0 \mid A + A \mid \sum_{i \in \mathbb{N}} A_i$

computation type $\quad \underline{B} ::= \quad FA \mid A \to \underline{B} \mid 1_{\Pi} \mid \underline{B} \,\Pi\, \underline{B} \mid \prod_{i \in \mathbb{N}} \underline{B}_i$

Call-By-Push-Value Types

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Strangely function types are computation types, and $\lambda \mathrm{x}.M$ is a computation.

Judgements

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A context Γ is a finite set of identifiers with associated value type

$$\mathtt{x}_0 : A_0, \ldots, \mathtt{x}_{m-1} : A_{m-1}$$

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Two judgements:

- A value $\Gamma \vdash^{\mathsf{v}} V : A$ denotes a function $[\![V]\!] : [\![\Gamma]\!] \longrightarrow [\![A]\!]$.
- A computation $\Gamma \vdash^{\mathsf{c}} M : \underline{B}$ denotes a function $[\![M]\!] : [\![\Gamma]\!] \longrightarrow [\![\underline{B}]\!]$.

The type FA

A computation in FA aims to return a value in A.

$$\frac{\Gamma \vdash^{\mathsf{v}} V : A}{\Gamma \vdash^{\mathsf{c}} \mathtt{return}\ V : FA} \qquad \frac{\Gamma \vdash^{\mathsf{c}} M : FA \quad \Gamma, \mathtt{x} : A \vdash^{\mathsf{c}} N : \underline{B}}{\Gamma \vdash^{\mathsf{c}} M\ \mathtt{to}\ \mathtt{x}.\ N : \underline{B}}$$

Sequencing in the style of Filinski's "Effect-PCF".

The type FA

A computation in FA aims to return a value in A.

$$\frac{\Gamma \vdash^{\mathsf{v}} V : A}{\Gamma \vdash^{\mathsf{c}} \mathtt{return}\ V : FA} \qquad \frac{\Gamma \vdash^{\mathsf{c}} M : FA \quad \Gamma, \mathtt{x} : A \vdash^{\mathsf{c}} N : \underline{B}}{\Gamma \vdash^{\mathsf{c}} M\ \mathtt{to}\ \mathtt{x}.\ N : \underline{B}}$$

Sequencing in the style of Filinski's "Effect-PCF".

$$\begin{array}{lcll} [\![\mathtt{return}\ V]\!] & : & \rho & \longmapsto \ \mathsf{inl}\ [\![V]\!]\rho \\ [\![M\ \mathtt{to}\ \mathtt{x}.\ N]\!] & : & \rho & \longmapsto \end{array}$$

$$\mathsf{match}\ [\![M]\!]\rho\ \mathsf{as} \begin{cases} \mathsf{inl}\ a. & [\![N]\!](\rho, \mathtt{x} \mapsto a) \\ \mathsf{inr}\ \mathrm{CRASH}. & c \\ \mathsf{inr}\ \mathrm{BANG}. & b \end{cases}$$

$$\text{where } [\![\underline{B}]\!] = (Y, c, b)$$

The type $U\underline{B}$

A value in $U\underline{B}$ is a thunk of a computation in $\underline{B}$.

$$\frac{\Gamma \vdash^{\mathsf{c}} M : \underline{B}}{\Gamma \vdash^{\mathsf{v}} \mathtt{thunk}\ M : U\underline{B}} \qquad \frac{\Gamma \vdash^{\mathsf{v}} V : U\underline{B}}{\Gamma \vdash^{\mathsf{c}} \mathtt{force}\ V : \underline{B}}$$

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$$[\![\mathtt{thunk}\ M]\!] = [\![M]\!]$$

$$[\![\mathtt{force}\ V]\!] = [\![V]\!]$$

Identifiers

An identifier is a value.

$$\frac{}{\Gamma \vdash^{\mathsf{v}} \mathtt{x} : A}\ (\mathtt{x} : A) \in \Gamma$$

$$\frac{\Gamma \vdash^{\mathsf{v}} V : A \quad \Gamma \vdash^{\mathsf{v}} W : B \quad \Gamma, \mathtt{x} : A, \mathtt{y} : B \vdash^{\mathsf{c}} M : \underline{C}}{\Gamma \vdash^{\mathsf{c}} \mathtt{let}\ (\mathtt{x}\ \mathtt{be}\ V, \mathtt{y}\ \mathtt{be}\ W).\ M : \underline{C}}$$

Tuples

$$\frac{\Gamma \vdash^{\mathsf{v}} V : A}{\Gamma \vdash^{\mathsf{v}} \mathtt{inl}\ V : A + A'} \qquad \frac{\Gamma \vdash^{\mathsf{v}} V : A'}{\Gamma \vdash^{\mathsf{v}} \mathtt{inr}\ V : A + A'}$$

$$\frac{\Gamma \vdash^{\mathsf{v}} V : A + A' \quad \Gamma, \mathtt{x} : A \vdash^{\mathsf{c}} M : \underline{B} \quad \Gamma, \mathtt{y} : A \vdash^{\mathsf{c}} M' : \underline{B}}{\Gamma \vdash^{\mathsf{c}} \mathtt{match}\ V\ \mathtt{as}\ \{\mathtt{inl}\ \mathtt{x}.\, M, \mathtt{inr}\ \mathtt{y}.\, M'\} : \underline{B}}$$

$$\frac{\Gamma \vdash^{\mathsf{v}} V : A \quad \Gamma \vdash^{\mathsf{v}} V' : A'}{\Gamma \vdash^{\mathsf{v}} \langle V, V' \rangle : A \times A'} \qquad \frac{\Gamma \vdash^{\mathsf{v}} V : A \times A' \quad \Gamma, \mathtt{x} : A, \mathtt{y} : A' \vdash^{\mathsf{c}} M : \underline{B}}{\Gamma \vdash^{\mathsf{c}} \mathtt{match}\ V\ \mathtt{as}\ \langle \mathtt{x}, \mathtt{y} \rangle . M : \underline{B}}$$

The rules for 1 are similar.

Functions

$$\frac{\Gamma, \mathtt{x} : A \vdash^{\mathtt{c}} M : \underline{B}}{\Gamma \vdash^{\mathtt{c}} \lambda \mathtt{x}.M : A \to \underline{B}} \qquad \frac{\Gamma \vdash^{\mathtt{c}} M : A \to \underline{B} \quad \Gamma \vdash^{\mathtt{v}} V : A}{\Gamma \vdash^{\mathtt{c}} MV : \underline{B}}$$

$$\frac{\Gamma \vdash^{\mathtt{c}} M : \underline{B} \quad \Gamma \vdash^{\mathtt{c}} M' : \underline{B'}}{\Gamma \vdash^{\mathtt{c}} \lambda\{^{\mathtt{l}}.\, M,\ ^{\mathtt{r}}.\, M'\} : \underline{B} \,\&\, \underline{B'}}$$

$$\frac{\Gamma \vdash^{\mathtt{c}} M : \underline{B} \,\&\, \underline{B'}}{\Gamma \vdash^{\mathtt{c}} M^{\mathtt{l}} : \underline{B}} \qquad \frac{\Gamma \vdash^{\mathtt{c}} M : \underline{B} \,\&\, \underline{B'}}{\Gamma \vdash^{\mathtt{c}} M^{\mathtt{r}} : \underline{B'}}$$

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$$\frac{\Gamma, \mathtt{x} : A \vdash^{\mathtt{c}} M : \underline{B}}{\Gamma \vdash^{\mathtt{c}} \lambda \mathtt{x}.M : A \to \underline{B}} \qquad \frac{\Gamma \vdash^{\mathtt{c}} M : A \to \underline{B} \quad \Gamma \vdash^{\mathtt{v}} V : A}{\Gamma \vdash^{\mathtt{c}} MV : \underline{B}}$$

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It is often convenient to write applications operand-first, as $V \text{`} M$ and $^{\mathtt{l}}\text{`} M$ and $^{\mathtt{r}}\text{`} M$.

Definitional interpreter for call-by-push-value

The terminals are computations: $\texttt{return}\ V$ $\quad$ $\lambda \mathtt{x}.M$ $\quad$ $\lambda\{^{\mathtt{l}}.\, M,\ ^{\mathtt{r}}.\, M'\}$

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To evaluate

- $\mathtt{return}\ V$: return $\mathtt{return}\ V$.
- $M\ \mathtt{to}\ \mathtt{x}.\ N$: evaluate M. If this returns $\mathtt{return}\ V$, then evaluate $N[V/\mathtt{x}]$.
- $\lambda \mathtt{x}.N$: return $\lambda \mathtt{x}.N$.
- MV: evaluate M. If this returns $\lambda \mathtt{x}.N$, evaluate $N[V/\mathtt{x}]$.
- $\lambda\{^{\mathtt{l}}.\, M,\ ^{\mathtt{r}}.\, M'\}$: return $\lambda\{^{\mathtt{l}}.\, M,\ ^{\mathtt{r}}.\, M'\}$.
- $M^{\mathtt{l}}$: evaluate M. If this returns $\lambda\{^{\mathtt{l}}.\, N,\ ^{\mathtt{r}}.\, N'\}$, evaluate N.
- $\mathtt{let}\ (\mathtt{x}\ \mathtt{be}\ V,\ \mathtt{y}\ \mathtt{be}\ W).\ M$: evaluate $M[V/\mathtt{x}, W/\mathtt{y}]$.
- $\mathtt{force}\ \mathtt{thunk}\ M$: evaluate M.
- $\mathtt{match}\ \mathtt{inl}\ V\ \mathtt{as}\ \{\mathtt{inl}\ \mathtt{x}.\, M, \mathtt{inr}\ \mathtt{y}.\, M'\}$: evaluate $M[V/\mathtt{x}]$.
- $\mathtt{match}\ \langle V, V'\rangle\ \mathtt{as}\ \langle \mathtt{x}, \mathtt{y}\rangle.M$: evaluate $M[V/\mathtt{x}, V'/\mathtt{y}]$.
- $\mathtt{error}\ e$, print error message e and stop.

Equational theory

β-laws

$$\begin{aligned} \texttt{force thunk}\ M &= M \\ \texttt{match}\ (\texttt{inl}\ V)\ \texttt{as}\ \{\texttt{true}.\ M, \texttt{false}.\ M'\} &= M[V/\texttt{x}] \\ (\lambda \texttt{x}.\ M)\ V &= M[V/\texttt{x}] \\ \texttt{let}\ (\texttt{x be}\ V,\ \texttt{y be}\ W).\ M &= M[V/\texttt{x}, W/\texttt{y}] \end{aligned}$$

η-laws

$$\begin{aligned} V &= \texttt{thunk force}\ V \\ M[V/\texttt{z}] &= \texttt{match}\ V\ \texttt{as}\ \{\texttt{inl x}.\ M[\texttt{inl x}/\texttt{z}],\ \texttt{inr y}.\ M[\texttt{inr x}/\texttt{z}]\} \\ M &= \lambda \texttt{x}.\ M\texttt{x} \end{aligned}$$

Sequencing laws

$$\begin{aligned} (\texttt{return}\ V)\ \texttt{to x}.\ M &= M[V/\texttt{x}] \\ M &= M\ \texttt{to x}.\ \texttt{return x} \\ (M\ \texttt{to x}.\ N)\ \texttt{to y}.\ P &= M\ \texttt{to x}.\ (N\ \texttt{to y}.\ P) \end{aligned}$$

Decomposing CBV into CBPV

A CBV type translates into a value type.

$$\begin{aligned} A \to B &\longmapsto U(A \to FB) \\ TB &\longmapsto UFB \end{aligned}$$

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A fine-grain CBV computation $\mathtt{x} : A, \mathtt{y} : B \vdash^{\mathsf{c}} M : C$ translates as $\mathtt{x} : A, \mathtt{y} : B \vdash^{\mathsf{c}} M : FC$.

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Therefore a CBV term $\mathtt{x} : A, \mathtt{y} : B \vdash M : C$ translates as $\mathtt{x} : A, \mathtt{y} : B \vdash^{\mathsf{c}} M : FC$

$$\begin{aligned} \mathtt{x} &\longmapsto \mathtt{return}\ \mathtt{x} \\ \lambda \mathtt{x}.\, M &\longmapsto \mathtt{return}\ \mathtt{thunk}\ \lambda \mathtt{x}.\, M \\ M\ N &\longmapsto M\ \mathtt{to}\ \mathtt{f}.\, N\ \mathtt{to}\ \mathtt{y}.\, ((\mathtt{force}\ \mathtt{f})\ \mathtt{y}) \end{aligned}$$

Decomposing CBN into CBPV

A CBN type translates into a computation type.

$$\begin{array}{rcl} \texttt{bool} & \longmapsto & F(1+1) \\ \underline{A}+\underline{B} & \longmapsto & F(U\,\underline{A}+U\,\underline{B}) \\ \underline{A}\to\underline{B} & \longmapsto & U\,\underline{A}\to\underline{B} \end{array}$$

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A CBN term $\mathtt{x}:\underline{A},\mathtt{y}:\underline{B}\vdash M:\underline{C}$ translates as $\mathtt{x}:U\underline{A},\mathtt{y}:U\underline{B}\vdash^{\mathsf{c}} M:\underline{C}$.

$$\begin{aligned} \mathtt{x} &\longmapsto \mathtt{force}\ \mathtt{x} \\ \mathtt{let}\,(\mathtt{x}\ \mathtt{be}\ M,\ \mathtt{y}\ \mathtt{be}\ M').\,N &\longmapsto \mathtt{let}\,(\mathtt{x}\ \mathtt{be}\ \mathtt{thunk}\ M,\ \mathtt{y}\ \mathtt{be}\ \mathtt{thunk}\ M').\,N \\ \lambda\mathtt{x}.\ M &\longmapsto \lambda\mathtt{x}.\ M \\ M\ N &\longmapsto M\ (\mathtt{thunk}\ N) \\ \mathtt{inl}\ M &\longmapsto \mathtt{return}\ \mathtt{inl}\ \mathtt{thunk}\ M \end{aligned}$$

Summary

We've seen

- the call-by-push-value calculus
- its operational semantics
- denotational semantics for errors.

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- the call-by-push-value calculus
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But

- our error semantics makes `thunk` and `force` invisible
- we still don't understand why a function is a computation.

CK-machine

An operational semantics due to Felleisen and Friedman (1986).
And Landin, Krivine, Streicher and Reus, Bierman, Pitts, ...

It is suitable for sequential languages whether CBV, CBN or CBPV.

At any time, there's a computation (C) and a stack of contexts (K).

Initially, K is empty.

Some authors make K into a single context, called an "evaluation context".

Transitions for sequencing

To evaluate M `to x.` N: evaluate M. If this returns `return` V, then evaluate $N[V/\mathtt{x}]$.

M `to x.` N		K	$\rightsquigarrow$
M	`to x.` N ::	K	

`return` V	`to x.` N ::	K	$\rightsquigarrow$
$N[V/\mathtt{x}]$		K	

Transitions for application

To evaluate $V`M$: evaluate M. If this returns $\lambda \mathtt{x}.N$, evaluate $N[V/\mathtt{x}]$.

$V`M$		K	$\rightsquigarrow$
M		$V :: K$	

$\lambda \mathtt{x}.N$		$V :: K$	$\rightsquigarrow$
$N[V/\mathtt{x}]$		K	

Those function rules again

$V \text{`} M$		K	$\rightsquigarrow$
M	V	$:: K$	

$\lambda \mathtt{x}.N$	V	$:: K$	$\rightsquigarrow$
$N[V/\mathtt{x}]$		K	

Those function rules again

$V`M$		K	$\rightsquigarrow$
M		$V :: K$	

$\lambda \mathtt{x}.N$		$V :: K$	$\rightsquigarrow$
$N[V/\mathtt{x}]$		K	

We can read $V`$ as an instruction "push V".

We can read $\lambda \mathtt{x}$ as an instruction "pop $\mathtt{x}$".

Those function rules again

$V \text{`} M$		K	$\rightsquigarrow$
M		$V :: K$	

$\lambda \mathtt{x}.N$		$V :: K$	$\rightsquigarrow$
$N[V/\mathtt{x}]$		K	

We can read $V\text{`}$ as an instruction "push V".

We can read $\lambda\mathtt{x}$ as an instruction "pop $\mathtt{x}$".

Revisiting some equations:

$$\begin{aligned} V \text{ ` } \lambda \mathtt{x}.\ M &= M[V/\mathtt{x}] \\ M &= \lambda \mathtt{x}.\ \mathtt{x} \text{ ` } M \qquad (\mathtt{x} \text{ fresh}) \\ \mathtt{error}\ e &= \lambda \mathtt{x}.\ \mathtt{error}\ e \\ \mathtt{print}\ c.\ \lambda \mathtt{x}.M &= \lambda \mathtt{x}.\ \mathtt{print}\ c.\ M \end{aligned}$$

Values and Computations

A value is, a computation does.

- A value of type $U\underline{B}$ is a thunk of a computation of type $\underline{B}$.
- A value of type $A + A'$ is a tagged value $\texttt{inl}\ V$ or $\texttt{inr}\ V$.
- A value of type $A \times A'$ is a pair $\langle V, V' \rangle$.

- A computation of type FA aims to return a value of type A.
- A computation of type $A \to \underline{B}$ aims
  to pop a value of type A and then behave in $\underline{B}$.
- A computation of type $\underline{B} \,\&\, \underline{B}'$ aims
  to pop the tag `l` and then behave in $\underline{B}$
  or pop the tag `r` and then behave in $\underline{B}'$.

What's in a stack?

A stack consists of

- arguments that are values
- arguments that are tags
- frames taking the form $\texttt{to x.}\ N$.

Example program of type $F\,\mathtt{nat}$ (with complex values)

```
print "hello0".
let (x be 3,
     y be thunk (
          print "hello1".
          λz.
          print "we just popped " + z.
          return x + z
     )).
print "hello2".
( print "hello3".
  7‘
  print "we just pushed 7".
  force y
) to w.
print "w is bound to " + w.
return w + 5
```

Typing the CK-machine

Initial configuration to evaluate $\Gamma \vdash^{\mathsf{c}} P : \underline{C}$

Γ	P	$\underline{C}$	$\mathtt{nil}$	$\underline{C}$	

Transitions

Γ	M to x. N	$\underline{B}$	K	$\underline{C}$	$\rightsquigarrow$
Γ	M	$F\,A$	to x. $N :: K$	$\underline{C}$	

Γ	return V	$F\,A$	to x. $N :: K$	$\underline{C}$	$\rightsquigarrow$
Γ	$N[V/\mathtt{x}]$	$\underline{B}$	K	$\underline{C}$	

Typically Γ would be empty and $\underline{C} = F\,\mathtt{bool}$.

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Γ	P	$\underline{C}$	$\mathtt{nil}$	$\underline{C}$	

Transitions

Γ	$M \mathtt{\ to\ x.\ } N$	$\underline{B}$	K	$\underline{C}$	$\rightsquigarrow$
Γ	M	$F A$	$\mathtt{to\ x.\ } N :: K$	$\underline{C}$	

Γ	$\mathtt{return}\ V$	$F A$	$\mathtt{to\ x.\ } N :: K$	$\underline{C}$	$\rightsquigarrow$
Γ	$N[V/\mathtt{x}]$	$\underline{B}$	K	$\underline{C}$	

Typically Γ would be empty and $\underline{C} = F\,\mathtt{bool}$.

We write $\Gamma \vdash^{\mathsf{k}} K : \underline{B} \Longrightarrow \underline{C}$ to mean that K can accompany a computation of type $\underline{B}$ during evaluation.

Typing rules, read off from the CK-machine

Typing a stack

$$\frac{}{\Gamma \vdash^{\mathsf{k}} \mathtt{nil} : \underline{C} \Longrightarrow \underline{C}} \qquad \frac{\Gamma, \mathtt{x} : A \vdash^{\mathsf{c}} M : \underline{B} \qquad \Gamma \vdash^{\mathsf{k}} K : \underline{B} \Longrightarrow \underline{C}}{\Gamma \vdash^{\mathsf{k}} \mathtt{to}\ \mathtt{x}.\ M :: K : FA \Longrightarrow \underline{C}}$$

$$\frac{\Gamma \vdash^{\mathsf{k}} K : \underline{B} \Longrightarrow \underline{C}}{\Gamma \vdash^{\mathsf{k}} \mathtt{1} :: K : \underline{B} \sqcap \underline{B'} \Longrightarrow \underline{C}} \qquad \frac{\Gamma \vdash^{\mathsf{v}} V : A \qquad \Gamma \vdash^{\mathsf{k}} K : \underline{B} \Longrightarrow \underline{C}}{\Gamma \vdash^{\mathsf{k}} V :: K : A \to \underline{B} \Longrightarrow \underline{C}}$$

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Typing a CK-configuration

$$\frac{\Gamma \vdash^{\mathsf{c}} M : \underline{B} \qquad \Gamma \vdash^{\mathsf{k}} K : \underline{B} \Longrightarrow \underline{C}}{\Gamma \vdash^{\mathsf{ck}} (M, K) : \underline{C}}$$

Operations on Stacks

1. Given a stack $\Gamma \vdash^{\mathsf{k}} K : \underline{B} \Longrightarrow \underline{C}$, we can weaken it or substitute values.

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3. Stacks $\Gamma \vdash^{\mathsf{k}} K : \underline{B} \Longrightarrow \underline{C}$ and $\Gamma \vdash^{\mathsf{k}} L : \underline{C} \Longrightarrow \underline{D}$ can be concatenated to give $\Gamma \vdash^{\mathsf{k}} K +\!\!+ L : \underline{B} \Longrightarrow \underline{D}$.

Special Stacks

Continuations

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Top-Level Stack

The top-level stack is $\Gamma \vdash^{\mathsf{k}} \texttt{nil} : \underline{C} \Longrightarrow \underline{C}$.
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The top-level stack is $\Gamma \vdash^{\mathsf{k}} \texttt{nil} : \underline{C} \Longrightarrow \underline{C}$.
The top-level type is $\underline{C}$.

If $\underline{C}$ is an F type, then `nil` is the top-level continuation:
it receives a value and returns it to the user.

Stacks denote homomorphisms

Consider a stack $\Gamma \vdash^{\mathsf{k}} K : \underline{B} \Longrightarrow \underline{C}$

where $[\![\underline{B}]\!] = (X, c, b)$ and $[\![\underline{C}]\!] = (Y, c', b')$.

What should K denote?

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because if M throws an error then so does $M \bullet K$.

We assume there's no exception handling.

Operations on stacks

We define $[\![K]\!]$ by induction on K.

Then we prove

- a weakening lemma
- a substitution lemma
- a dismantling lemma
- a concatenation lemma

providing a semantic counterpart for each operation on stacks.

Soundness of CK-machine

What should a CK-configuration $\Gamma \vdash^{\mathsf{ck}} (M, K) : \underline{C}$ denote?

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What should a CK-configuration $\Gamma \vdash^{\mathsf{ck}} (M, K) : \underline{C}$ denote?

$$\begin{array}{rcccc} [\![(M,K)]\!] & : & [\![\Gamma]\!] & \longrightarrow & [\![\underline{C}]\!] \\ & & \rho & \longmapsto & [\![K]\!](\rho, [\![M]\!]\rho) \end{array}$$

Properties:

1. If $(M, K) \rightsquigarrow (M', K')$ then $[\![(M, K)]\!] = [\![(M', K')]\!]$.
2. $[\![(\texttt{error CRASH}, K)]\!]\rho = c'$.
3. $[\![(\texttt{error BANG}, K)]\!]\rho = b'$.

Adjunction between values and stacks

We have an adjunction between the category of values (sets and functions) and the category of stacks (E-pointed sets and homomorphisms).

$$\mathbf{Set} \underset{U^E}{\overset{F^E}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} E/\mathbf{Set}$$

This resolves the exception monad $X \longmapsto X + E$ on $\mathbf{Set}$.

State

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$\mathtt{l}$ stores a natural number, and $\mathtt{l}'$ stores a boolean.

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$$\frac{\Gamma \vdash^{\mathsf{v}} V : \mathtt{nat} \quad \Gamma \vdash^{\mathsf{c}} M : \underline{B}}{\Gamma \vdash^{\mathsf{c}} \mathtt{l} := V.\ M : \underline{B}} \qquad \frac{\Gamma, \mathtt{x} : \mathtt{nat} \vdash^{\mathsf{c}} M : \underline{B}}{\Gamma \vdash^{\mathsf{c}} \mathtt{read\ l\ as\ x}.\ M : \underline{B}}$$

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A state is $\mathtt{l} \mapsto n, \mathtt{l}' \mapsto b$.

The set of states is $S \cong \mathbb{N} \times \mathbb{B}$.

Big-step semantics for state

The big-step semantics takes the form $s, M \Downarrow s', T$.

A pair (s, M) is called an SC-configuration.

We can type these using

$$\frac{\Gamma \vdash^{\mathsf{c}} M : \underline{B}}{\Gamma \vdash^{\mathsf{sc}} (s, M) : \underline{B}}\, s \in S$$

Denotational semantics of state

How can we give a denotational semantics for call-by-push-value with state?

- Algebra semantics.
- Intrinsic semantics.

Algebra semantics for state (briefly)

Moggi's monad for state is $S \to (S \times -)$.
Its Eilenberg-Moore algebras were characterized by Plotkin and Power.

Algebra semantics for state (briefly)

Moggi's monad for state is $S \to (S \times -)$.
Its Eilenberg-Moore algebras were characterized by Plotkin and Power.

A value type A denotes a set $[\![A]\!]$, a semantic domain for values.

A computation type $\underline{B}$ denotes an Eilenberg-Moore algebra $[\![\underline{B}]\!]_{\mathsf{alg}}$,
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We complete the story with an adequacy theorem:

$$\text{If } s, M \Downarrow s', T \text{ then } [\![s, M]\!]\varepsilon = [\![s', T]\!]\varepsilon$$

This requires an SC-configuration to have a denotation.

Intrinsic semantics of state

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The behaviour of a computation $\Gamma \vdash^{\mathsf{c}} M : \underline{B}$ depends on the state and environment:

$$[\![M]\!] : S \times [\![\Gamma]\!] \longrightarrow [\![\underline{B}]\!]$$

State: semantics of types

An SC-configuration of type FA will terminate as $s, \mathtt{return}\ V$.

$$[\![FA]\!] = S \times [\![A]\!]$$

An SC-configuration of type $A \to \underline{B}$ will pop $\mathtt{x} : A$ and then behave in $\underline{B}$.

$$[\![A \to \underline{B}]\!] = [\![A]\!] \to [\![\underline{B}]\!]$$

An SC-configuration of type $\underline{B} \,\&\, \underline{B}'$ will pop $\mathtt{l}$ and then behave in $\underline{B}$, or pop $\mathtt{r}$ and then behave in $\underline{B}'$.

$$[\![\underline{B} \,\&\, \underline{B}']\!] = [\![\underline{B}]\!] \times [\![\underline{B}']\!]$$

A value $\Gamma \vdash^{\mathsf{v}} V : U\underline{B}$ can be forced in any state s, giving an SC-configuration $s, \mathtt{force}\ V$.

$$[\![U\underline{B}]\!] = S \to [\![\underline{B}]\!]$$

State: the value/stack adjunction

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$$\mathbf{Set} \underset{S \to -}{\overset{S \times -}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{Set}$$

between values and stacks.

State in call-by-value and call-by-name

For call-by-value we recover

$$\begin{aligned} [\![\mathtt{bool}_{\mathbf{CBV}}]\!] &= 1+1 \\ [\![A \to_{\mathbf{CBV}} B]\!] &= [\![U(A \to FB)]\!] \\ &= S \to ([\![A]\!] \to (S \times [\![B]\!])) \end{aligned}$$

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For call-by-name we recover

$$\begin{aligned} [\![\mathtt{bool}_{\mathbf{CBN}}]\!] &= [\![F(1+1)]\!] \\ &= S \times (1+1) \\ [\![\underline{A} \to_{\mathbf{CBN}} \underline{B}]\!] &= [\![U\,\underline{A} \to \underline{B}]\!] \\ &= (S \to [\![\underline{A}]\!]) \to [\![\underline{B}]\!] \end{aligned}$$

This is O'Hearn's semantics of types for a stateful CBN language.

Naming and changing the current stack

Extend the language with two instructions:

- `letstk` α means let α be the current stack.
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Execution takes places in a bigger language.

Γ	`letstk` $\alpha.\ M$	$\underline{B}$	K	$\underline{C} \mid \Delta$	$\rightsquigarrow$
Γ	$M[K/\alpha]$	$\underline{B}$	K	$\underline{C} \mid \Delta$	

Γ	`changestk` $K.\ M$	$\underline{B'}$	L	$\underline{C} \mid \Delta$	$\rightsquigarrow$
Γ	M	$\underline{B}$	K	$\underline{C} \mid \Delta$	

Similar to Crolard's syntax. Numerous variations in the literature.

Typing judgements for control

We have typing judgements:

$$\Gamma \vdash^{\mathsf{v}} V : A \mid \Delta \qquad\qquad \Gamma \vdash^{\mathsf{c}} M : \underline{B} \mid \Delta$$

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Example typing rules

$$\frac{\Gamma \vdash^{\mathsf{c}} M : \underline{B} \mid \Delta, \alpha : \underline{B}}{\Gamma \vdash^{\mathsf{c}} \mathtt{letstk}\ \alpha.\ M \mid \Delta}$$

$$\frac{\Gamma \vdash^{\mathsf{c}} M : \underline{B} \mid \Delta}{\Gamma \vdash^{\mathsf{c}} \mathtt{changestk}\ \alpha.\ M : \underline{B'} \mid \Delta}\ (\alpha : \underline{B}) \in \Delta$$

Typing judgements for execution language

During execution, the top-level type $\underline{C}$ must be indicated:

$$\Gamma \vdash^{\mathsf{v}} V : A \; [\underline{C}] \; \Delta \qquad \Gamma \vdash^{\mathsf{c}} M : \underline{B} \; [\underline{C}] \; \Delta$$

$$\Gamma \vdash^{\mathsf{k}} K : \underline{B} \Longrightarrow \underline{C} \mid \Delta \qquad \Gamma \vdash^{\mathsf{ck}} (M, K) : \underline{C} \mid \Delta$$

Typically Γ and Δ would be empty and $\underline{C} = F\,\mathtt{bool}$.

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Algebra semantics of control

Fix a set R, the semantic domain for CK-configurations.

That means: a hypothetical extremely closed CK-configuration,
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Moggi's monad for control operators ("continuations") is $(- \to R) \to R$.

Maybe we can build a denotational semantics
where a computation type $\underline{B}$ denotes an Eilenberg-Moore algebra $[\![\underline{B}]\!]_{\mathsf{alg}}$,
a semantic domain for computations.

Intrinsic semantics of control

The denotation of $\underline{B}$ is a semantic domain for stacks from $\underline{B}$.

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The behaviour of a computation $\Gamma \vdash^{\mathsf{c}} M : \underline{B} \mid \Delta$ depends on the environment, current stack and stack environment:

$$[\![M]\!] : [\![\Gamma]\!] \times [\![\underline{B}]\!] \times [\![\Delta]\!] \longrightarrow R$$

A value $\Gamma \vdash^{\mathsf{v}} V : A \mid \Delta$ denotes

$$[\![V]\!] : [\![\Gamma]\!] \times [\![\Delta]\!] \longrightarrow [\![A]\!]$$

Control: semantics of types

A stack from FA receives a value $\mathtt{x} : A$ and then behaves as a configuration.

$$[\![FA]\!] = [\![A]\!] \to R$$

A stack from $A \to \underline{B}$ is a pair $V :: K$.

$$[\![A \to \underline{B}]\!] = [\![A]\!] \times [\![\underline{B}]\!]$$

A stack from $\underline{B} \amalg \underline{B}'$ is a tagged stack $^{\mathtt{l}} :: K$ or $^{\mathtt{r}} :: K$.

$$[\![\underline{B} \amalg \underline{B}']\!] = [\![\underline{B}]\!] + [\![\underline{B}']\!]$$

A value of type $U\underline{B}$ can be forced alongside any stack K, giving a configuration.

$$[\![U\underline{B}]\!] = [\![\underline{B}]\!] \to R$$

Semantics of the execution language

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depends not only on the environment and the stack environment
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That gives an adjunction

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between values and stacks.

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This is Streicher and Reus' semantics
for a CBN language with control operators.

Summary: adjunctions between values and stacks

For a set E, the adjunction $\mathbf{Set} \underset{U^E}{\overset{F^E}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} E/\mathbf{Set}$

models call-by-push-value with errors.

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