The price of mathematical scepticism

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Abstract

This paper argues that, insofar as we doubt the bivalence of the Continuum Hypothesis or the truth of the Axiom of Choice, we should also doubt the consistency of third-order arithmetic, both the classical and intuitionistic versions.

Underlying this argument is the following philosophical view. Mathematical belief springs from certain intuitions, each of which can be either accepted or doubted in its entirety, but not half-accepted. Therefore, our beliefs about reality, bivalence, choice and consistency should all be aligned.

1 Introduction

1.1 Theory vs reality

According to a widely held "classical" view of mathematics, the Continuum Hypothesis (CH) is *bivalent*, i.e. either objectively true or objectively false, even if it is absolutely unknowable which of these is the case. Furthermore, the Axiom of Choice (AC) is true, and therefore also the weaker principle known as Dependent Choice (DC).

There are many other views, however. Here are two that you may have encountered:

- "There is no canonical universe of mathematical reality, but rather many universes of equal status. All of them satisfy the ZFC axioms, but CH holds in some of them and fails in others."
- "AC is unacceptable because it leads to the Banach–Tarski theorem. Therefore ZF+DC should be adopted as a foundational theory."

Each favours a strong foundational theory (at least ZF), yet at the same time is sceptical of the classical conception.

In this article, I shall present an argument that we cannot "have our cake and eat it" in this way. Although scepticism is legitimate, it comes at a price.

Before this is spelt out, we need some technical preliminaries.

Firstly, let us note that both CH and the Banach–Tarski theorem are *third-order arithmetical* sentences, meaning that—with suitable coding—each quantifier ranges over $\mathbb{N}$ or $\mathcal{P}\mathbb{N}$ or $\mathcal{P}\mathcal{P}\mathbb{N}$, but nothing more complex. Accordingly, to discuss these sentences, we need not consider advanced theories such as ZF. Let us merely consider Z_3, the

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theory of third-order arithmetic. (Technically, it is a 3-sorted first-order theory, with Extensionality axioms and unrestricted Comprehension and Induction schemes.)

Secondly, say that a relation R from a set A to a set B is *entire* when, for all $a \in A$, there is $b \in B$ such that $R(a, b)$. Then AC and DC are stated as follows.

AC For any sets A and B, and any entire relation R from A to B, there is a function $f \in B^A$ such that, for all $a \in A$, we have $R(a, f(a))$.

DC For any set B, and any $b \in B$ and entire endorelation R on B (i.e. relation from B to itself), there is a sequence $(x_n)_{n \in \mathbb{N}} \in B^{\mathbb{N}}$ such that $x_0 = b$ and, for all $n \in \mathbb{N}$, we have $R(x_n, x_{n+1})$.

The contention of this article is that, insofar as we doubt CH bivalence or AC, we should also doubt the consistency of Z_3. Likewise, doubting DC leads to doubt in the consistency of Z_2, the theory of second-order arithmetic.

Note In some fields of mathematics, such as topos theory, it is common to avoid using AC and other classical principles, in order to gain information about interesting models where these principles fail. (See [28] for a recent example that actually relies on AC being true in reality.) Since this practice is not motivated by scepticism, it is philosophically uncontroversial and does not bear on our discussion.

1.2 Structure of paper

We proceed as follows. We begin (Section 2) with a general discussion of belief and doubt, and I set out the paper's fundamental principles—which are open to dispute, of course. We then classify mathematical beliefs (Section 3), based on the bivalence of different kinds of sentence. We consider the intuitions that may give rise to each of the positions (Section 4) and discuss which of them are reliable (Section 5). I then argue that the various ways of answering this question lead to the claimed consequences (Section 6). I also consider intuitionistic theories (Section 7), and critique the view that "reality is indeterminate" (Section 8). The issue of encoding is addressed in Section 9.

We then review some literature, looking at positions similar to the one I am proposing (Section 10) and ones that conflict with it (Section 11).

Section 12 concludes, with a mention of future work.

2 Principles of justified belief

The words "doubt" and "scepticism" have various shades of meaning in English. In this article, they refer to a lack of belief in X, not to a belief that X is false. So please do not interpret me as saying that CH bivalence or AC sceptics should believe Z_3 to be inconsistent; they should not.

The following examples illustrate our basic principles concerning belief and doubt. To avoid irrelevant infinity issues, let $\mathbb{N}_{\mathsf{G}}$ be the set of *Googolplex-bounded numbers*, i.e. natural numbers less than $10^{10^{100}}$.

1. Consider the statement "Cleopatra ate an even number of grapes". There is no evidence for or against. A person who believes this may happen to be right, but

their belief is nevertheless arbitrary and unjustified. So the correct position is to doubt it.

2. The *Googolplex Goldbach* conjecture says that every even Googolplex-bounded number other than 0 and 2 is a sum of two primes. We do not know whether this is true, so we doubt it.

   What would cause us to believe it? Either a proof, or intuition, or a combination of the two. These are (we shall suppose) the only acceptable grounds for belief. Furthermore, appeals to intuition raise the tricky question of which intuitions are reliable.

   We might be tempted towards belief by the fact [65] that the Goldbach property has been verified up to 4×10^{18}. And it is especially tempting to believe that the property holds for the least even number that has not yet been checked.[1] But even this proposition—call it *Liminal Goldbach*—might be false for all we know. So we doubt it.

   This illustrates a general principle: inductive evidence, however strong, is not sufficient grounds for belief. Mathematicians throughout the ages have largely agreed on this point.

   (A possible objection: it is common mathematical practice to trust a proof checked by a computer or another person, and this relies on inductive inference. We are ignoring such issues.)

3. Consistency statements are not essentially different from statements about prime numbers. Recall that a theory T is *consistent* when False has no T-proof. Likewise, let us say that T is *Googolplex consistent* when False has no T-proof whose length is Googolplex-bounded. I shall assume without further comment that proof length has been precisely defined for each of our theories.

   Consider the statement "Z_3 is Googolplex consistent". As before, our default position is to doubt it, and only proof or intuition will give us adequate grounds to believe it. Gödel's second incompleteness theorem and similar results do not justify relaxing this policy.

   One sometimes hears the following argument for consistency: "Many clever people have used this theory and studied its foundations for years, and found no contradiction." Since this is an inductive inference, it is not sufficient grounds for belief.[2]

To summarize:

- For any statement, our default position is doubt.
- Only proof and/or intuition will move us to a state of belief.
- We need to decide which intuitions are reliable.

[1] As of 16 January 2022, this number is $4.01 \times 10^{18} + 4$. (Personal communication from Tomás Oliveira e Silva.)

[2] As Hamkins [40] points out: the negation of Fermat's Last Theorem turned out to be inconsistent, even though, before Wiles, many clever people had looked seriously and been unable to refute it.

- Inductive inference is not accepted.
- These principles apply, in particular, to consistency statements.

3 The bivalence questionnaire

Our investigation of mathematical reality begins with a list of sentences whose truth value is unknown. The details of each sentence are not so important, but please pay attention to the *logical form*. The list is as follows.

Physical sentences which concern the physical universe.

- The *Cleopatra Hypothesis*: Cleopatra ate an even number of grapes.

Computational sentences where quantifier ranges are finite.

- The *Googolplex Goldbach conjecture*: Every even Googolplex-bounded number other than 0 and 2 is a sum of two primes.

Arithmetical sentences where quantifiers range over all natural numbers.

- The *Goldbach conjecture* [97]: Every even natural number other than 0 and 2 is a sum of two primes. This has the form $\forall n \in \mathbb{N}.\, \phi(n)$, where ϕ is computational.
- The *twin prime conjecture* [101]: There are infinitely many $n \in \mathbb{N}$ such that both n and $n+2$ are prime. This has the form $\forall n \in \mathbb{N}.\, \exists m \in \mathbb{N}. \phi(m, n)$, where ϕ is computational.

Second-order arithmetical sentences where quantifiers range over all sets of natural numbers.

- The *Littlewood conjecture* [99]: For any real numbers α and β, we have $\liminf_{n\to\infty} n\|n\alpha\|\|n\beta\| = 0$, where $\|\ \|$ is the distance to the nearest integer. This has the form $\forall x \in \mathcal{P}\mathbb{N}.\, \phi(x)$, where ϕ is arithmetical, because a real number can be encoded as a set of natural numbers, and a pair of subsets of $\mathbb{N}$ can be encoded as a single subset.
- The *Toeplitz conjecture* [98]: Every simple closed curve contains all four vertices of some square. This has the form $\forall x \in \mathcal{P}\mathbb{N}.\, \exists y \in \mathcal{P}\mathbb{N}.\, \phi(x, y)$, where ϕ is arithmetical, because a continuous function on $\mathbb{R}$ can be encoded as a continuous function on $\mathbb{Q}$.

Third-order arithmetical sentences where quantifiers range over all sets of sets of natural numbers.

- The *Continuum Hypothesis* [96]: There is a bijection from $\aleph_1$ to $2^{\mathbb{N}}$. This has the form $\exists x \in \mathcal{P}\mathcal{P}\mathbb{N}.\, \phi(x)$, where ϕ is second-order arithmetical, because an element of $\aleph_1$ can be encoded (non-uniquely) as a well-ordered subset of $\mathbb{N}$.

- The *Suslin Hypothesis* [100, 36]: The tree $\{0,1\}^{<\omega_1}$ has no subtree in which every chain and every antichain is countable. This has the form $\neg\exists x \in \mathcal{P}\mathcal{P}\mathbb{N}. \forall y \in \mathcal{P}\mathcal{P}\mathbb{N}.\, \phi(x,y)$, where ϕ is second-order arithmetical, because an element of the tree can be encoded (non-uniquely) as a tuple $\langle X, \prec, Y\rangle$, where $\langle X, \prec\rangle$ is a well-ordered subset of $\mathbb{N}$, and $Y \subseteq X$ indicates which elements of X are mapped to 1.

Unrestricted set-theoretic sentences where quantifiers range over all sets or all ordinals.

- The *Generalized Continuum Hypothesis* [96]: Every infinite cardinal κ satisfies $\kappa^+ = 2^\kappa$. This has the form $\forall\alpha \in \mathsf{Ord}.\, \phi(\alpha)$, where ϕ uses only restricted quantifiers,[3] since cardinals can be encoded as initial ordinals.
- The *Eventually Generalized Continuum Hypothesis*: There is an infinite cardinal λ such that every cardinal $\kappa \geqslant \lambda$ satisfies $\kappa^+ = 2^\kappa$. This has the form $\exists\alpha \in \mathsf{Ord}.\, \forall\beta \in \mathsf{Ord}.\, \phi(\alpha,\beta)$, where ϕ uses only restricted quantifiers.

Class-theoretic sentences where quantifiers range over all classes.

- The *Club-Failure Hypothesis* considered by Schlutzenberg [81]: Every club class of infinite cardinals has a member whose successor cardinal κ is a GCH failure, i.e. satisfies $\kappa^+ < 2^\kappa$. Writing $\mathsf{Class}(\mathsf{Ord})$ for the collection of classes of ordinals, this has the form $\forall X \in \mathsf{Class}(\mathsf{Ord}).\, \phi(X)$, where ϕ is set-theoretic.
- The *Ord–Suslin Hypothesis* considered by Hamkins and Switzer [88]: The tree $\{0,1\}^{<\mathsf{Ord}}$ has no subtree in which every chain and every antichain is a set. This has the form $\neg\exists X \in \mathsf{Class}(\mathsf{Ord}).\, \forall Y \in \mathsf{Class}(\mathsf{Ord}).\, \phi(X,Y)$, where ϕ is set-theoretic.

For each kind of sentence other than physical and computational, I have given two examples. The first has just one quantifier ($\forall$ or $\exists$) of the specified kind, and the second has two ($\forall\exists$ or $\exists\forall$). Such sentence forms are often described in the terminology of logical complexity: the Goldbach conjecture is Π^0_1, the twin prime conjecture Π^0_2, the Littlewood conjecture Π^1_1 and so forth. Note that the Goldbach conjecture (like every Π^0_1 sentence) is "falsifiable", meaning that someone who asserts it runs the risk of being hit by a counterexample.

Now I am going to interrogate you. Assume pessimistically that these sentences cannot be proved or refuted *in any convincing way* within the lifetime of the universe. Under this assumption (which may be correct for all we know), which of these sentences do you consider to be bivalent? In other words, do you think that—despite our hopeless ignorance—there is a fact of the matter whether Cleopatra ate an even number of grapes? Whether every even Googolplex-bounded number other than 0 and 2 is a sum of two primes? And so forth.

Let me stress that "bivalence ambivalence" is allowed and even encouraged. I dare say that few people would be certain of their answer for all the sentences.

Questionnaires of this kind have often appeared [31, 93]. They provide a crude but useful device to measure a person's belief in objective reality, a belief known as "realism" or "platonism". (These words will be used interchangeably.)

[3] The sense is more liberal than that of the limited ZF syntax, and allows $\mathcal{P}$.

It is time to name various philosophical positions.

- An *ultrafinitist* [17, 66] doubts that computational sentences are bivalent.
- A *finitist* accepts this, but doubts that arithmetical sentences are bivalent.
- A *countabilist* accepts this, but doubts that second-order arithmetical sentences are bivalent.
- A *sequentialist* accepts this and also DC, but doubts that third-order arithmetical sentences are bivalent.
- A *particularist* accepts this and also AC, but doubts that sentences that quantify over all sets or ordinals are bivalent.
- A *totalist* accepts all the above, but doubts that sentences that quantify over all classes are bivalent.

The above taxonomy immediately raises questions. Is this all just a choice between various coloured pills? Why not have an option for someone who accepts that 17th order arithmetical sentences are bivalent but not 18th order? Or for someone who accepts that Π^0_{52} sentences are bivalent but not Π^0_{53} ones?

To answer these questions, recall that belief should not be arbitrary. Furthermore (according to our principles), it should be justified by proof or intuition. So we cannot be mere "truth value realists", believing for no reason that sentences of a certain kind are bivalent. What, then, are the intuitions that support the various positions?

4 Intuitions of mathematical reality

I now present five intuitions that I experience, and hopefully you do too. They are little people inside our head, and each of them is going to speak. For the moment, just listen to them. We postpone the question of whether they are reliable.

Googolplex "I perceive the notion of Googolplex-bounded number. Since this is a clearly defined notion, quantification over the set $\mathbb{N}_{\mathsf{G}}$ yields an objective truth value."

Arbitrary Natural Number "I perceive the notion of a natural number, given by zero and successor and nothing more. This is a clearly defined notion, as restrictive as possible. So quantification over the set $\mathbb{N}$ yields an objective truth value."

Arbitrary Sequence "Given a set B, I perceive the notion of a sequence $(x_n)_{n\in\mathbb{N}}$ in B, which consists of successive arbitrary choices of an element of B. This is a clearly defined notion, as liberal as possible. So quantification over the set $B^{\mathbb{N}}$ yields an objective truth value. Since a sequence consists of successive arbitrary choices, DC holds."

Arbitrary Function "Given sets A and B, I perceive the notion of a function f from A to B, which consists of independent arbitrary choices $f(a) \in B$, one for each $a \in A$. This is a clearly defined notion, as liberal as possible. So quantification over the set B^A yields an objective truth value. Since a function consists of independent arbitrary choices, AC holds."

Arbitrary Ordinal "I perceive the notion of an ordinal. This is a clearly defined notion, as liberal as possible. So quantification over the class Ord yields an objective truth value."

Let me stress that AC is integral to the intuition of a function as consisting of independent arbitrary choices; it is not a separate intuition. Likewise, DC is integral to the intuition of a sequence as consisting of successive arbitrary choices.

Ultrafinitists accept none of these intuitions. Finitists accept the first one, countabilists the first two, sequentialists the first three, particularists the first four, and totalists all five. To see the link to Z_2 and Z_3, note that for any set A we have a bijection $\mathcal{P}A \cong \{0,1\}^A$ that represents each subset C of A by its characteristic function.[4] So we have bijections $\mathcal{P}\mathbb{N} \cong \{0,1\}^{\mathbb{N}}$ and $\mathcal{P}\mathcal{P}\mathbb{N} \cong \{0,1\}^{\{0,1\}^{\mathbb{N}}}$.

The taxonomy I have given is crude, and ignores many important distinctions. For example, it is usual [94] to distinguish finitists from constructivists (also called intuitionists). The latter believe in *constructions* on natural numbers, and in higher-order constructions; but such notions are beyond the scope of this paper. I have also avoided the question of what ultrafinitists *do* believe, but presumably calculations are acceptable.[5]

Finitism is linked to Primitive Recursive Arithmetic (PRA), a subsystem of PA [90]. Countabilism is linked to the "predicative" line of work initiated by Weyl [95]. Sequentialism is linked to descriptive set theory [75, Section 3.2.3].

5 Reliability of the intuitions

We now need to consider which of these intuitions are reliable. Each one claims to have (limited) access to an objective, "platonic" realm, much larger than we can directly apprehend. So the truly sceptical answer is that none are reliable. *"How can a human mind have access to an immense platonic realm? The idea is absurd!"*[6] The price of such an attitude is ultrafinitism.

This is a contentious point, because finitists and constructivists sometimes argue in exactly this way against more credulous positions. But the anti-platonist argument has nothing to do with infinity *per se*. The set $\mathbb{N}_{\mathsf{G}}$ is no more capable of direct apprehension, by an actual human or computer, than $\mathbb{N}$. To rescue finitism from the charge of being a platonist philosophy, some may say that Googolplex Goldbach can "in principle" be decided by checking, while others may "prove" its bivalence by an induction up to

[4] Constructivists would say that I am assuming C to be a "decidable" subset, meaning that every $a \in A$ is either a member or a non-member of C. Since our concern at this point is bivalence and classical theories, that is not an issue.

[5] Friedman [30] recalls meeting the ultrafinitist Esenin-Volpin: "I then proceeded to start with 2^1 and asked him whether this is 'real' or something to that effect. He virtually immediately said yes. Then I asked about 2^2, and he again said yes, but with a perceptible delay. Then 2^3, and yes, but with more delay. This continued for a couple of more times, till it was obvious how he was handling this objection. [...] There is no way that I could get very far with this."

[6] Such arguments have been considered by Benacerraf [6], Field [27] and others.

Googolplex. But surely each of these defences relies on a prior belief in the very set $\mathbb{N}_{\mathsf{G}}$ that ultrafinitists doubt.

In summary, anyone who believes in the bivalence of Googolplex Goldbach is a platonist. Welcome to the club!

Leaving aside ultrafinitists, then, we all accept $\mathbb{N}_{\mathsf{G}}$ and are platonists. Now we must decide how far to go, and it is not an easy question. The first four intuitions have the following profound differences.

- The set $\mathbb{N}_{\mathsf{G}}$ can in principle be grasped.
- Each element of $\mathbb{N}$ can in principle be grasped.
- An element of $B^{\mathbb{N}}$ is given by just one choice at a time, and each time-point can in principle be grasped.
- An element of B^A is given by A-many choices at the same time.

Suppose we accept Arbitrary Natural Number and Arbitrary Sequence. Shall we accept Arbitrary Function? Two arguments have been made against it.

Firstly, some have suggested that the independence results [38, 15, 53] provide evidence against CH bivalence. I do not see why that should be so, even if the truth value of CH is absolutely unknowable. Whether Cleopatra ate an even number of grapes is unknowable, but that is not an argument against bivalence.[7]

In fact, none of the intuitions claim to have *complete* knowledge of the entities they perceive. On the contrary, they profess extreme ignorance, merely claiming to know the most basic properties. Most of our knowledge about $\mathbb{N}$, for example, comes from proof, not directly from the Arbitrary Natural Number intuition. Whatever limits may exist on our proof ability, they do not call into question the reliability of that intuition.

This point also applies in reverse. If the CH mystery is solved at some future time, this will not give us a reason to consider the Arbitrary Function intuition reliable. The one has nothing to do with the other.

Secondly, some have suggested that the Banach–Tarski theorem provides evidence against AC. But this criticism is based on geometric intuition, which mathematicians have learnt to distrust. Furthermore, it has been argued that there are also theorems provable without AC that violate geometric intuition [24].

Discounting these arguments against the Arbitrary Function intuition, we are still left with the question of whether to accept it. Personally I find the intuition strong enough to accept, but am not free of ambivalence, and can understand others being more cautious.

Lastly we come to the Arbitrary Ordinal intuition. It is highly controversial [67, 21, 77, 29, 80] because of the Burali-Forti paradox: it claims to perceive a notion of ordinal that is "as liberal as possible" and yet excludes the order-type of the well-ordered class Ord. Although it has been defended [9, 12], I personally am sceptical. In any case, the issue is beyond the scope of this article, which is concerned with higher-order arithmetic.

[7] Gaifman [31] makes a similar point: "I toss a fair coin and, without noting the side it landed, I toss it again. Nobody knows and nobody will be able to know the outcome of the first toss. [...] Yet people have no problem in believing that there is a true answer to the question 'which side did the coin land?'".

6 Consequences of belief and doubt

6.1 Introduction

Here is the summary so far. The starting position was that the only acceptable grounds for belief are proof and intuition. I then listed some intuitions that I experience (and am assuming that there are no others that would undermine my argument). The key question was which of these are reliable, and we have noted various possible answers. In this section, we consider their consequences.

6.2 Peano Arithmetic and beyond

We begin with the Arbitrary Natural Number intuition. If we accept it, then we believe in a platonic realm of natural numbers, and the bivalence of every arithmetical statement. Every PA axiom is true, and every inference rule preserves truth. So every theorem is true, and PA is consistent.

If, on the other hand, we doubt Arbitrary Natural Number, i.e. we are finitists, then this simple path to PA consistency is blocked. But perhaps some other proof will convince us. So we turn to the literature. Gentzen [33, 14] proved PA consistency using *induction up to* ε_0, and Gödel's Dialectica argument [4, 37] proved it using *higher-order constructions*. Unless we accept one of these principles, we have to doubt the Googolplex consistency of PA.

Now consider the theory Z_2. We believe it to be consistent if we accept Arbitrary Natural Number and Arbitrary Sequence. On the other hand, if we stop at Arbitrary Natural Number, i.e. we are countabilists, then the simple consistency argument does not work. But perhaps some other proof will convince us. So we turn to the literature. Tait [89] and Girard [35] proved it using a predicate on $\mathbb{N}$ defined "impredicatively" via quantification over $\mathcal{P}\mathbb{N}$, but a countabilist would surely not accept such a definition. There is also Spector's proof of Z_2 consistency, which uses *higher-typed bar recursion* [25, 86]. Unless we accept this principle—which is rather unlikely—we have to doubt the Googolplex consistency of Z_2.

Moving on, if we accept Arbitrary Function, then we conclude that CH is bivalent and Z_3 is consistent. But if we stop at Arbitrary Sequence, i.e. we are sequentialists, then we have to doubt the Googolplex consistency of Z_3. There is no middle ground. A formal consistency proof might convince us, but the ones in the literature—e.g. [72, 91]—use a predicate on $\mathcal{P}\mathbb{N}$ defined "impredicatively" via quantification over $\mathcal{P}\mathcal{P}\mathbb{N}$, which a sequentialist would surely not accept.

My key point is that, although we can either accept or doubt an intuition, we cannot half-accept. If we consider an intuition to be unreliable, then we should fully discard it. For a historical example: once the mathematical community came to view geometrical intuition as unreliable, it was fully discarded, in the sense that appealing to it in a proof was no longer allowed.

Thus, accepting Z_3 but not AC is not an option, since AC is asserted by Arbitrary Function. So if the Banach–Tarski theorem is anything less than an objectively true statement, then either Arbitrary Natural Number or Arbitrary Function is an unreliable intuition, and the Googolplex consistency of Z_3 is in doubt.

Likewise, it cannot be said that $\mathcal{P}^{73}\mathbb{N}$ is "less credible" than $\mathcal{P}\mathcal{P}\mathbb{N}$. This is a statistical way of thinking, appropriate only when inductive inference is used. If $\mathcal{P}^{73}\mathbb{N}$ is not real, then either Arbitrary Natural Number or Arbitrary Function is an unreliable intuition, so the reality of $\mathcal{P}\mathcal{P}\mathbb{N}$ is in doubt.

In the same way, our line of thinking does not allow the "positivist" view of Kahrs [45], which accepts the bivalence of Π^0_1 sentences but doubts that of the twin prime conjecture. For if Arbitrary Natural Number is an unreliable intuition, then even the bivalence of the Goldbach conjecture is in doubt. Likewise for each pair of sentences in our questionnaire: if we doubt the bivalence of the second sentence, then we should also doubt that of the first.

6.3 Finitism and ultrafinitism

Let us revisit the two most sceptical schools, which differ in their view of PRA. For a finitist, each PRA axiom is true and each proof rule preserves truth. So each theorem is true and PRA is consistent. But an ultrafinitist cannot accept this argument. Presumably they will also be unconvinced by the formal consistency proof, which uses induction up to ω^ω. So they will doubt the Googolplex consistency of PRA.

Finally, let us note that a finitist cannot accept the bivalence of Π^0_1 statements such as "PA is consistent" or "ZFC is consistent". And an ultrafinitist cannot even accept the bivalence of computational statements such as "PRA is Googolplex consistent" or "ZFC is Googolplex consistent".

7 Intuitionistic theories

Up to this point, the theories we have seen are *classical*, i.e. they include the law of Excluded Middle $\phi \vee \neg\phi$. Dropping this law from PA gives the intuitionistic theory known as *Heyting arithmetic* (HA). Dropping it from Z_2 gives *intuitionistic second-order arithmetic* (IZ_2), and dropping it from Z_3 gives *intuitionistic third-order arithmetic* (IZ_3). Furthermore, we can obtain *intensional* versions of IZ_2 and IZ_3 by dropping Extensionality. The following results are provable in PRA [92, page 170].

- PA and HA are equiconsistent.
- Z_2 and intensional IZ_2 are equiconsistent.
- Z_3 and intensional IZ_3 are equiconsistent.

We accordingly ask: is there an intuition—other than the ones on our list—that would yield the consistency of these intuitionistic theories (and therefore also the classical ones)? In the case of HA, the previously mentioned notion of higher-order construction may be considered such an intuition.

But for IZ_2 (or intensional IZ_2), the answer seems to be no. The problem is that the Comprehension scheme allows quantification ranging over $\mathcal{P}\mathbb{N}$, which is impredicative. It is hard to see how someone who doubts the Arbitrary Natural Number or Arbitrary Sequence intuition can justify this.[8]

[8]This concern does not apply to theories without powerset, such as dependent type theory with predicative universes [61] or CZF [16]. These theories are weaker in consistency strength than Z_2, and are supported by proof-theoretic results [82] and constructive intuitions. One can of course debate whether these provide sufficient justification, but the issue is beyond the scope of this paper.

Likewise for IZ_3 (or intensional IZ_3), the answer seems to be no, because the Comprehension scheme allows quantification ranging over $\mathcal{P}\mathcal{P}\mathbb{N}$, which is impredicative. It is hard to see how someone who doubts the Arbitrary Natural Number or Arbitrary Function intuition can justify this.

Some authors have proposed *free topos theory* (the axiomatic theory of an elementary topos with a natural numbers object) as a foundation of mathematics [50, 51]. It contains IZ_3, so—according to our argument—we should doubt its consistency if we doubt CH bivalence or AC.[9]

8 Multiversism

Let us next examine a particular kind of bivalence scepticism, called *multiversism* [3, 41]. It asserts that there are many mathematical universes, all of equal status. In short, "reality is indeterminate". Supposedly, a non-bivalent sentence is one that holds in one universe and not in another. Sometimes an analogy is drawn with Euclid's Fifth Axiom.

I shall now raise two concerns with multiversism.[10]

Firstly, there is usually a theory (such as ZFC) that all the universes are supposed to model. This theory needs to be consistent, or else the multiverse will be a "nulliverse". As discussed above, it is hard to see how the belief in consistency can be justified.

Secondly, it seems that multiversism fails to properly account for bivalence doubt. Let me give some examples of this.

- A finitist's doubt in the bivalence of the Goldbach conjecture stems from a fear that $\mathbb{N}$ may be unreal, not from a fear that it may be indeterminate. In other words, the finitist does not fear that there may be several versions of $\mathbb{N}$, with the Goldbach conjecture holding in one and failing in another. For in their view, if the conjecture holds in some "version of $\mathbb{N}$" that is at least a model of Robinson arithmetic (say), then it is simply true.

- A countabilist's doubt in the bivalence of the Littlewood conjecture stems from a fear that $\mathcal{P}\mathbb{N}$ may be unreal, not from a fear that it may be indeterminate. For in their view, if the conjecture fails in some "version of $\mathcal{P}\mathbb{N}$" that is at least a collection of subsets of $\mathbb{N}$, then it is simply false.

- A sequentialist's doubt in CH bivalence stems from a fear that $\mathcal{P}\mathcal{P}\mathbb{N}$ may be unreal, not from a fear that it may be indeterminate. For in their view, if CH holds in some "version of $\mathcal{P}\mathcal{P}\mathbb{N}$" that is at least a collection of subsets of $\mathcal{P}\mathbb{N}$, then it is simply true.

- A particularist's doubt in GCH bivalence stems from a fear that Ord may be unreal, not from a fear that it may be indeterminate. For in their view, if GCH fails in some "version of Ord" that is at least a collection of ordinals, then it is simply false.

[9] Cf. the discussion in [51, Section 1.7.2].

[10] A multiverse theory can be construed either as a philosophical view of reality or as a mathematical account of a class of models. My comments only concern the former.

- A totalist's doubt in the bivalence of the Club-Failure Hypothesis stems from a fear that $\mathsf{Class}(\mathsf{Ord})$ may be unreal, not from a fear that it may be indeterminate. For in their view, if the hypothesis fails in some "version of $\mathsf{Class}(\mathsf{Ord})$" that is at least a collection of classes of ordinals, then it is simply false.

Each of these examples concerns a single-quantifier sentence just beyond the boundary of platonist belief. Doubt in the bivalence of such sentences cannot be attributed to a fear of indeterminacy.[11]

9 Platonism is not essentialism

I have now almost finished presenting my argument. In brief, it says that—with the possible exception of formal consistency proofs—the only adequate basis for consistency belief is platonism. My final task is to correct a certain misunderstanding of platonism that makes my position seem more demanding than it actually is. The issue (or a version of it) has been raised by Benacerraf [5], Reynolds [79] and others.

To illustrate the point, here are two injections from $\mathbb{Q}$ to $\mathbb{N}$. The *red encoding* sends 0 to 0, and $\frac{m}{n}$ (for coprime $m, n > 0$) to $2^m \times 3^n$, and $-\frac{m}{n}$ (for coprime $m, n > 0$) to $2^m \times 3^n \times 5$. The *yellow encoding* sends 0 to 17, and $\frac{m}{n}$ (for coprime $m, n > 0$) to $2^m \times 3^n \times 5$, and $-\frac{m}{n}$ (for coprime $m, n > 0$) to $2^m \times 3^n$. The choice between them is arbitrary, as there is no reason to prefer one to the other. Now consider the following statements:

- "Via the red encoding, any believer in the reality of $\mathbb{N}$ must also believe in the reality of $\mathbb{Q}$."
- "Via the yellow encoding, any believer in the reality of $\mathbb{N}$ must also believe in the reality of $\mathbb{Q}$."

According to a naive construal of platonism—let us call it *essentialism*—neither statement is acceptable, and certainly not both. That is because "believing in the reality of $\mathbb{Q}$" supposedly requires a metaphysical commitment that goes beyond merely noting an arbitrary encoding of $\mathbb{Q}$ into another totality that one believes to be real. For example, the rational number $\frac{1}{2}$ cannot be captured "in its essence" by its red encoding (18) or its yellow encoding (90).

Yet surely everyone would accept the above statements. That is why, although people sometimes say "I believe in the reality of $\mathbb{N}$ but have doubts about $\mathbb{R}$," nobody ever says "I believe in the reality of $\mathbb{N}$ but have doubts about $\mathbb{Q}$," and we would be astonished to hear this. So the construal of platonism as essentialism is evidently incorrect. Platonists do not, in fact, think that the rational number $\frac{1}{2}$ has some kind of transcendent essence, and would be content with either encoding of $\mathbb{Q}$.

In summary, we must take care in construing the phrase "believing in the reality of X", where X is a totality such as $\mathbb{N}_{\mathsf{G}}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{C}$, $\mathcal{P}\mathcal{P}\mathbb{N}$ or Ord. Mere truth value realism is too little, but essentialism is too much.

[11] Cf. the discussion of Π^0_1 sentences in Koellner [47].

10 Similar views

Let us now review some literature. Section 11 will consider various critical views and challenges, but first I shall point to views that agree with aspects of my argument.

10.1 Finitism vs ultrafinitism

In Section 5, I described finitism as a platonist philosophy, because of its belief in the reality of the set $\mathbb{N}_{\mathsf{G}}$. Here are some similar views, beginning with van Dantzig [17]:

> Unless one is willing to admit fictitious "superior minds" [...] it is necessary, in the foundations of mathematics like in other sciences, to take account of the limited possibilities of the human mind and of mechanical devices replacing it.

Bernadete [7, page 210]:

> This standard [finitist] concept of even the potential infinite is no less dubious than our standard concept of the actual infinite, when called to account by [ultrafinitist] proto-mathematics.

Wang [94]:

> Finitism [...] is an idealization.

Kreisel [49]:

> Finitism is, of course, an idealization.

Troelstra and van Dalen [92, page 6]:

> All the constructivist schools described in section 1 [including finitism] contain elements of idealization.

Nelson [63]:

> Finitism is the last refuge of the platonist.

10.2 Doubting consistency

I can hardly claim that this paper's main contention—lack of platonist belief leads to consistency doubt—is new.

Firstly, it is the very basis of Hilbert's programme and subsequent work on formal consistency proof [76]. That programme was an attempt to convince reality sceptics (specifically, finitists) that a theory expressing certain platonist beliefs (PA) is in fact consistent. The underlying assumption is that their lack of platonist belief leaves the sceptics with insufficient basis to believe that the theory is consistent. If that were not so, why bother to look for a consistency proof? Gödel's second incompleteness theorem tells us that the search is futile (unless the sceptics accept some proof principle that is outside the theory), but does not change the predicament.

Secondly, here are some similar views, beginning with Dummett [22]:

> Why, then, does he [the nominalist Field] believe ZF to be consistent? Most people do, indeed: but then most people are not nominalists. They believe ZF to be consistent because they suppose themselves in possession of a perhaps hazily conceived intuitive model of the theory; but Field can have no such reason.

Parsons [68, page 58]:

> It is [...] hard to see what grounds other than inductive the nominalist can have for believing consistency statements for theories having only infinite models to be true.

Koellner [48]:

> I think that the concept [suggested by Feferman] of being clear enough to secure consistency (and what the structure is *supposed* to be like) but not clear enough to secure definiteness is itself inherently unclear.

Džamonja [46]:

> The ZFC axioms [...] have other models too. But, somehow, believing in the consistency comes back to thinking if there is this universe of sets or not.

Potter [71]:

> As soon as we accept the image of God constructing set theory, and exercising free choice in how He cosntructed it, we must allow the possibility that He chose not to construct it at all.

Lastly, here are some specific consistency doubts.

- The consistency of Z_3 was doubted by Silver [85].
- The consistency of Z_2 was doubted by Gentzen [34], Lorenzen [55] and Péter [70, page 233].
- The consistency of PA was doubted by the finitist Goodstein [39].
- The consistency of PRA was doubted by the ultrafinitist Nelson [64].

10.3 Axiom of Choice

In Section 4, I described AC as an integral part of the Arbitrary Function intuition that underpins belief in the bivalence of third-order arithmetical sentences. Here are some similar views, beginning with Jourdain [44]:

> The multiplicative axiom [AC] is necessary in order to be able to say that $x, y, z, \ldots$ (an infinity) has any meaning at all.

Bernays [8]:

> The axiom of choice is an immediate application of the quasi-combinatorial concepts in question.

Shoenfield [84]:

> If we interpret a collection as being an arbitrary division of the objects available into members and non-members of the collection, it is reasonable to claim that such a collection [representing a choice function] exists.

Ferreirós [26]:

> From the standpoint of a principled acceptance of arbitrary subsets, it is obvious that one should accept choice sets.

Lavine [52, page 4]:

> The principle [AC] really is inherent in the notion of an arbitrary function.

11 Comparisons and challenges

We shall now look at some points of disagreement (or apparent disagreement) between my argument and other views in the literature.

11.1 Appeals to intuition

Firstly, my argument maintains that mathematics is based (at least in part) on intuition; but many authors are wary of appeals to intuition. This wariness sometimes leads them to seek other kinds of justification for mathematics. Some of the resulting projects are mentioned in the following sections.

11.2 The need for object realism

Apart from the role given to intuition, another potentially controversial aspect of my argument is that it lumps together various kinds of realism. As explained in [54], there is a distinction between *truth value realism*, the belief that sentences of a certain kind are bivalent, and *object realism*, the belief that "abstract mathematical objects exist".[12] My narrative maintains that the two go together: for each kind of sentence, we should either hold both these beliefs or hold neither.

For a contrasting view, let us take a look at the "modal structuralist" project, which attempts to justify truth value realism without relying on object realism.[13] In Hellman's account [42], the truth of an arithmetical sentence ϕ is explained as the validity of (a version of) the sentence: "Necessarily, in any second-order PA model, ϕ is true."

Although this does result in a bivalence principle, I shall raise two concerns.

Firstly, for a translation between two languages to count as an explanation, the target language needs to be more meaningful *a priori* than the source. Is this really the case for the modal structuralist interpretation of PA, whose target language includes modalities, first-order quantifiers of unspecified range (or perhaps universal range), and second-order quantifiers?

[12] Cf. the view attributed by Dummett to Kreisel [87]: "The problem is not the existence of mathematical objects but the objectivity of mathematical sentences."

[13] A somewhat similar project is presented in Chihara [13].

Secondly, the modal structuralist theory adopts as an axiom the possible existence of a second-order PA model. But how can this be justified without appealing to the Arbitrary Natural Number intuition?

Because of such concerns, I am unpersuaded by this kind of attempt to sustain truth value realism without object realism. But let me repeat: construing the latter as something less than essentialism (Section 9) makes it easier to accept than it would otherwise be. For related discussion, see e.g. [68, 54, 59, 78, 83].

11.3 Anti-realist theory of meaning

We turn next to Dummett's "anti-realist" position,[14] which concerns not only mathematics but many other areas of interest. See [56] for an overview.

According to this philosophy, the bivalence of Googolplex Goldbach should be accepted, but not that of the Goldbach conjecture or even the Cleopatra hypothesis [19]. These views are based not on a finitistic or constructive ontology (as Section 4 might suggest), but on a verificationist "theory of meaning" aligned with intuitionistic logic. An early paper [18] declares:

> We are entitled to say that a statement P must be either true or false [...] only when P is a statement of such a kind that we could in a finite time bring ourselves into a position in which we were justified either in asserting or denying P.

This position seems to undermine the premise of our questionnaire. For it says that, if a sentence cannot be known "in a finite time" to be true or to be false, then *ipso facto* its bivalence cannot be asserted. So there is no need to wonder whether (for example) the totality $\mathcal{PP}\mathbb{N}$ is real; the theory of meaning does all the work.

I find this rather hard to swallow. Surely questions of mathematical reality are genuine and cannot be avoided by mere linguistic convention? In any case, the anti-realist position and its elaboration by Dummett have given rise to a substantial literature, both critical and supportive, e.g. [23, 62, 10, 102, 73, 11]. Note, in particular, Raatikainen's comparative analysis of intuitionistic notions of truth and the challenges they face [74], and Rumfitt's defence of certain kinds of classical reasoning that Dummett did not accept [80].

11.4 The ability to reflect

In Section 6, I stated that a finitist will accept the consistency of PRA, and a countabilist that of PA. This appears to be at odds with Tait's view [90] that finitistic reasoning is limited to PRA, and with Isaacson's view [43] that countabilistic ("arithmetical") reasoning is limited to PA.

The source of the discrepancy is that I have tacitly granted everyone the ability to reflect on their own language and reasoning. An *unreflective* finitist would accept every PRA proof presented to them in their mother tongue, but (according to Tait) would consider PRA as a whole to be an unintelligible formal system whose consistency is

[14] I present an extreme version, for expository convenience. In Dummett's writings, many variations are considered.

doubtful. It is *unreflective* finitistic reasoning that, in Tait's view, is limited to PRA. Likewise it is *unreflective* countabilistic reasoning that, in Isaacson's view, is limited to PA. When the positions of Tait and Isaacson are interpreted in the way just described, there is no disagreement.

11.5 Inductive inference

In Section 2, I stated that inductive inference is not accepted, not even for Liminal Goldbach, let alone for consistency statements. Although this is the standard attitude of the mathematical community, there is a substantial literature that takes inductive evidence seriously. For example, such evidence has been used in set theory to defend the $\mathrm{AD}^{\mathrm{L}(\mathbb{R})}$ hypothesis [57, 58, 60], in computational complexity theory to defend the $\mathbf{P} \neq \mathbf{NP}$ hypothesis [1], and in many other fields of mathematics [2].

See Paseau [69] for a discussion that specifically considers the inductive evidence for consistency statements.

12 Conclusions

We began by taking the view that only proof and intuition can provide adequate grounds for belief. This led us from the broad question of what to believe to the more focused question of which intuitions to accept.

As a result, we have only a few options. We can doubt the very notion of human access to platonic reality, but the price of such extreme scepticism is doubting the bivalence of Googolplex Goldbach—the ultrafinitist view. Moving on, we can doubt the bivalence of CH or the truth of AC, but then the Googolplex consistency of Z_3 is in doubt. So we cannot adopt a foundational theory that includes Z_3, such as ZF; nor even one that includes IZ_3, such as IZF [16] or free topos theory.

The focus of this article is higher-order arithmetic, rather than set theory. So we have not examined particularism and totalism. Particularists (like me) need to know: what is the price of doubting the bivalence of GCH? Totalists need to know: what is the price of doubting the bivalence of the Club-Failure Hypothesis? These questions are left to future work.

Although the story is unfinished, then, let me sum up. When setting out our fundamental mathematical beliefs, we are free to decide how credulous or sceptical to be. Our decision will depend on the strength of the intuitions we experience, and on our degree of caution. But this freedom has limits, as we must fully accept or fully doubt each intuition. Thus, believing in the consistency of everything and the reality of nothing is not an option. Scepticism always comes at a price.

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