Coproducts of Monads on Set

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***Abstract*—Coproducts of monads on Set have arisen in both the study of computational effects and universal algebra.**

We describe coproducts of consistent monads on Set by an initial algebra formula, and prove also the converse: if the coproduct exists, so do the required initial algebras. That formula was, in the case of ideal monads, also used by Ghani and Uustalu. We deduce that coproduct embeddings of consistent monads are injective; and that a coproduct of injective monad morphisms is injective.

Two consistent monads have a coproduct iff either they have arbitrarily large common fixpoints, or one is an exception monad, possibly modified to preserve the empty set. Hence a consistent monad has a coproduct with every monad iff it is an exception monad, possibly modified to preserve the empty set. We also show other fixpoint results, including that a functor (not constant on nonempty sets) is finitary iff every sufficiently large cardinal is a fixpoint.

I. Introduction

The notion of *monad*, in particular on the category of sets, has numerous applications. In computer science the following two are prominent.

1) It is used to give semantics of *computational effects* [14], such as non-deterministic choice, exceptions, I/O, reading and assigning to memory cells, and control effects that capture the current continuation.
2) It provides an abstract account of the notion of "algebraic theory". For example, a *finitary* algebraic theory Th consists of a signature—a set of operations with a finite arity—and a set of equations between terms. Then the monad $\mathbb{T}_{\mathsf{Th}}$ sends X to the set of terms with variables drawn from X, modulo equivalence.

The *coproduct* of monads $\mathbb{S}$ and $\mathbb{T}$ was studied by Kelly [11], who showed that an algebra for the coproduct is a *bialgebra*: a set A with both an $\mathbb{S}$-algebra structure $\sigma : SA \to A$ and a $\mathbb{T}$-algebra structure $\tau : TA \to A$. These coproducts have arisen in both application areas:

1) The exception monad transformer [6], applied to a monad $\mathbb{T}$, gives $X \mapsto T(X + E)$. This is a coproduct of $\mathbb{T}$ with the exception monad $X \mapsto X + E$. More generally, Hyland, Plotkin and Power [10] gave a formula for the coproduct of a free monad $\mathbb{F}_H$ with a general monad $\mathbb{T}$. This provides semantics combining I/O effects, represented by $\mathbb{F}_H$, with some other effects, represented by $\mathbb{T}$.
2) Given two theories Th and Th', we form their *sum* [15] by taking the disjoint union of the signatures and the union of the equation sets. The monad $\mathbb{T}_{\mathsf{Th}+\mathsf{Th}'}$ is then a coproduct of $\mathbb{T}_{\mathsf{Th}}$ and $\mathbb{T}_{\mathsf{Th}'}$. The sum of theories has received much attention in the field of term rewriting [3]. In particular it is shown [3, Prop. 4.14] that $\mathsf{Th}+\mathsf{Th}'$ is *conservative* over the summands, provided each summand is consistent i.e. does not prove $\forall x, y.\ x = y$. This amounts to injectivity of the coproduct embeddings for the monads, and is a surprisingly nontrivial result.

In each field some basic questions have remained.

1) Are there other monad transformers given by coproducts with a certain monad $\mathbb{T}$? We give an almost negative answer: up to isomorphism, $\mathbb{T}$ must be either an exception monad or the terminal monad, possibly modified in each case to preserve the empty set. No other monad has a coproduct with the powerset monad or with a (nontrivial) continuation monad. This contrasts sharply with the recent result of [9] that *every* monad has a tensor with the powerset and continuation monads.
2) We can consider theories whose operations have *countable* arities, or more generally arities of size $< \lambda$, for a regular cardinal $\lambda \geqslant \aleph_0$. (Regularity ensures that, if the operations have arity $< \lambda$, then terms will too.) These theories, and their corresponding monads, are called λ-*accessible*. Does the conservativity result hold for these? More problematically still, there are monads, such as the powerset and continuation monads, that are not accessible (i.e. not λ-accessible for any λ). We show that coproduct embeddings for consistent monads are always injective. This subsumes the conservativity result for finitary and accessible theories.

Kelly [11] showed that giving a coproduct $\mathbb{S} \oplus \mathbb{T}$ amounts to giving a free bialgebra on every set. Three specific constructions of these coproducts appear in the literature. Each deals in a different way with the problem of the "shared units": trivial terms—those that are just variables—are common to the two summands.

(1) Kelly [11] gives a multi-step construction that uses quotienting to identify the shared units. Because it does not directly describe what gets equated, it does not enable us to prove results such as conservativity.
(2) Hyland, Power and Plotkin [10] treat the case where $\mathbb{S}$ is a *free* monad, for example one arising from a theory with no equations. Here a term in the sum consists of layers alternating between terms of $\mathbb{T}$ and operations of $\mathbb{S}$, as

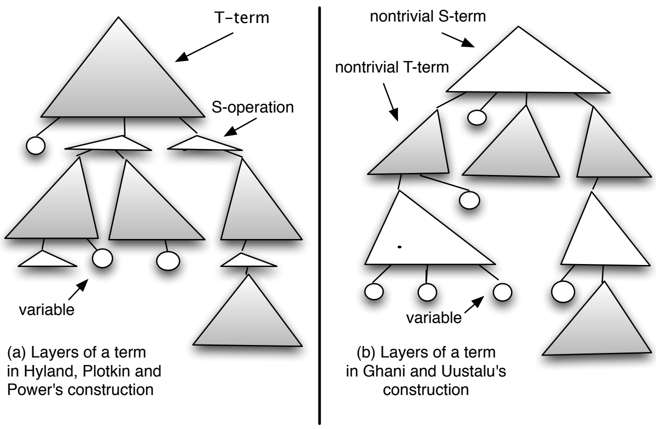


Fig. 1. Layers of a term in two coproduct constructions

depicted in Fig. 1(a), with a $\mathbb{T}$-layer uppermost.

(3) Ghani and Uustalu [8] treat the case where both $\mathbb{S}$ and $\mathbb{T}$ are *ideal monads* (see Elgot [7]), corresponding to a theory whose equations are all between nontrivial terms. A nontrivial term in the sum consists of layers alternating between nontrivial terms of $\mathbb{S}$ and those of $\mathbb{T}$, as depicted in Fig. 1(b). The uppermost layer may be of either kind. However, the majority of important monads, e.g. list, powerset, finite powerset, state and continuation monads, fail to be ideal.

Our first contribution is to show that Ghani and Uustalu's coproduct formula works for all consistent monads, not just ideal ones. That seems surprising; the formula makes use of the "ideal", an endofunctor on Set representing the nontrivial terms, which only an ideal monad possesses. Our solution is to replace that ideal by the *unit complement*, an endofunctor on the category Inj of sets and injections, possessed by every consistent monad on Set, as we shall see.

In the setting of accessible monads, the initial algebras in the coproduct formula are guaranteed to exist, so we are done. But in the general setting, it is only half the story: *if* the initial algebras exist, we obtain a free bialgebra. Our second contribution is to show the converse. We therefore have a formula for a coproduct of monads whenever that coproduct exists.

This leads to our third contribution: a characterization of when a coproduct of monads exists in terms of their cardinal fixpoints: they must either have arbitrarily large common fixpoints, or else one of them is an exception monad, possibly modified to preserve the empty set. This has many corollaries about the existence of coproducts between different kinds of monads. En route we give several new results about fixpoints, including the surprising fact that a set functor (not constant on nonempty sets) is finitary iff every sufficiently large cardinal is a fixpoint of it. The last result depends on earlier work by Trnková [18] and Koubek [12] about properties of set functors.

Acknowledgments. Ohad Kammar and Gordon Plotkin proved Lemma VI.7 for finitary monads, using Knuth-Bendix rewriting. We thank Ohad Kammar for discussions on this topic. The third author was supported by EPSRC Advanced Research Fellowship EP/E056091/1.

II. Universal Property of a Monad vs. Free Algebras

Notation II.1. We write $\mathbb{S}, \mathbb{T}, \mathbb{U}$ for monads and S, T, U for endofunctors of a category $\mathcal{C}$, thus $\mathbb{S} = (S, \eta^S, \mu^S)$. Accordingly, an "$\mathbb{S}$-algebra" must satisfy the Eilenberg-Moore axioms, whereas an "S-algebra" need not.

Remark II.2. The *transport* of an $\mathbb{S}$-algebra (X, α) along an isomorphism $i : X \to Y$ is the $\mathbb{S}$-algebra $(Y, i \cdot \alpha \cdot Si^{-1})$. It is easy to verify that the axioms of Eilenberg-Moore algebras are fulfilled.

In particular, given an isomorphism $i : SA \to Y$, then Y is a free $\mathbb{S}$-algebra on A w.r.t. the transport

$$SY \xrightarrow{Si^{-1}} SSA \xrightarrow{\mu_A} SA \xrightarrow{i} Y$$

and the universal arrow $i \cdot \eta_A : A \to Y$.

In this section we review the general notions of *free monad* and *coproduct of monads*. The key point is that both of these notions have two descriptions: one using a universal property on a monad, and one using free algebras. Happily, on Set, they turn out to be equivalent. The proof exploits the following fact about continuation monads $R^{(R^-)}$.

Lemma II.3 (Kelly [11])**.** *(1) Let H be an endofunctor on* Set *and R a set. There is a bijection Γ^H_R from H-algebra structures $HR \to R$ to natural transformations $H \to R^{(R^-)}$, whose inverse assigns to $\alpha\colon H \to R^{(R^-)}$ the algebra $\alpha_R(\mathrm{id}_R)\colon HR \to R$.*

(2) Let $\mathbb{S}$ be a monad on Set *and R a set. Then Γ^S_R gives a bijection from $\mathbb{S}$-algebra structures $SR \to R$ to monad morphisms $\mathbb{S} \to R^{(R^-)}$.*

In the case of free monads, the two definitions are as follows.

Definition II.4. Let H be an endofunctor on a category $\mathcal{C}$.

(1) A *free monad* on H is a monad $\mathbb{F}_H$ and natural transformation $H \xrightarrow{\gamma} F_H$ that is initial among all such pairs $(\mathbb{S}, H \xrightarrow{\lambda} \mathbb{S})$.

(2) Suppose every $\mathcal{C}$-object A generates a free H-algebra[1] $(F_H A, \rho_A)$ with unit $A \xrightarrow{\eta_A} F_H A$. (Equivalently: the forgetful functor from the H-algebras category to $\mathcal{C}$ has a left adjoint.) Then the resulting monad on $\mathcal{C}$ is an *algebraic free monad* on H.

Proposition II.5 (Barr [4])**.** *(1) Let H be an endofunctor on $\mathcal{C}$. An algebraic free monad on H is a free monad with embedding γ given at A by*

$$HA \xrightarrow{H\eta_A} HF_H A \xrightarrow{\rho_A} F_H A$$

[1] If $\mathcal{C}$ has finite coproducts, a free H-algebra on A is the same thing as an initial algebra for $X \mapsto HX + A$.

(2) Conversely, for $\mathcal{C}$ with products, any free monad arises in this way.

Corollary II.6. *For set functors H the free monad $\mathbb{F}_H$ on H fulfils $F_H A \cong HF_H A + A$ for every set A.*

In the case of coproducts, the two definitions are as follows:

Definition II.7. Let $\mathbb{S}$ and $\mathbb{T}$ be monads on a category $\mathcal{C}$.

(1) A *coproduct* of $\mathbb{S}$ and $\mathbb{T}$ is a coproduct $\mathbb{S} \oplus \mathbb{T}$ in the category of monads and monad morphisms.
(2) An $(\mathbb{S}, \mathbb{T})$-*bialgebra* (X, σ, τ) is an object X with Eilenberg-Moore algebra structures $SX \xrightarrow{\sigma} X$ for $\mathbb{S}$ and $TX \xrightarrow{\tau} X$ for $\mathbb{T}$.
(3) Suppose that every $A \in \mathcal{C}$ generates a free $(\mathbb{S}, \mathbb{T})$-bialgebra $((\mathbb{S} \oplus \mathbb{T})A, p^S_A, p^T_A)$ with unit $A \xrightarrow{\eta_A} (\mathbb{S} \oplus \mathbb{T})A$. (Equivalently: the forgetful functor from the $(\mathbb{S}, \mathbb{T})$-bialgebra category to $\mathcal{C}$ has a left adjoint.) Then the resulting monad is an *algebraic coproduct* of $\mathbb{S}$ and $\mathbb{T}$.

Proposition II.8 (Kelly [11]). *(1) Let $\mathbb{S}$ and $\mathbb{T}$ be monads on $\mathcal{C}$. An algebraic coproduct of $\mathbb{S}$ and $\mathbb{T}$ is $\mathbb{S} \oplus \mathbb{T}$ with embeddings given at A by*

$$SA \xrightarrow{S\eta_A} S(\mathbb{S} \oplus \mathbb{T})A \xrightarrow{p^S_A} (\mathbb{S} \oplus \mathbb{T})A$$

$$TA \xrightarrow{T\eta_A} T(\mathbb{S} \oplus \mathbb{T})A \xrightarrow{p^T_A} (\mathbb{S} \oplus \mathbb{T})A$$

(2) Conversely, for $\mathcal{C}$ with products, any coproduct of monads arises in this way.

Thus, whilst it is the "algebraic coproduct" notion that corresponds to the joining of two theories, in Set we do not need to distinguish between the two notions.

We can easily generalize this to a coproduct of a family of monads $(\mathbb{S}_i)_{i \in I}$. Here an $(\mathbb{S}_i)$-*multialgebra* is a set X with an Eilenberg-Moore algebra structure $S_i X \xrightarrow{\sigma_i} X$ for $\mathbb{S}_i$ (where i ranges through I). And the monad of free $(\mathbb{S}_i)$-multialgebras is the coproduct of the family $(\mathbb{S}_i)_{i \in I}$.

We illustrate coproducts of monads on Set with some examples.

Example II.9 (Hyland et al. [10]). We have for the exception monad $\mathbb{M}_E : X \mapsto X + E$

$$\mathbb{T} \oplus \mathbb{M}_E = \mathbb{T}(- + E)$$

for all monads $\mathbb{T}$. More generally, the coproduct of $\mathbb{T}$ with a family $(\mathbb{M}_{E_p})_{p \in P}$ of exception monads is $\mathbb{T}(- + \coprod_{p \in P} E_p)$.

Example II.10. We have, for the terminal monad $\mathbb{1} : X \mapsto 1$

$$\mathbb{1} \oplus \mathbb{T} = \mathbb{1}$$

for all monads $\mathbb{T}$. Indeed, $\mathbb{1}$ has just one Eilenberg-Moore algebra (up to isomorphism), hence, there is only one bialgebra. More generally, the coproduct of $\mathbb{1}$ with any family of monads is $\mathbb{1}$. For the submonad $\mathbb{1}_0$ of the terminal monad given by $0 \mapsto 0$ and $X \mapsto 1$ else all coproducts exist also (and are equal to $\mathbb{1}$ or $\mathbb{1}_0$).

III. Initial Algebras in Inj^I

In order to examine monads on Set, we shall also have to consider categories of the form Set^I, where I is a set. An object is an I-tuple of sets, often called a "many-sorted set". We also need to work with Inj, the category of sets and injections, and Inj^I. We now look at initial algebras on Inj^I

Definition III.1 ([1]). Let H be an endofunctor on a category $\mathcal{C}$ with colimits of chains.

(1) The *initial chain* of H, depicted

$$0 \longrightarrow H0 \longrightarrow H^2 0 \longrightarrow H^3 0 \longrightarrow \cdots$$

is a functor from $\mathbf{Ord}$ to $\mathcal{A}$ with objects $H^i 0$ and connecting morphisms $H^i 0 \xrightarrow{h_{i,j}} H^j 0$ ($i \le j$ in $\mathbf{Ord}$). It is defined by transfinite induction on objects by

$$H^0 0 = 0, \qquad H^{i+1} 0 = H(H^i 0),$$

and

$$H^i 0 = \operatorname*{colim}_{k<i} H^k 0 \text{ for limit ordinals } i.$$

Analogously for morphisms:

$$h_{i+1,j+1} = Hh_{i,j}$$

and for limit ordinals i the cocone $(h_{k,i})_{k<i}$ is a colimit.
(2) The initial chain *converges* at an ordinal α if the connecting map $h_{\alpha,\alpha+1}$ is invertible.
(3) For any H-algebra $A = (X, \theta)$, we define the *canonical cocone* $(b^A_i)_{i \in \mathbf{Ord}}$ from the initial chain to X by setting b^A_{i+1} to be

$$H^{i+1} 0 \xrightarrow{Hb^A_i} HX \xrightarrow{\theta} X$$

Lemma III.2. *Let H be an endofunctor on Inj^I. Any H-algebra homomorphism $f \colon A \to B$ is a morphism of canonical cocones i.e.*

$$f \cdot b^A_i = b^B_i.$$

Proposition III.3 (Trnková et al. [19]). *Let H be an endofunctor on Inj^I.*

(1) If the initial chain of H converges at i, then the H-algebra $(H^i 0, h^{-1}_{i,i+1})$ is initial.
(2) Conversely, if there exists an H-algebra, then the initial chain of H converges at some ordinal.

Proof:

(1) Standard, and similar to the proof of Proposition III.6 below.
(2) Let A be an H-algebra. Since $\mathsf{range}(b^A_j) \subseteq X$ increases with j, we have for some ordinal i

$$\mathsf{range}(b^A_i) = \mathsf{range}(b^A_{i+1})$$

making $h_{i,i+1}$ an isomorphism.

■

If H is finitary (i.e. preserves filtered colimits), then the initial chain converges at ω. More generally, for a regular cardinal $\lambda \geqslant \aleph_0$, if H is λ-accessible (i.e. preserves λ-filtered colimits), then the initial chain converges at λ.

For convenience, we shall frequently describe functors on Inj^I, and also on Set^I, by means of a system of equations. For example, if F and G are endofunctors on Inj, then an "algebra of" the system

$$\begin{aligned} X &= FY \\ Y &= GX \end{aligned}$$

means an algebra for the endofunctor on Inj^2 mapping (X, Y) to (FY, GX). In this case the two components of the initial chain take the form

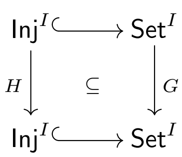
$$0 \longrightarrow F0 \longrightarrow FG0 \longrightarrow FGF0 \longrightarrow \cdots \tag{III.1}$$

and

$$0 \longrightarrow G0 \longrightarrow GF0 \longrightarrow GFG0 \longrightarrow \cdots \tag{III.2}$$

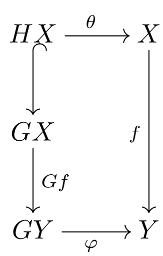
We now consider the relationship between endofunctors on Inj^I and those on Set^I.

Definition III.4. Let G be an endofunctor on Set^I. Suppose for each object X we have a subobject $HX \subseteq GX$, in such a way that Gm restricts to an injection $HX \rightarrowtail HY$ for each injection $X \overset{m}{\rightarrowtail} Y$. We say that H is a *subfunctor on* Inj^I of G. This can be depicted as



Definition III.5. Let G be an endofunctor on Set^I, with a subfunctor H on Inj^I.

(1) For an H-algebra (X, θ) and a G-algebra (Y, φ), an *H-G-algebra morphism* is a function $X \overset{f}{\longrightarrow} Y$ satisfying



(III.3)

(2) For a G-algebra $A = (Y, \varphi)$, we define the *canonical cocone* $c_i^A : H^i0 \to Y$ ($i \in \mathbf{Ord}$) from the initial chain of H to Y by setting c_{i+1}^A to be

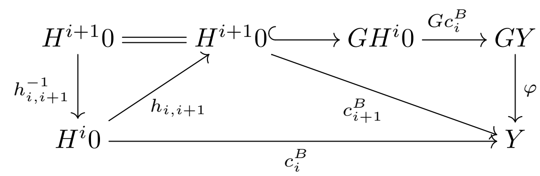
$$H^{i+1}0 \hookrightarrow GH^i0 \xrightarrow{Gc_i^A} GY \xrightarrow{\varphi} Y$$

The cocone property is established by an easy transfinite induction.

We conclude this section by the following "recursive function definition" principle.

Proposition III.6. *Let G be an endofunctor on Set^I, with subfunctor H on Inj^I. If $(\mu H, r)$ is an initial H-algebra, then for every G-algebra (Y, φ) there is a unique H-G-algebra morphism $(\mu H, r) \longrightarrow (Y, \varphi)$.*

Proof: By Proposition III.3(2) the initial chain of H converges at some ordinal i. Without loss of generality we may assume $(\mu H, r) = (H^i0, h_{i,i+1}^{-1})$. For $B = (Y, \varphi)$ we see that $(H^i0, h_{i,i+1}^{-1}) \xrightarrow{c_i^B} (Y, \varphi)$ is a homomorphism by inspecting the commutative diagram below:



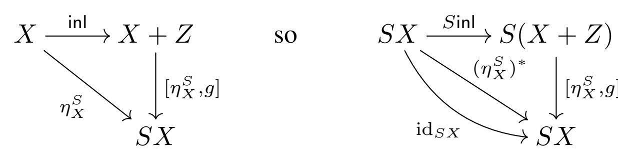
For any H-G-algebra morphism $f\colon A \to B$ it is easy to prove by transfinite induction on $j \leq i$ that $f \cdot c_j^A = c_j^B$ (cf. Lemma III.2). For $A = (\mu H, r)$ we have $c_j^A = h_{j,i}$, which implies $c_i^A = \mathrm{id}$. Thus, $f = c_i^B$ is a unique homomorphism. ■

IV. The Unit Complement of a Monad

We present some basic properties of monads on Set.

Lemma IV.1. *Every monad $\mathbb{S}$ on* Set *preserves injections.*

Proof: It suffices to show that $X \xrightarrow{\mathsf{inl}} X + Z$ is sent to an injection. Let $p, q \in SX$ be such that $(S\mathsf{inl})p = (S\mathsf{inl})q$. Writing $Z \xrightarrow{g} SX$ for the constant function to p,



where we write x^* for $\mu_X \cdot Sx$. ■

Definition IV.2. A monad $\mathbb{S}$ on Set is *consistent* when η_X^S is injective for all sets X.

Up to isomorphism, there are only two inconsistent monads.

Lemma IV.3. *If $\mathbb{S}$ is inconsistent then it is isomorphic to either $\mathbb{1}$ or $\mathbb{1}_0$.*

Proof: Suppose $\eta_X(x) = \eta_X(x')$ for some $x \neq x' \in X$. We show that $|SY| \leqslant 1$ for any set Y; hence $|SY| = 1$ if Y is nonempty since $Y \xrightarrow{\eta_X} SY$ cannot have empty codomain. Given elements $p, p' \in SY$, let $X \xrightarrow{f} SY$ be a function sending x to p and x' to p'. Since f is

$$X \xrightarrow{\eta_X} SX \xrightarrow{Sf} SSY \xrightarrow{\mu_Y} SY$$

it identifies x and x', giving $p = p'$, so $SY = 1$. ■

Since we already know how to form a coproduct with $\mathbb{1}$ or with $\mathbb{1}_0$, we lose nothing by restricting attention to consistent monads. We can then perform a fundamental construction.

Definition IV.4. Let $\mathbb{S}$ be a consistent monad on Set. For any set X, we set

$$\bar{S}X = SX \setminus \mathsf{range}(\eta_X).$$

In the example of a monad arising from a consistent theory, $\bar{S}X$ is the set of *nontrivial* equivalence classes of terms on X, i.e those classes that do not contain a variable.

Proposition IV.5. *Let $\mathbb{S}$ be a consistent monad on* Set. *Then $\bar{S}$ is a subfunctor of $\mathbb{S}$ on* Inj.

Proof: It suffices to show that if $p \in SX$ is sent by $S(X \xrightarrow{\mathsf{inl}} X+Y)$ into the range of η^S_{X+Y} then $p \in \mathsf{range}(\eta^S_X)$. We reason as follows: either

$$\begin{array}{rcl}(S\mathsf{inl})p & = & (\eta^S_{X+Y}\mathsf{inl})x \\ & = & (S\mathsf{inl}\,\eta^S_X)x\end{array}$$

giving $p = \eta^S_X x$ by Lemma IV.1, or

$$(S\mathsf{inl})p = \eta^S_{X+Y}\mathsf{inr}\ y. \tag{IV.4}$$

In the latter case, we apply $S(\ X+Y \xrightarrow{[\mathsf{in}_0,\mathsf{in}_1]} X+Y+Y\ )$ to (IV.4) giving

$$(S\mathsf{in}_0)p = \eta^S_{X+Y+Y}\mathsf{in}_1 y.$$

We also apply $S(\ X+Y \xrightarrow{[\mathsf{in}_0,\mathsf{in}_2]} X+Y+Y\ )$ to (IV.4) giving

$$(S\mathsf{in}_0)p = \eta^S_{X+Y+Y}\mathsf{in}_2 y.$$

Injectivity of η^S_{X+Y+Y} gives $\mathsf{in}_1 y = \mathsf{in}_2 y$, which is impossible. ■

We call $\bar{S}$ the *unit complement* of $\mathbb{S}$. By contrast with the "ideal monad" framework of [8], $\bar{S}$ might not extend to an endofunctor on Set:

Examples IV.6. (1) If $\mathbb{S}$ is the finite powerset monad, then $\bar{S}X$ is the set of all non-singleton finite subsets of X. For the (non-injective) function $2 \xrightarrow{g} 1$, we cannot define $\bar{S}2 \xrightarrow{\bar{S}g} \bar{S}1$ consistently with Sg.

(2) If $\mathbb{S}$ is the finite list monad $X \mapsto X^*$, then $\bar{S}X$ is the set of all words of length $\neq 1$. In this case $\bar{S}$ *does* extend to an endofunctor on Set. Nevertheless $\mathbb{S}$ is not an ideal monad—μ^S does not map $\bar{S}S$ to $\bar{S}$.

Lemma IV.7. *Let $\mathbb{S}$ be a consistent monad on* Set. *For any regular cardinal $\lambda \geqslant \aleph_0$, if S is λ-accessible, so is $\bar{S}$.*

V. Initial Bialgebras and Multialgebras

We saw in Sect. II that, to find the coproduct of two monads $\mathbb{S}$ and $\mathbb{T}$, we need a free bialgebra on each set A. In this section, we study the simpler problem of finding an initial bialgebra (i.e. $A = \emptyset$). We shall see in Sect.VI that this enables us to solve the general problem. When writing $+$ we always mean coproduct in Set.

To find an initial bialgebra for $\mathbb{S}$ and $\mathbb{T}$, we seek an initial algebra in Inj for the system

$$\begin{array}{rcl} X & = & \bar{S}Y \\ Y & = & \bar{T}X \end{array} \tag{V.5}$$

If it exists, we call it (S^*, T^*). The algebra structure is called $r^S \colon \bar{S}T^* \overset{\cong}{\to} S^*$ and $r^T \colon \bar{T}S^* \overset{\cong}{\to} T^*$. By Proposition III.3 this exists whenever (V.5) has a solution. This is in particular the case if $\mathbb{S}$ and $\mathbb{T}$ are λ-accessible.

Theorem V.1. *Let $\mathbb{S}$ and $\mathbb{T}$ be consistent monads on* Set.

(1) If (S^*, T^*) exists, then

$$(S^* + T^*, p^S, p^T) \tag{V.6}$$

is an initial $(\mathbb{S}, \mathbb{T})$-bialgebra, where $p^S \colon S(S^* + T^*) \to S^* + T^*$ is the free $\mathbb{S}$-algebra on T^* transported (see Remark II.2) along the isomorphism

$$ST^* \cong \bar{S}T^* + T^* \xrightarrow{r^S + T^*} S^* + T^*$$

and p^T is defined similarly.

(2) Conversely, any initial $(\mathbb{S}, \mathbb{T})$-bialgebra arises in this way.

Explicitly, the unique bialgebra morphism from (V.6) to an $(\mathbb{S}, \mathbb{T})$-bialgebra (B, σ, τ) is constructed as follows. The functor given by (V.5) is a subfunctor on Inj^2 of the functor

$$\begin{array}{rcl} X & = & SY \\ Y & = & TX \end{array} \tag{V.7}$$

on Set^2 in the sense of Definition III.4. Now (B, σ, τ) is an algebra of (V.7), so by Proposition III.6, we obtain unique $S^* \xrightarrow{f^S} B$ and $T^* \xrightarrow{f^T} B$ such that the squares

$$\begin{array}{ccccccc} \bar{S}T^* & \xrightarrow{r^S} & S^* & \quad & \bar{T}S^* & \xrightarrow{r^T} & T^* \\ \downarrow & & & & \downarrow & & \\ ST^* & & \downarrow f^S & & TS^* & & \downarrow f^T \\ \downarrow Sf^T & & & & \downarrow Tf^S & & \\ SB & \xrightarrow[\sigma]{} & B & & TB & \xrightarrow[\tau]{} & B \end{array} \tag{V.8}$$

commute. Then the bialgebra morphism is given by

$$S^* + T^* \xrightarrow{[f^S, f^T]} B\,.$$

Sketch of proof: For (1) we prove by diagram chasing that $[f^S, f^T]$ is a homomorphism for both monads S and T.

For (2), assuming that an initial bialgebra on a set A is given, we prove that the initial chain (S^*_i, T^*_i) converges by verifying that the canonical cocone (Definition III.1) has all components injective from which the statement easily follows. The main technical trick of the proof is that for every sufficiently large ordinal i we construct a bialgebra such that the canonical cocones have their components at i injective. ■

Remark V.2. The carrier $S^* + T^*$ of the initial $(\mathbb{S}, \mathbb{T})$-bialgebra can be written as

$$\mu\bar{S}\bar{T} + \mu\bar{T}\bar{S}$$

Indeed, in the chain (III.1) all even members form the initial chain of FG, analogously with (III.2).

Lemma V.3. *Let* $\mathbb{S}\xrightarrow{\alpha}\mathbb{S}'$ *and* $\mathbb{T}\xrightarrow{\beta}\mathbb{T}'$ *be injective monad morphisms. If there is an initial* $(\mathbb{S}',\mathbb{T}')$*-bialgebra* (I',m',n')*, then there is an initial* $(\mathbb{S},\mathbb{T})$*-bialgebra* (I,m,n)*, and the unique* $(\mathbb{S},\mathbb{T})$*-bialgebra homomorphism*

$$(I,m,n)\xrightarrow{f}(I',m'\cdot\alpha_{I'},n'\cdot\beta_{I'})$$

is injective.

Remark V.4. To find an initial *multi*algebra for more than two monads, we have to adapt (V.5).

- In the case of three consistent monads $\mathbb{S},\mathbb{T},\mathbb{U}$ we take in Inj the initial algebra (S^*,T^*,U^*) of the equations

$$\begin{array}{rcl} X &=& \bar{S}(Y+Z)\\ Y &=& \bar{T}(X+Z)\\ Z &=& \bar{U}(X+Y)\end{array}\qquad\text{(V.9)}$$

  and then the initial trialgebra is carried by $S^*+T^*+U^*$.
- In the case of a family $(\mathbb{S}_p)_{p\in P}$ of consistent monads, we take in Inj the initial algebra $(S_p^*)_{p\in P}$ of the equations

$$X_p=\bar{S}_p(\sum_{q\in P\setminus\{p\}}X_q)\qquad(p\in P)$$

  with structure $(r_p)_{p\in P}$ and the initial multialgebra is carried by $\sum_{p\in P}S_p^*$. The $\mathbb{S}_p$-structure is given by the free $\mathbb{S}_p$-algebra structure on

$$S_p^+\overset{\text{def}}{=}\sum_{q\in P\setminus\{p\}}S_q^*$$

  transported along the isomorphism

$$\sum_{p\in P}S_p^*\cong S_p^*+S_p^+\xrightarrow{r_p^{-1}+S_p^+}\bar{S}_pS_p^++S_p^+\cong S_p(S_p^+)$$

All the results of this section (except Remark V.2) go through in this more general setting.

VI. Coproducts of Monads

In this section a formula for coproducts of monads on Set is presented. We denote by $+$ coproducts in Set and by $\oplus$ coproducts of monads.

Remark VI.1. Suppose we have consistent monads $\mathbb{S}$ and $\mathbb{T}$, and we want a free $(\mathbb{S},\mathbb{T})$-bialgebra on a set A. This is the same thing as an initial $(\mathbb{S},\mathbb{T},\mathbb{M}_A)$-trialgebra, where $M_AX=X+A$ is the exception monad, since an $\mathbb{M}_A$-algebra on X corresponds to a morphism $A\to X$. We know that this initial trialgebra is given by an initial algebra of (V.9), which in this case takes the form

$$\begin{array}{rcl} X &=& \bar{S}(Y+Z)\\ Y &=& \bar{T}(X+Z)\\ Z &=& A\end{array}$$

By an elementary argument this corresponds to an initial algebra of

$$\begin{array}{rcl} X &=& \bar{S}(Y+A)\\ Y &=& \bar{T}(X+A)\end{array}\qquad\text{(VI.10)}$$

Recall that these initial algebras are taken in Inj.

Definition VI.2. Let $\mathbb{S}$ and $\mathbb{T}$ be consistent monads on Set.

(1) For any set A, we define (S^*A,T^*A) to be an initial algebra of (VI.10) if it exists. The algebra structure is called

$$s_A^*\colon\bar{S}(T^*A+A)\overset{\cong}{\to}S^*A\text{ and }t_A^*\colon\bar{T}(S^*A+A)\overset{\cong}{\to}T^*A.$$

(2) Consider S^*A+T^*A+A to be a bialgebra as follows. Denote by $p_A^S\colon S(S^*A+T^*A+A)\to S^*A+T^*A+A$ the free $\mathbb{S}$-algebra on T^*A+A transported (see Remark II.2) along the isomorphism

$$\begin{array}{ccc} S(T^*A+A) & \cong & \bar{S}(T^*A+A)+T^*A+A\\ & & \downarrow{\scriptstyle s_A^*+T^*A+A}\\ S^*A+T^*A+A & \cong & S^*A+(T^*A+A)\end{array}$$

and p_A^T is the free $\mathbb{T}$-algebra on S^*A+A transported along the analogous isomorphism.

Proposition VI.3. *Let* $\mathbb{S}$ *and* $\mathbb{T}$ *be consistent monads on* Set*. Let* A *be a set.*

(1) If (S^*A,T^*A) *exists, then*

$$(S^*A+T^*A+A,p_A^S,p_A^T)\qquad\text{(VI.11)}$$

with unit $\mathsf{inr}\colon A\to S^*A+T^*A+A$ *is a free* $(\mathbb{S},\mathbb{T})$*-bialgebra on* A*.*

(2) Conversely, any free $(\mathbb{S},\mathbb{T})$*-algebra on* A *arises in this way.*

Explicitly, the unique bialgebra morphism from the above algebra to an $(\mathbb{S},\mathbb{T})$-bialgebra (B,σ,τ) extending $A\xrightarrow{h}B$ is constructed as follows. By Proposition III.6, we obtain unique $S^*A\xrightarrow{f^S}B$ and $T^*A\xrightarrow{f^T}B$ such that

$$\begin{array}{ccccc} \bar{S}(T^*A+A) & & \xrightarrow{\quad s_A^*\quad} & & S^*A\\ \downarrow & & & & \\ S(T^*A+A) & & & & \downarrow{\scriptstyle f^S}\\ \downarrow{\scriptstyle Sf^T} & & & & \\ S(B+A) & \xrightarrow[S[\mathrm{id},h]]{} & SB & \xrightarrow[\sigma]{} & B\end{array}$$

and

$$\begin{array}{ccccc} \bar{T}(S^*A+A) & & \xrightarrow{\quad t_A^*\quad} & & T^*A\\ \downarrow & & & & \\ T(S^*A+A) & & & & \downarrow{\scriptstyle f^T}\\ \downarrow{\scriptstyle Tf^S} & & & & \\ T(B+S) & \xrightarrow[T[\mathrm{id},h]]{} & TB & \xrightarrow[\tau]{} & B\end{array}$$

commute. Then the bialgebra morphism is given by

$$S^* + T^* + A \xrightarrow{[f^S, f^T, h]} B$$

It is easily checked that this is the construction derived from that in Theorem V.1 and Remark V.4.

Theorem VI.4. *A coproduct of monads $\mathbb{S}$ and $\mathbb{T}$ on Set exists iff one of the monads is inconsistent or an initial algebra (S^*A, T^*A) for (VI.10) exist in Inj for all A. Under these circumstances:*

(1) $(\mathbb{S} \oplus \mathbb{T})A$ is given by $(S^*A + T^*A) + A$ for every set A
(2) the unit of $\mathbb{S} \oplus \mathbb{T}$ is given at A by

$$A \xrightarrow{\mathsf{inr}} S^*A + T^*A + A$$

Remark VI.5. The coproduct embedding $\mathbb{S} \longrightarrow \mathbb{S} \oplus \mathbb{T}$ is given at A by

$$\begin{array}{ccc} SA \cong \bar{S}A + A & \xrightarrow{\bar{S}\mathsf{inr}+A} & \bar{S}(T^*A + A) + A \\ & & \downarrow{\scriptstyle s^*_A + A} \\ S^*A + T^*A + A & \xleftarrow[\mathsf{inl}+A]{} & S^*A + A \end{array}$$

and likewise for the embedding $\mathbb{T} \longrightarrow \mathbb{S} \oplus \mathbb{T}$.

Corollary VI.6. *If $\mathbb{S}$ and $\mathbb{T}$ are consistent monads and $\mathbb{S} \oplus \mathbb{T}$ exists, then $\mathbb{S} \oplus \mathbb{T}$ is consistent and the coproduct embeddings*

$$\mathbb{S} \xrightarrow{\mathsf{inl}} \mathbb{S} \oplus \mathbb{T} \xleftarrow{\mathsf{inr}} \mathbb{T}$$

are injective.

Lemma VI.7. *Let $\mathbb{S}', \mathbb{T}'$ be consistent monads such that $\mathbb{S}' \oplus \mathbb{T}'$ exists. For any injective monad morphisms $\mathbb{S} \xrightarrow{i} \mathbb{S}'$ and $\mathbb{T} \xrightarrow{j} \mathbb{T}'$*

- *$\mathbb{S} \oplus \mathbb{T}$ exists*
- *the monad morphism $\mathbb{S} \oplus \mathbb{T} \xrightarrow{i \oplus j} \mathbb{S}' \oplus \mathbb{T}'$ is injective.*

Proof: Analogous to Remark VI.1, for each set A, the initial $(\mathbb{S}', \mathbb{T}', \mathbb{M}_A)$-trialgebra (I', m', n', a') exists. Therefore by Lemma V.3 the initial trialgebra (I, m, n, a) of $\mathbb{S}$, $\mathbb{T}$ and $\mathbb{M}_A$, exists, i.e. the free $(\mathbb{S}, \mathbb{T})$-bialgebra on A, giving $(\mathbb{S} \oplus \mathbb{T})A$. Moreover, Lemma V.3 gives the injectivity of the unique trialgebra morphism from (I, m, n, a) to $(I', m' \cdot \alpha_{I'}, n' \cdot \beta_{I'}, a')$, i.e. the unique bialgebra morphism commuting with the units, which is precisely $(i \oplus j)_A$. ■

To form the coproduct of a family $(\mathbb{S}_p)_{p \in P}$ of consistent monads, we take for each set A the initial algebra $(S^*_p A)_{p \in P}$ of the equations

$$X_p = \bar{S}_p\Big(\sum_{q \in P \setminus \{p\}} X_q + A\Big) \qquad (p \in P).$$

in Inj. The free $(\mathbb{S}_p)_{p \in P}$-multialgebra on A exists iff $(S^*_p A)_{p \in P}$ exists, and is then carried by $\sum_{p \in P} S^*_p A + A$. All the results of the section then adapt in the evident way.

VII. Functors and Monads on Set

In this section we will discuss properties of endofunctors and monads on Set needed for the technical development in the next section.

Theorem VII.1 (Trnková [18])**.** *For every set functor H there exists a set functor $\widehat{H}$ preserving finite intersections and agreeing with H on all nonempty sets and functions.*

In fact, Trnková gave a construction of $\widehat{H}$ as follows: consider the two subobjects $t, f\colon 1 \to 2$. Their intersection is the empty function $e : \emptyset \to 1$. Since $\widehat{H}$ must preserve this intersection it follows that $\widehat{H}e$ is injective and forms (not only a pullback but also) an equalizer of $\widehat{H}t = Ht$ and $\widehat{H}f = Hf$. Thus $\widehat{H}$ must be defined on $\emptyset$ (and e) as the equalizer

$$\widehat{H}\emptyset \xrightarrow{\widehat{H}e} \widehat{H}1 = H1 \underset{Hf}{\overset{Ht}{\rightrightarrows}} H2.$$

Trnková proved that this defines a set functor preserving finite intersections.

Corollary VII.2. *The full subcategory of $[\mathsf{Set}, \mathsf{Set}]$ given by all endofunctors preserving finite intersections is reflective.*

More formally, we have a natural transformation $r\colon H \to \widehat{H}$ such that for any natural transformation $s\colon H \to K$, where K preservers intersections, there is a unique natural transformation $s^\sharp\colon \widehat{H} \to K$ such that $s^\sharp \cdot r = s$.

Proof: From $t \cdot e = f \cdot e$ we obtain $Ht \cdot He = Ht \cdot He$. Therefore, the universal property of the equalizer induces a unique map $r_\emptyset\colon H\emptyset \to \widehat{H}\emptyset$ such that $He = \widehat{H}e \cdot r_\emptyset$. This yields a natural transformation

$$r\colon H \to \widehat{H}$$

with the component $r_\emptyset$ and with $r_X = \mathrm{id}_{HX}$ for all $X \neq \emptyset$.

Now let K be an endofunctor preserving finite intersections and let $s\colon H \to K$ be any natural transformation. Then Ke is the equalizer of Kt and Kf, and so we obtain a unique map $s^\sharp_\emptyset$ as displayed below:

$$\begin{array}{ccccc} \widehat{H}\emptyset & \xrightarrow{\widehat{H}e} & H1 & \underset{Hf}{\overset{Ht}{\rightrightarrows}} & H2 \\ {\scriptstyle s^\sharp_\emptyset}\downarrow & & \downarrow{\scriptstyle s_1} & & \downarrow{\scriptstyle s_2} \\ K\emptyset & \xrightarrow[Ke]{} & K1 & \underset{Kf}{\overset{Kt}{\rightrightarrows}} & K2 \end{array}$$

Together with $s^\sharp_X = s_X$ for all $X \neq \emptyset$ this defines a natural transformation $s^\sharp\colon \widehat{H} \to K$ with $s^\sharp \cdot r = s$. It is now easy to show that $s^\sharp$ is unique with this property. Thus, $r\colon H \to \widehat{H}$ is a reflection as desired. ■

Definition VII.3. We call the above reflection $\widehat{H}$ of H (which is unique up to unique natural isomorphism) the *Trnková closure* of H. For a functor H preserving finite intersections we can always choose $\widehat{H} = H$.

Example VII.4. Let C_M be the constant functor on M, and C^0_M its modification given by $\emptyset \mapsto \emptyset$ and $X \mapsto M$ for all

$X \neq \emptyset$. Then the Trnková closure of C_M^0 is the embedding $r\colon C_M^0 \to C_M$.

Remark VII.5. Trnková closure extends "naturally" to monads: for every monad $\mathbb{S} = (S, \eta, \mu)$ there is a unique monad structure on $\widehat{S}$ for which r is a monad morphism. We denote this monad by $\widehat{S}$ and call it the *Trnková closure of the monad* $\mathbb{S}$.

Notation VII.6. For every monad $\mathbb{S}$ on Set we denote by $\mathbb{S}^0$ its submonad agreeing with $\mathbb{S}$ on all nonempty sets (and functions) and with $S^0\emptyset = \emptyset$.

Proposition VII.7. *Every monad $\mathbb{S}$ on* Set *fulfils either $\mathbb{S} \cong \widehat{\mathbb{S}}$ or $\mathbb{S} \cong (\widehat{\mathbb{S}})^0$.*

Example VII.8. The exception monad

$$\mathbb{M}_E X = X + E$$

has the submonad $\mathbb{M}_E^0$ (given by $\emptyset \mapsto \emptyset$ and $X \mapsto X + E$ for all $X \neq \emptyset$).

Remark VII.9. We say that a set functor H *substantially fulfils* some property if its Trnková closure $\widehat{H}$ fulfils it. For example, C_M^0 is a substantially constant functor. And $\mathbb{M}_E^0$ is a substantially exceptional monad.

Example VII.10. Substantially exceptional monads have a coproduct with every monad on Set. This follows for $\mathbb{M}_E^0$ by an argument analogous to that of Example II.9.

We finish this section by a result of Koubek [12] about behaviours of set functors on cardinalities. Using similar ideas, we prove an analogous result for the above endofunctor $\bar{S}$.

Proposition VII.11 (Koubek [12])**.** *If a set functor H is not substantially constant (see Remark VII.9), then there exists a cardinal λ with $\operatorname{card} HX \geq \operatorname{card} X$ for all sets X with cardinality at least λ.*

Theorem VII.12. *For every consistent monad $\mathbb{S}$ on* Set *which is not substantially exceptional there exists an infinite cardinal λ with*

$$\operatorname{card} \bar{S}X \geq \operatorname{card} X$$

for all sets X of cardinality at least λ.

Sketch of proof: Since $\mathbb{S}$ is not substantially exceptional, there exists an infinite cardinal λ such that for every set X of cardinality at least λ there exists an element x in SX such that the coproduct embeddings $v_i\colon X \to X \times X$ (a coproduct of X copies of X) fulfil: $\bar{S}v_i(x)$ are pairwise distinct elements. Since $X \times X$ is isomorphic to X this proves $\operatorname{card} \bar{S}X \geq \operatorname{card} X$. ■

VIII. A Fixpoint Characterization of Coproducts

In this section we see a remarkable phenomenon, first studied by Koubek [12]: that many properties of functors and monads on Set may be recovered from merely knowing their behaviour on cardinals. As we shall see, an instance of this is the existence of coproducts of monads. Recall that every cardinal λ is considered to be the set of all smaller ordinals.

Definition VIII.1. By a *fixpoint* of a set functor H is meant a cardinal λ such that $\operatorname{card} H\lambda = \lambda$.

Recall from Remark VII.9 that a set functor is *substantially constant* iff its domain restriction to all nonempty sets is naturally isomorphic to a constant functor. Analogously for *substantially exceptional monads*.

Proposition VIII.2 (Trnková et al. [19])**.** *A set functor generates a free monad iff it has arbitrarily large fixpoints or is substantially constant.*

Lemma VIII.3. *Let H be a set functor with arbitrarily large fixpoints. There exists a cardinal λ such that F_H and H have among larger cardinals the same fixpoints.*

Next we characterize finitarity of set functors completely via fixpoints. Recall that a set functor is finitary iff for every set X and every element $x \in HX$ there exists a finite subset $m : Y \hookrightarrow X$ with $x \in \mathsf{range}(Hm)$. This is equivalent to H preserving filtered colimits, see [2].

Lemma VIII.4. *Let $n > \alpha$ be infinite cardinals of the same cofinality. Then there exists a collection of more than n subsets of n which are almost α-disjoint (i. e., have cardinality α and the intersection of any distinct pair has smaller cardinality).*

Remark VIII.5. Almost disjoint collections were introduced by Tarski [17]. The present result can be found in Baumgartner [5].

The proof of the following proposition uses ideas of Koubek in [12].

Theorem VIII.6. *Let H be a set functor that is not substantially constant. Then H is finitary iff all cardinals from a certain cardinal onwards are fixpoints of H.*

Sketch of proof: If H is finitary, and λ is an upper bound on $\operatorname{card} Hn$, $n \in \mathbb{N}$, then every cardinal greater or equal to λ is a fixpoint. Conversely, if H is not finitary, there exists an infinite cardinal α and an element $x \in H\alpha$ not reachable from smaller cardinals. Then no cardinal n cofinal with α is a fixpoint of H. To see this, choose an almost α-disjoint collection as in Lemma VIII.4 and express it as a family of injections $m_i\colon \alpha \to n$. By using Trnková closure $\widehat{H}$ we see that the elements $Hm_i(x)$ are pairwise distinct. This proves $\operatorname{card} Hn > n$. ■

Proposition VIII.7. *Let H be an accessible set functor that is not substantially constant. Then there exists a cardinal λ_0 such that all cardinals 2^κ with $\kappa \geq \lambda_0$ are fixpoints of H.*

Theorem VIII.8. *Two consistent monads $\mathbb{S}$ and $\mathbb{T}$ on* Set *have a coproduct iff one is substantially exceptional or they have arbitrarily large joint fixpoints ($\lambda = \operatorname{card} S\lambda = \operatorname{card} T\lambda$).*

Proof: (1) Necessity follows from Theorem VI.4. If both monads are not substantially constant, choose a cardinal λ that works for $\mathbb{S}$ as well as $\mathbb{T}$ in Theorem VII.12. For every set A of cardinality at least λ we choose sets $X \cong \bar{S}(Y + A)$ and $Y \cong \bar{T}(X + A)$ and prove that X is a joint fixpoint of $\bar{S}$

and $\bar{T}$ of cardinality at least $\operatorname{card} A$. The latter is clear from Theorem VII.12:

$$\operatorname{card} X = \operatorname{card} \bar{S}(Y + A) \geq \operatorname{card}(Y + A) \geq \operatorname{card} A.$$

Analogously, $\operatorname{card} Y \geq \operatorname{card} A$. Thus, $X + A \cong X$ and $Y + A \cong Y$, from which we conclude

$$X \cong \bar{S}Y \text{ and } Y \cong \bar{T}X.$$

We have $\operatorname{card} \bar{T}X \geq \operatorname{card} X$ by Theorem VII.12, and another application of Theorem VII.12 yields

$$\operatorname{card} X = \operatorname{card} \bar{S}\bar{T}X \geq \operatorname{card} \bar{T}X,$$

thus the cardinal of X is a fixpoint of $\bar{T}$. Then from $Y \cong \bar{T}X$ we conclude $X \cong Y$ and this yields, by symmetry, a fixpoint of $\bar{S}$. Since in Set we have $SZ = \bar{S}Z + Z$, it follows that also S and T have arbitrarily large joint fixpoints.

(2) Sufficiency. By Example VII.10 we need to prove that if $\mathbb{S}$ and $\mathbb{T}$ are not substantially constant and have arbitrarily large joint fixpoints, then $\mathbb{S} \oplus \mathbb{T}$ exists. Due to Theorem VII.12 $\bar{S}$ and $\bar{T}$ have arbitrarily large joint fixpoints too. For every set A let X be an infinite set of cardinality $\operatorname{card} X \geq \operatorname{card} A$ which is a fixpoint of $\bar{S}$ and $\bar{T}$. Then $X \cong \bar{S}(X + A)$ and $X \cong \bar{T}(X + A)$ yields a solution of Equation (VI.10). Consequently, $\mathbb{S} \oplus \mathbb{T}$ exists by Proposition III.3 and Theorem VI.4. ■

Notation VIII.9. $\mathcal{P}$ denotes the power-set monad (i. e. the monad of the computational effect of non-determinism). And $\mathcal{P}_f$ the finite-power-set submonad (of finitely branching non-determinism).

Corollary VIII.10. *For every consistent monad $\mathbb{S}$ on* Set *the following conditions are equivalent:*

(a) all coproducts $\mathbb{S} \oplus \mathbb{T}$ with monads $\mathbb{T}$ exist,
(b) $\mathbb{S}$ is substantially exceptional,
(c) the coproduct $\mathbb{S} \oplus \mathcal{P}$ exists.

Indeed, since $\mathcal{P}$ has no fixpoint, $(c) \to (a)$ follows from the above theorem, $(a) \to (b)$ is Example VII.10 and $(b) \to (c)$ is clear.

Corollary VIII.11. *For every monad $\mathbb{S}$ on* Set *the following conditions are equivalent:*

(a) $\mathbb{S}$ has coproducts with all finitary monads,
(b) the functor S generates a free monad,
(c) the coproduct $\mathbb{S} \oplus \mathcal{P}_f$ exists.

Indeed $(b) \to (a)$ follows from Theorems VIII.6 and VIII.8 by using Proposition VIII.2.

(a) → (c) is obvious, and (c) → (b) also follows from Theorems VIII.6 and VIII.8.

Remark VIII.12. In Corollary VIII.10 we could use in lieu of $\mathcal{P}$ any monad without fixpoints (e. g. the continuation monad). And in Corollary VIII.11 in lieu of $\mathcal{P}_f$ we could use any finitary monad that is not substantially exceptional (by applying Theorem VII.12).

Corollary VIII.13. *Let $\mathbb{S}$ be a consistent monad and $\mathbb{F}_H$ a free monad. Then a coproduct $\mathbb{S} \oplus \mathbb{F}_H$ exists iff S and H have arbitrarily large joint fixpoints or one of the monads is substantially exceptional.*

This follows from Theorem VIII.8 and Lemma VIII.3.

Corollary VIII.14. *For every finitary monad $\mathbb{S}$ on* Set *all coproducts with free monads exist.*

Open Problem VIII.15. Does every accessible monad on Set have coproducts with all free monads?

The following result nicely “complements” the preceding corollary:

Corollary VIII.16. *A monad $\mathbb{S}$ has coproducts with all finitary monads iff a free monad on S exists.*

Example VIII.17. We present two free monads on Set whose coproduct does not exist. In other words, two set functors H and K generating a free monad but such that $H + K$ does not generate one. This is a variation on an example, constructed in [13] under the assumption of generalized continuum hypothesis, of a non-accessible functor generating a free monad.

Given a class A of cardinal numbers, we can define a functor $\mathcal{P}_A$ on Set by

$$\mathcal{P}_A X = \{M \subseteq X; \operatorname{card} M \in A \text{ or } M = \emptyset\}.$$

For every function $f : X \to Y$ put

$$\mathcal{P}_A f(M) = \begin{cases} f[M] & \text{if } f \text{ restricted to } M \text{ is injective} \\ \emptyset & \text{else} \end{cases}$$

Suppose the complement $\bar{A} = \mathsf{Card} \setminus A$ contains, for some infinite cardinal λ, the interval $(\lambda, 2^\lambda]$ (of all cardinals $\lambda < \alpha \leq 2^\lambda$). Then 2^λ is a fixpoint of $\mathcal{P}_A$:

$$\operatorname{card} \mathcal{P}_A(2^\lambda) \leq \sum_{\alpha \in A, \alpha \leq 2^\lambda} (2^\lambda)^\alpha \leq \sum_{\alpha \leq \lambda} 2^{\alpha\lambda} = 2^\lambda.$$

Let A be a class of cardinals such that both A and $\bar{A}$ contain the intervals $(\lambda, 2^\lambda]$ for arbitrary large cardinals λ. Then $\mathcal{P}_A$ and $\mathcal{P}_{\overline{A}}$ generate free monads by Theorem II.6. However, $\mathcal{P}_A + \mathcal{P}_{\overline{A}}$ has no fixpoints, thus, it does not generate a free monad.

Finally, we can generalize Theorem VIII.8 to a family of monads:

Theorem VIII.18. *A family of consistent monads on* Set *has a coproduct iff*

(1) all those monads that are not substantially exceptional have arbitrarily large joint fixpoints or
(2) all monads but at most one are substantially exceptional.

IX. Conclusions

We have described coproducts of monads on Set. If one of the monads is inconsistent (i. e. a submonad of the terminal monad), then so is the coproduct. For consistent monads we have shown that coproducts of monads on Set are well-behaved and can be concretely described:

(1) If two consistent monads have a coproduct, then the coproduct injections are injective.
(2) A consistent monad has coproducts with all monads iff it is substantially exceptional (that is, a submonad of an exception monad).
(3) Two consistent monads have a coproduct iff they have arbitrarily large joint fixpoints or one is substantially exceptional.

Moreover, for every consistent monad (S, η, μ) we proved that complements of the unit form an endofunctor $\bar{S}$ on the category Inj of sets and injections. We used the functor $\bar{S}$ to present a formula for coproducts: Consistent monads $\mathbb{S}$ and $\mathbb{T}$ have a coproduct iff for every set A the recursive equations

$$X = \bar{S}(Y + A) \qquad \text{and} \qquad Y = \bar{T}(X + A)$$

have an initial solution S^*A, T^*A; the coproduct monad then sends A to $S^*A + T^*A + A$. This formula was used by Ghani and Uustalu [8] for ideal monads. We also obtain an iterative construction of the coproduct: S^*A and T^*A are the colimits of the chains S_i^*A and T_i^*A starting with $\emptyset$ and given by $S_{i+1}^*A = \bar{S}(T_i^*A + A)$ and $T_{i+1}^*A = \bar{T}(S_i^*A + A)$. This is a substantially easier and clearer construction than that presented previously by Kelly [11].

From the above result we derived that the coproduct of finitary monads is given by the formula $A \mapsto S_\omega^*A + T_\omega^*A + A$, and that every finitary monad has a coproduct with all free monads. Coproducts of a monad and a free monad were described by Hyland, Plotkin and Power [10], our results imply that a consistent monad $\mathbb{S} = (S, \eta, \mu)$ has a coproduct with the free monad on a functor H iff S and H have arbitrarily large joint fixpoints or $\mathbb{S}$ is substantially exceptional.

It is an open problem whether every accessible monad has a coproduct with every free monad.

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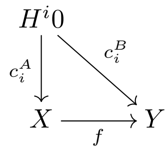
Appendix

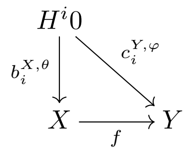
In order to prove the unicity in Proposition III.6 we use the following

Lemma A.1. *Let G be an endofunctor on Set^I, with a subfunctor H on Inj^I.*

(1) Any G-algebra morphism $f\colon A \to B$ is a morphism of canonical cocones, i.e.

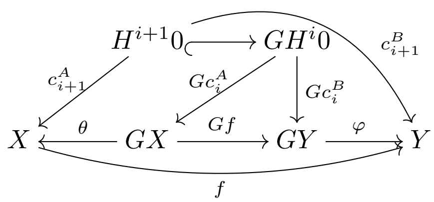


(2) Any H-G-algebra morphism $(X,\theta)\xrightarrow{f}(Y,\varphi)$ is a morphism of canonical cocones, i.e.

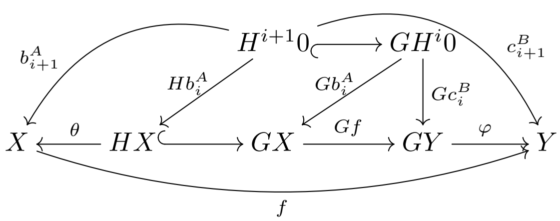


Proof:

(1) The inductive step is given by $A = (X,\theta)$ and $B = (Y,\varphi)$ by



(2) The inductive step is given for $A = (X,\theta)$ and $B = (Y,\varphi)$ by



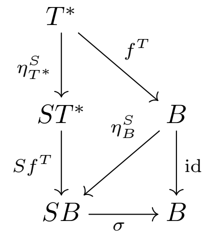
■

Proof of Lemma IV.7: Let $D : \mathcal{I} \longrightarrow \mathsf{Inj}$ be a diagram, where $\mathcal{I}$ is a λ-filtered small category, with colimit $(V, (\mathsf{in}_i)_{i\in\mathcal{I}})$. Since S is λ-accessible, $(SV, (S\mathsf{in}_i)_{i\in\mathcal{I}})$ is a colimit of SD. For $x \in \bar{S}V$, we have $x \in SV$ so $x = (S\mathsf{in}_i)y$ for some $i \in \mathcal{I}$ and $y \in SD_i$. Suppose $y = (\eta^S D_i)z$ for some $z \in D_i$. Then

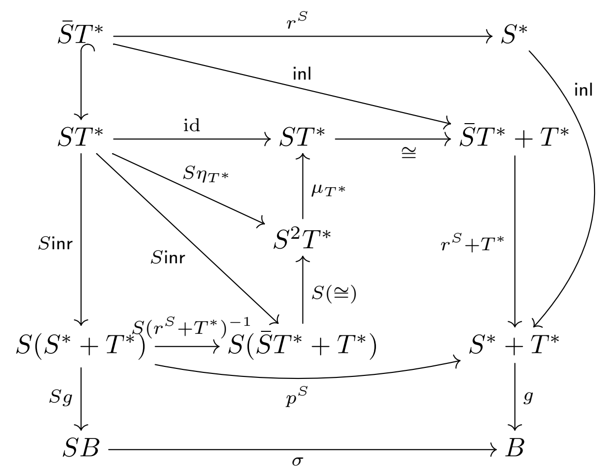
$$\begin{aligned} x &= (S\mathsf{in}_i)y \\ &= (S\mathsf{in}_i)(\eta^S D_i)z \\ &= (\eta^S V)\mathsf{in}_i\, z \end{aligned}$$

contradicting $x \in \bar{S}V$; hence $y \in \bar{S}D_i$. We conclude that $(\bar{S}V, (\bar{S}\mathsf{in}_i)_{i\in\mathcal{I}})$ is a colimit of $\bar{S}D$. ■

Proof of Theorem V.1: For (1) we show that $[f^S, f^T]$ is a an $\mathbb{S}$-algebra homomorphism in Fig. 2, and it is likewise a $\mathbb{T}$-algebra homomorphism. We need only prove part (ii) of the figure, since (i) is S applied to (ii) and all the other parts are obvious. The left-hand component of (ii) is the left-hand diagram in (V.8) and the right-hand component is given by



For uniqueness, let g be a bialgebra morphism from $(S^* + T^*, p^S, p^T)$ to (B, σ, τ). The components of g are $g \cdot \mathsf{inl} = f^S$ and $g \cdot \mathsf{inr} = f^T$. This follows from the commutative diagram below:



and the analogous diagram for $g \cdot \mathsf{inr}$. Indeed, these diagrams commute since p^S is defined as $(r^S{+}T^*)\cdot\mu^S_{T^*}\cdot S(R^S{+}T^*)^{-1}$, see Remark II.2, and analogously for p^T.

For (2), assuming an initial bialgebra (A, σ^0, τ^0), we have to show the initial chain (S_i^*, T_i^*) of (V.5) converges. Let $(\sigma_j^0, \tau_j^0)_{j\in\mathbf{Ord}}$ be the canonical cocone (Definition III.1) from the initial chain of (V.5) to the algebra (A, A, σ^0, τ^0) of (V.7). If we can show σ_j^0 and τ_j^0 to be injective for all j, we will be done, as in the proof of Prop. III.3(2).

We are going to find, for every ordinal i, a bialgebra (B, σ, τ) such that the canonical cocone (σ_j, τ_j) from the initial chain of (V.5) to the algebra (B, B, σ, τ) of (V.7) fulfils:

$$\sigma_i : S_i^* \to B \text{ and } \tau_i : T_i^* \to B \text{ are both injective.}$$

This suffices, because the unique bialgebra morphism $h : A \to B$ is also a morphism of algebras for (V.7), giving by Lemma A.1(1)

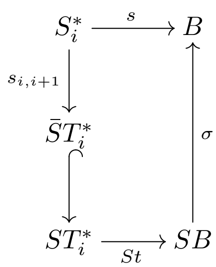
$$\sigma_i = h \cdot \sigma_i^0 \text{ and } \tau_i = h \cdot \tau_i^0$$

which makes σ_i^0 and τ_i^0 injective.

(b1) We first prove that there exists a $\mathbb{S}$-algebra (B, σ) of size $\geqslant 2$ and disjoint subobjects

$$s : S_i^* \to B \text{ and } t : T_i^* \to B$$

such that the square



commutes. Here $s_{i,i+1}\colon S_i^* \to S_{i+1}^* = ST_i$ is the connecting morphism of the chain (III.1) for $F = \bar{S}$ and $G = \bar{T}$. Analogously $t_{i,i+1} : T_i^* \to \bar{T}S_i^*$. Indeed, let (B, σ) the free algebra on $T_i^* + 2$, with

$$s : S_i^* \xrightarrow{s_{i,i+1}} \bar{S}T_i^* \hookrightarrow ST_i^* \xrightarrow{S\mathsf{inl}} S(T_i^* + 2)$$

and

$$t : T_i^* \xrightarrow{\eta^S} ST_i^* \xrightarrow{S\mathsf{inl}} S(T_i^* + 2)$$

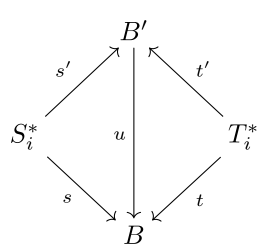
These injections are disjoint by definition of $\bar{S}$, and the square commutes due to $\mu^S \cdot S\eta^S = \mathrm{id}$.

(b2) For every infinite cardinal $\kappa \geq \mathrm{card}(ST_i^*)$ we can, additionally, require in (b1) that B has cardinality 2^κ. Indeed, starting with an algebra B_0 as in (b1), form its power $B = B_0^\kappa$ in $\mathsf{Set}^{\mathbb{S}}$ and take the subobjects $\triangle \cdot s : S_i^* \to B$ and $\triangle \cdot t : T_i^* \to B$ (for s and t as in (b1)). They are disjoint, and the above square clearly commutes. Since S_i^* has at least two elements, so does $B_0 = ST_i^*$ due to the injection $s : S_i^* \to B_0$. Thus, from $2^\kappa = \kappa^\kappa$ we conclude $\mathrm{card}(B) = \mathrm{card}((ST_i^*)^\kappa) = 2^\kappa$.

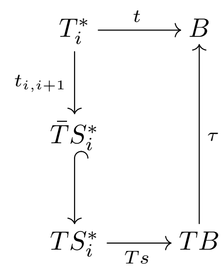
(b3) By symmetry, given an infinite cardinal κ greater or equal to the cardinalities of ST_I^* and TS_i^*, there exists a $\mathbb{T}$-algebra (B', τ') and disjoint subobjects

$$s' : S_i^* \to B' \text{ and } t' : T_i^* \to B'$$

such that the corresponding square commutes and B' has cardinality 2^κ. Since $B \cong B'$ and s, t are also disjoint subobjects, we can find an isomorphism $u : B' \to B$ such that the diagram



commutes. We let (B, τ) be the transport of (B', τ') along u (see Remark II.2). Consequently, the bialgebra (B, σ, τ) has the property that besides the above square also the square



commutes.

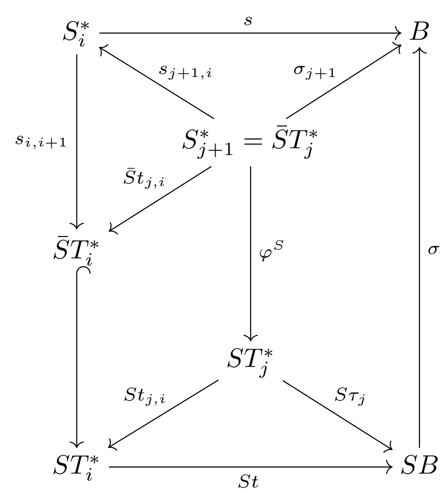
(b4) We now prove for all $j \leqslant i$ that

$$\sigma_j = s \cdot s_{j,i} \text{ and } \tau_j = t \cdot t_{j,i}$$

The case $j = i$ implies

$$\sigma_i = s \text{ and } \tau_i = t,$$

which concludes the proof. We use induction on j, with $j = 0$ and the limit case trivial. For the induction step, where $j < i$, we use the following diagram



(and the corresponding diagram for t). It is our task to prove that the upper triangle commutes. Since the outside commutes, see (b1), it is sufficient to observe that all the remaining inner parts commute. For the lower triangle use the induction hypothesis, the right-hand part is the definition of σ_{j+1}, the left-hand triangle is the definition of $s_{j+1,i+1}$ (as $\bar{S}t_{i,j}$), and the part under it commutes by the naturality of $S' \hookrightarrow S$

■

Proof of Lemma V.3: Since α is injective, it restricts to a natural transformation $\bar{\alpha} : \bar{S} \longrightarrow \bar{T}$, and likewise β restricts to $\bar{\beta} : \bar{S}' \longrightarrow \bar{T}'$. By Theorem V.1(2), the system

$$\begin{aligned} X &= \bar{S}'Y \\ Y &= \bar{T}'X \end{aligned}$$

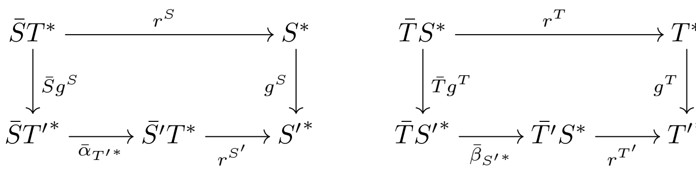
has an initial algebra $((S'^*, T'^*), (r^{S'}, r^{T'}))$. So the system

$$\begin{aligned} X &= \bar{S}Y \\ Y &= \bar{T}X \end{aligned}$$

has an algebra

$$P \overset{\text{def}}{=} ((S'^*, T'^*), (r^{S'} \cdot \bar{\alpha}_{T'^*}, r^{T'} \cdot \bar{\beta}_{S'^*}))$$

Therefore, by Prop. III.3, it has an initial algebra $((S^*, T^*), r^S, r^T)$, and we obtain a unique algebra morphism (g^S, g^T) from it to P, i.e.



Now $S^* + T^*$ carries an initial $(\mathbb{S}, \mathbb{T})$-bialgebra as described in Theorem V.1(1). We show that $g^S + g^T$ is an $\mathbb{S}$-algebra morphism in Fig. 3 which commutes: recall the definition of p^S and $p^{S'}$ from Remark II.2 and use the naturality of α and $\overline{\alpha}$. Analogously, $g^S + g^T$ for the exception monad $\mathbb{M}_A$ is likewise a $\mathbb{T}$-algebra morphism. Therefore it is the desired bialgebra morphism, and it is injective since g^S and g^T are. ∎

Proof of Theorem VI.4: The main statement and (1)–(2) are immediate from Proposition VI.3. For the remark: recall that, since $\eta^{S\oplus T} = \mathsf{inr}$, the coproduct embeddings in Proposition II.8 are $p^S_A \cdot S\mathsf{inr}$ and $p^T_A \cdot T\mathsf{inr}$, respectively. From the definition of p^S_A and p^T_A, see Remark II.2, we conclude that the diagram in Fig. 4 commutes. And we have an analogous diagram from $p^T_A \cdot T\mathsf{inr}$. This finishes the proof. ∎

Proof of Proposition VII.7: If $S\emptyset = \emptyset$, then $\mathbb{S} \cong \widehat{\mathbb{S}}^0$ follows from the fact that $r^S : \mathbb{S} \to \widehat{\mathbb{S}}$ has all components on nonempty sets invertible.

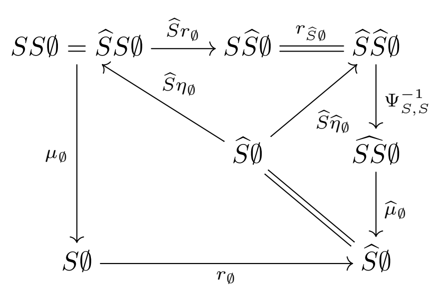
Now suppose that $S\emptyset \neq \emptyset$. We want to prove that $r_\emptyset$ in Corollary VII.2 is invertible. Since $e : \emptyset \to 1$ is injective and S preserves injections we conclude from

$$Se = r_1 \cdot Se = \widehat{S}e \cdot r_\emptyset$$

that $r_\emptyset$ is injective. We will prove that it is a split epic by verifying

$$r_\emptyset \cdot \mu_\emptyset \cdot \widehat{S}\eta_\emptyset = \mathrm{id}_{\widehat{S}\emptyset}.$$

To this end note that $S\emptyset \neq \emptyset$ implies $\widehat{S}S\emptyset = SS\emptyset$ and $Sr_\emptyset = \widehat{S}r_\emptyset$ and consider the diagram below:



Its outside square commutes since r preserves multiplication, the upper triangle does since r preserves the unit and the right-hand one does by the monad laws of $\widehat{S}$. Thus, the left-hand inner part commutes which yields the desired equation. ∎

Proof of Theorem VII.12: (a) We first prove that if a consistent monad $\mathbb{S}$ fulfils $Sf(y) = y$ for all endomorphisms $f : Y \to Y$ and all $y \in \bar{S}Y$, then $\mathbb{S}$ is substantially exceptional. Let $E = S1 \setminus \mathsf{range}(\eta_1)$. We will find a natural isomorphism

$$r_X : X + E \to \mathbb{S}X \qquad (\text{for all } X \neq \emptyset).$$

From that Proposition VII.7 implies that $\mathbb{S} \cong \mathbb{M}_E$ or $\mathbb{M}^0_E$. Given $e \in E$, the element

$$r_X(e) \overset{\mathsf{def}}{=} Sg(e) \text{ where } g \colon 1 \to X$$

is independent of the choice of g. To see this use the assumption $Sf(y) = y$ for all $y \in \bar{S}X$ to obtain for every given $g' \colon 1 \to X$ an $f : X \to X$ with $g' = f \cdot g$. This defines the right-hand component of r_X, the left-hand one is η_X. Naturality is obvious. The map r_X is injective: η_X is injective by assumption, Sg is injective because g is a split monomorphism, and for every $e \in E$ we have $Sg(e) \notin \mathsf{range}(\eta_X)$ (indeed, $g \cdot h = \mathrm{id}_1$ for $h : X \to 1$, and we have $e = Sh(Sg(e)) \notin \mathsf{range}(\eta_1)$). And r_X is also surjective: for every $x \in SX - \eta_X[X]$ apply the above property to the endomorphism $f = g \cdot h$:

$$x = S\mathrm{id}_X(x) = Sf(Sh(x)) = Sf(e), \text{ where } e = Sh(x)$$

(b) To prove the lemma, choose some set Y and an endomorphism $f : Y \to Y$ with

$$Sf(y) \neq y \text{ for some } y \in \bar{S}Y.$$

Put

$$\lambda = \operatorname{card} Y + \aleph_0.$$

Given a set X of cardinality at least λ, there exists $x \in \bar{S}X$ such that the coproduct embeddings $v_1, v_2 : X \to X + X$ fulfil $\bar{S}v_1(x) \neq \bar{S}v_2(x)$; to see this choose $m : Y \to X$ and $e : X \to Y$ with $e \cdot m = \mathrm{id}$, and let $x = Sm(y)$. We prove the above property by contradiction: Suppose that $Sv_1(x) = Sv_2(x)$. Since $g = m \cdot f \cdot e + \mathrm{id} : X + X \to X + X$ fulfils $g \cdot v_1 = m \cdot f \cdot e$ and $g \cdot v_2 = \mathrm{id}$, thus, $S(mfe)(x) = x$ which, since $x = Sm(y)$, implies that

$$Sm(Sf(y)) = x = Sm(y).$$

We know from Lemma IV.1 that Sm is injective, thus, $Sf(y) = y$, a contradiction.

We are prepared to prove $\operatorname{card} \bar{S}X \geq \operatorname{card} X$. Since X is infinite, we have pairwise disjoint injections $\sigma_i : X \to X, i \in I$, where $\operatorname{card} I = \operatorname{card} X$. Arguing as above for $\bar{S}v_1(x) \neq \bar{S}v_2(x)$, we see that, for the coproduct injections $v_i : X \to \coprod_{i \in I} X$, $\bar{S}v_i(x)$ are pairwise distinct for $i \in I$. Since since $\operatorname{card} I = \operatorname{card} X$ we have $X \cong \coprod_{i\in I} X$ and therefore $\operatorname{card} \bar{S}X = \operatorname{card} \bar{S}(\coprod_{i \in I} X) \geq \operatorname{card} I = \operatorname{card} X$. ∎

Proof of Lemma VIII.3: We can assume without loss of generality that H preserves injections. (If it does not, use Trnková closure (Definition VII.3) which has essentially the same fixpoints as H, and generates a free monad iff H does.) Since H is not essentially constant, there exists an infinite cardinal λ as in Proposition VII.11.

We verify that H and F_H have the same fixpoints among sets A of at least λ elements.

(a) If $F_H A \cong A$, then A is a fixpoint of H due to $A \cong F_H A \cong H(F_H A) + A \cong HA + A$ (see Corollary II.6) and $\operatorname{card} HA \geq \operatorname{card} A$ due to the choice of λ.

(b) If $HA \cong A$, then since A is infinite there exists an isomorphism
$$a : HA + A \to A$$
We define a cocone $f_i : (H{+}A)^i 0 \to A$ of the initial chain of $H{+}A$, see Definition III.1, by transfinite induction. The first step and limit steps are clear. For isolated steps put $f_{i+1} = a \cdot (Hf_i + A)$.
It is easy to see by transfinite induction that all f_i's are injective, hence, the free algebra $F_H A$ (which has the form $(H + A)^i 0$ for some ordinal by Proposition III.3) has cardinality at most $\operatorname{card}(HA + A) = \operatorname{card} A$. Since $F_H A \cong HF_H A + A$, we conclude $F_H A \cong A$.

■

Proof of Lemma VIII.4: By Zorn's lemma there exists a maximal almost α-disjoint system $\mathcal{C}$ of subsets of n. Assuming $\operatorname{card} \mathcal{C} \leq n$, we derive a contradiction. Put $\mathcal{C} = \{X_i; i < n\}$.

Since $n > \alpha$ and $\mathsf{cof}\, n = \mathsf{cof}\, \alpha$, there exists a strictly increasing sequence of cardinals n_j, $j < \alpha$, with
$$\alpha < n_j \text{ for all } j \text{ and } n = \sup_{j<\alpha} n_j.$$
For every $j < \alpha$ we see, since $\operatorname{card} X_i = \alpha < n_j$ that $\operatorname{card} \bigcup_{i<n_j} X_i \leq n_j$, therefore there exists
$$x_j \in n_{j+1} - \bigcup_{i<n_j} X_i.$$
The set $A = \{x_j; j < \alpha\}$ meets every member X_i of $\mathcal{C}$ in a subset of $\{x_j; j < \alpha \text{ and } n_j \leq i\}$ which is a set of cardinality less than α, thus, $\mathcal{C} \cup \{A\}$ is almost α-disjoint, a contradiction.

■

Proof of Theorem VIII.6: Sufficiency. Let H be finitary and λ_0 be an infinite upper bound on the cardinalities of Hn, $n < \omega$. For every set X of cardinality at least λ_0 we have, since
$$HX = \bigcup_{n<\omega} \bigcup_{f:n\to X} \mathsf{range}(Hf), \qquad \text{(A.12)}$$
that
$$\operatorname{card} HX \leq \sum_{n<\omega} \lambda_0 \cdot \operatorname{card} X^n = \operatorname{card} X.$$
Combining this with Proposition VII.11 finishes the proof.

Necessity. We use the fact that H preserves finite nonempty intersections (see Theorem VII.1.

The equation (A.12) characterizes finitary set functors, see [2]. Suppose that H is non-finitary. Then we can choose the smallest cardinal α such that $H\alpha$ is not equal to $\bigcup_{n<\omega} \bigcup_{f:n\to\alpha} \mathsf{range}(Hf)$. It follows that
$$H\alpha \neq \bigcup_{\beta<\alpha} \bigcup_{f:\beta\to\alpha} \mathsf{range}(Hf).$$
For otherwise each element of $H\alpha$ is in the range of some Hf, where $f = n \xrightarrow{f'} \beta \xrightarrow{f''} \alpha$, since, by minimality of α,
$$H\beta = \bigcup_{n<\omega} \bigcup_{f:n\to\beta} \mathsf{range}(Hf).$$
We are going to prove that for every set X of cofinality equal to that of α we have $\operatorname{card} HX > \operatorname{card} X$. Since there exists arbitrarily large such sets X, this concludes the proof.

Choose an almost α-disjoint family $X_i, i \in I$, as in Lemma VIII.4; thus the index set I fulfils $\operatorname{card} I > \operatorname{card} X$. Let $m_i : \alpha \to X$ be the corresponding injections with images X_i ($i \in I$). Without loss of generality $X_i \cap X_j \neq \emptyset$ for all $i \neq j$. Choose an element
$$a \in H\alpha - \bigcup_{\beta<\alpha} \bigcup_{f:\beta\to\alpha} Hf[H\beta]. \qquad \text{(A.13)}$$
Then the elements $Hm_i(a)$ are for $i \in I$ pairwise distinct: indeed, from $Hm_i(a) = Hm_j(a)$ it follows that a lies in $Hf[H(X_i \cap X_j)]$ (where $m_i \cdot f = m_j \cdot g$ is the pullback): recall $Hm_i \cap Hm_j = H(m_i \cap m_j)$. This is a contradiction because, since $\beta := \operatorname{card}(X_i \cap X_j) < \alpha$, we have that a lies in the right-hand union (A.13) above. Consequently,
$$\operatorname{card} HX \geq \operatorname{card} I > \operatorname{card} X.$$

■

Proof of Proposition VIII.7: This is analogous to the proof of Theorem VIII.6. Let H be λ-accessible. As proved in [2] this means that the formula in (A.12) holds provided that the first union ranges over all $n < \lambda$. Let $\lambda_0 \geq \lambda$ be an upper bound on cardinalities of Hn, $n < \lambda$. For every set X of cardinality 2^κ, $\kappa \geq \lambda_0$, we have
$$\begin{aligned} \operatorname{card} HX &\leq \sum_{n<\lambda} \lambda_0 \cdot \operatorname{card} X^n \\ &= \lambda \cdot \lambda_0 \cdot 2^\kappa \\ &= \operatorname{card} X. \end{aligned}$$

■

Proof of Corollary VIII.14: The statement is trivial in the case where $\mathbb{S}$ is consistent or substantially exceptional. So assume that it is not. Let λ be a cardinal with $\operatorname{card} SX \geq \operatorname{card} X$ for all sets X of cardinality at least λ, see Theorem VIII.6. Then either $\mathbb{S}$ is not consistent or cardinals $\kappa \geq \lambda$ are fixpoints of S. For every free monad $\mathbb{F}_H$ either H has arbitrarily large fixpoints, or it is essentially constant, see Proposition VIII.2. In the first case $\mathbb{S} \oplus \mathbb{F}_H$ exists because HS generates a free monad: all fixpoints of H from λ onwards are fixpoints of HS. In the latter case $\mathbb{F}_H$ is substantially exceptional. ■

Proof of Corollary VIII.16: If $\mathbb{F}_S$ exists, then by Lemma VIII.3, Proposition VIII.2 and Theorem VIII.6 S has arbitrarily large joint fixpoints with every finitary monad. Now apply Theorem VIII.8. Conversely, if $\mathbb{S} \oplus \mathbb{T}$ exists for every finitary monad $\mathbb{T}$, then S has arbitrarily large fixpoints: for every cardinal λ the monad $\mathbb{T}X = \lambda^* \times X$ of λ unary operations has, by Theorem VIII.8, a joint fixpoint κ with S, and clearly $\kappa \geq \lambda$. ■

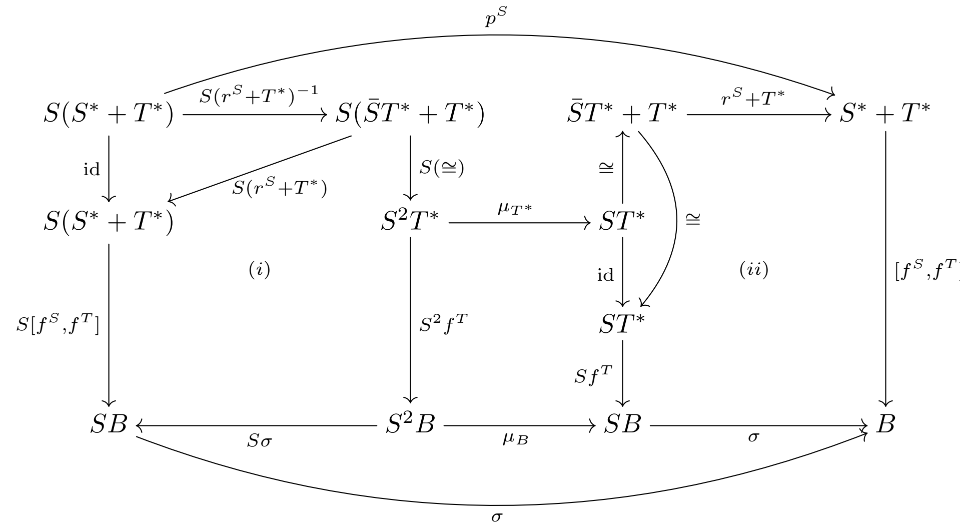


Fig. 2. Showing $[f^S, f^T]$ is an $\mathbb{S}$-algebra morphism in the proof of Theorem V.1(1)

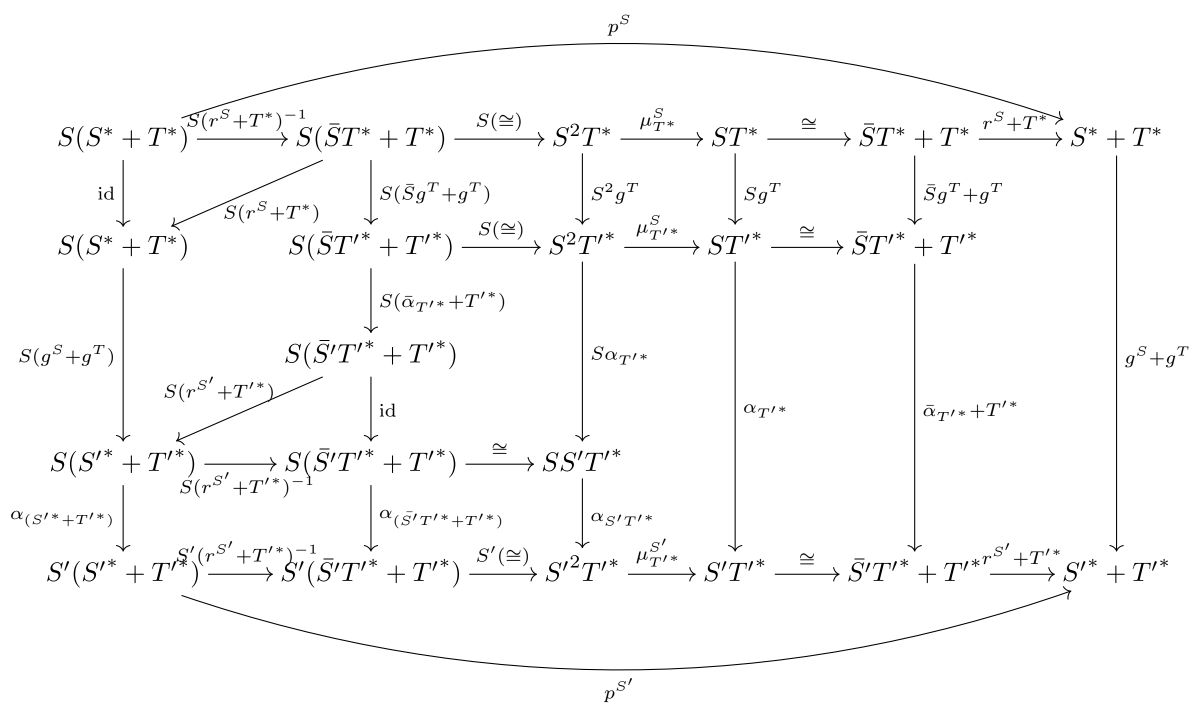


Fig. 3. Showing $g^S + g^T$ is an $\mathbb{S}$-algebra morphism in the proof of Lemma V.3

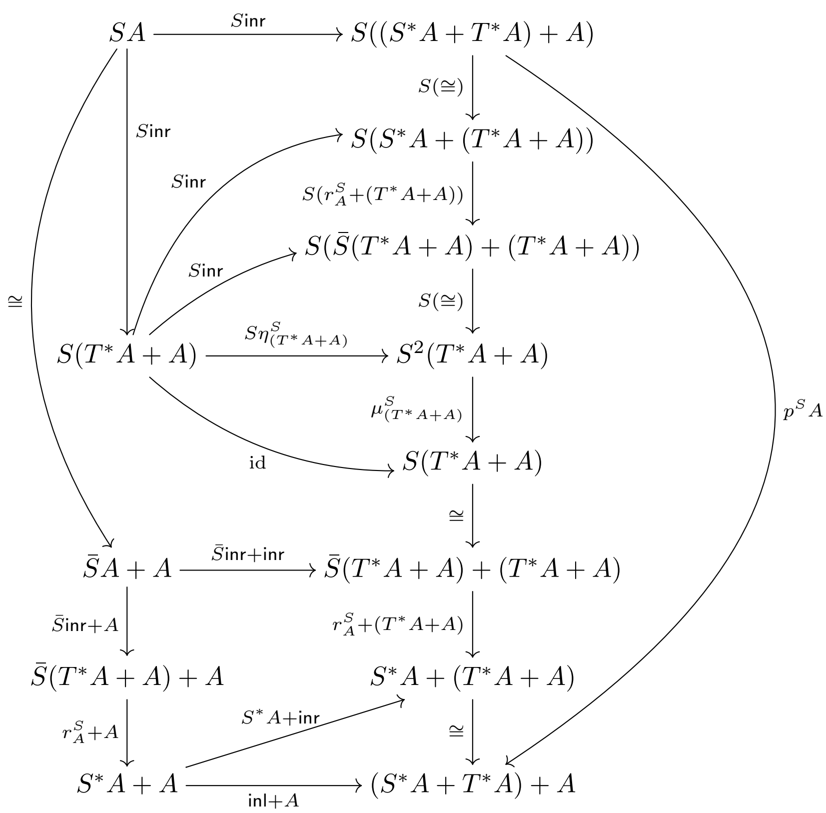


Fig. 4. Showing embedding description in proof of Theorem VI.4