Morphisms between plays

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Examples:

1. Innocence
2. Single-threadedness
3. Asynchronous games

The first two examples use alternating plays.

Plays

A play s consists of

- an initial segment moves $s \subseteq \mathbb{N}$
- (justification pointers) a map moves $s \longrightarrow \{*\} +$ moves s
- (arena elements) a map moves $s \longrightarrow R$ (arena element).

with the usual conditions.

Strategies recap

A strategy σ (Opponent first) on R is a set of even-length plays that

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Traces and divergences

- Finite trace is $s \in \sigma$
- Divergence is $sm\bot$ where $s \in \sigma$ but $\nexists n.\ smn \in \sigma$.
- Infinite trace is infinite play whose even-length prefixes are all in σ.

Innocent Strategies

- A P-visible play is one in which Proponent always points to a move in the current P-view.
- A P-visible strategy is one in which all plays are P-visible.
- An innocent strategy is a special kind of P-visible strategy.
- It can be represented as a set of P-views. (After the first move, Opponent always points to the previous move.)

Innocent strategies are used to model state-free terms.

Single-Threaded Strategies

- A thread-visible play is one in which Proponent doesn't change thread.
- A thread-visible strategy is one in which all plays are thread-visible.
- A single-threaded strategy is a special kind of thread-visible strategy.
- It can be represented as a set of well-opened plays. (After the first move, Opponent never again points to $*$).

Single-threaded strategies are used to model values.

Some problems

Traditional innocence

Doesn't work for certain language features:

- Nondeterminism, under linear time semantics
- Name generation (ν-calculus)
- Polymorphism

In each case there are state-free terms whose denotation isn't "innocent".

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Traditional single-threadedness

Doesn't work for polymorphism.
There are (state-free) values whose denotation isn't "single-threaded".

A category for innocence

We form a category of P-visible plays and P-viewing morphisms.

P-viewing morphism $s \xrightarrow{f} t$

- A function moves $s \longrightarrow$ moves t.
- Preserves arena elements and justification pointers.
- If n^O is followed by m^P then $f(n)$ is followed by $f(m)$.
- If n^O is followed by $\perp$ then $f(n)$ is followed by $\perp$.

Characterizing innocence

Theorem

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There's an analogous theorem for OP-visible strategies.

Variations

Adapt the category to other settings.

In particular names (f should preserve and reflect name equality) and polymorphism.

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Conjecture: these are precisely the strategies definable by a state-free term.

A category for single-threadedness

We form a category of thread-visible plays and threading morphisms.

Threading morphism $s \xrightarrow{f} t$

- A function moves $s \longrightarrow$ moves t.
- Preserves arena elements and justification pointers.
- If n^O is followed by m^P then $f(n)$ is followed by $f(m)$.
- If n^O is followed by $\bot$ then $f(n)$ is followed by $\bot$.
- If m^P is thread-followed by n^O (there might be moves in between, but not from the same thread) then $f(m)$ is thread-followed by $f(n)$.

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Asynchronous Games

Laird, Ghica, Murawski gave a game semantics for concurrent calculi, up to may-testing, as follows.

A strategy σ on R is a set of non-alternating plays

- lower wrt the prefix order
- containing the empty play
- (P-swap) if $sm^P nt \in \sigma$ then $snm^P t \in \sigma$ (assuming it's a play)
- (O-swap)if $smn^O t \in \sigma$ then $sm^O nt \in \sigma$ (assuming it's a play)
- (O-completeness) if $s \in \sigma$ then $sm^O \in \sigma$.

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- Challenging to prove basic categorical properties, e.g. left and right identity laws.
- Challenging to adapt to infinite plays, where we might want to perform infinitely many swaps simultaneously

A category for asynchrony

We form a category of non-alternating plays and asynchrony morphisms.

Asynchrony morphism $s \xrightarrow{f} t$

- A partial injection moves $s \longrightarrow$ moves t.
- Preserves arena elements and justification pointers.
- For any $m^P \in$ moves s,
  - $f(m)$ is defined
  - for any $n^O < f(m)$, there is $n' < m$ such that $f(n') = n$.

Characterizing asynchronous strategies

Theorem

A set of non-alternating plays is an asynchronous strategy iff

it is lower wrt asynchrony morphisms and contains the empty play.

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A set of non-alternating plays is an asynchronous strategy iff

it is lower wrt asynchrony morphisms and contains the empty play.

A similar result was proved by Mohamed Menaa, and used to establish the categorical properties of the asynchronous game model.