Broad Infinity and Generation Principles

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Abstract

We introduce Broad Infinity, a new and arguably intuitive set-theoretic axiom scheme. It states that "broad numbers", which are three-dimensional trees whose growth is controlled by a specified class function, form a set. The Broad Infinity scheme is equivalent (assuming the Axiom of Choice) to the widely studied Ord-is-Mahlo scheme: every closed unbounded class of ordinals contains a regular ordinal.

Whereas the axiom of Infinity leads to generation principles for sets and families and ordinals, Broad Infinity leads to more advanced versions of these principles. We relate these principles under various prior assumptions: the Axiom of Choice, the Law of Excluded Middle, and weaker assumptions.

1 Introduction

1.1 Summary of the Paper

This paper has three main contributions:

- To introduce a new axiom scheme, called Broad Infinity.
- To show that Broad Infinity provides generation principles for families, and—provided the Axiom of Choice (AC) is assumed—for sets and ordinals.
- To show that—again provided AC is assumed—Broad Infinity is equivalent to a previously studied principle called Ord-is-Mahlo [20, 39, 16].

In the course of this introduction, we present the new axiom scheme (Section 1.4), and the generation priniciples for sets (Section 1.5), families (Section 1.6) and ordinals (Section 1.7). Section 1.8 explains the connection to Ord-is-Mahlo.

Following the introduction, we come to the main part of the paper. It begins (Section 2) by formulating a base theory that is weak enough to let us track various assumptions in our proofs, such as the Law of Excluded Middle. Then we develop some methods that will be useful throughout the paper (Section 3–4, and systematically set out the relationships that will be proved, under the various assumptions (Section 5).

This completes the "setting up". We then establish our results in three parts: results about sets and families (Section 6), basic results about ordinals (Section 7), and finally, advanced results about ordinals using powersets (Section 8).

Section 9 illustrates the principles, by showing how they imply the existence of universes, both Grothendieck and Tarski-style. Finally, Section 10 concludes with some topics for further research.

1.2 Order theory

The following concepts are used throughout the paper.

Let A be a poset (or partially ordered class, collection of classes, etc.). An element $a \in A$ is *least* when for all $x \in A$ we have $a \leqslant x$, and *minimal* when every $x \in A$ that is $\leqslant a$ is equal to a. Any least element is the unique minimal element. We say that A is a *meet-semilattice* when every pair of elements has a meet (greatest lower bound). In this case, any minimal element is least. The dual properties hold for *greatest* and *maximal* elements, and *join-semilattices*.

For posets A and B, a map $f : A \to B$ is *monotone* when, for all $x, y \in A$, if $x \leqslant y$, then $f(x) \leqslant f(y)$.

For a poset A and monotone endomap f on A, an element $x \in A$ is a *prefixpoint* when $f(x) \leqslant x$, and a *postfixpoint* when $x \leqslant f(x)$. Note that any minimal prefixpoint is a fixpoint. Moreover, a greatest lower bound of a set of prefixpoints is a prefixpoint, so, if A is a meet-semilattice, then any minimal prefixpoint is a least prefixpoint. The dual properties hold for postfixpoints.

1.3 Sets and Urelements

Recall that ZF assumes that everything is a set, and the axiom of Foundation. The variant that allows urelements is called *ZFA* (where "A" stands for "atoms"), and uses the formula $\mathsf{IsSet}(a)$ to assert that a is a set. The variant that excludes the Axiom of Foundation is called *ZFN* (where "N" stands for "non-well-founded").

Throughout the introduction, we work in either ZF or ZFA or ZFN or ZFAN, according to the reader's preference. In Section 1.4.1 only, we do not assume the axiom of Infinity.

The class of all things (universal class) is denoted $\mathfrak{T}$, and the class of all sets $\mathfrak{S}$. In ZF and ZFN, they are the same.

1.4 From Infinity to Broad Infinity

We present four principles that assert the existence of certain infinite sets, starting with the Axiom of Infinity and ending with Broad Infinity.

1.4.1 Infinity

Although there are various ways of formulating Infinity, the following is most suitable for our purposes. The first step is to define $\mathsf{Zero} \in \mathfrak{T}$ and $\mathsf{Succ} : \mathfrak{T} \to \mathfrak{T}$ in such a way that Succ is injective and never yields Zero. Zermelo's definition [41] achieves this:

$$\begin{aligned} \mathsf{Zero} &\stackrel{\text{def}}{=} \emptyset \\ \mathsf{Succ}(x) &\stackrel{\text{def}}{=} \{x\} \end{aligned}$$

A set X is said to be *nat-inductive* when it satisfies the following.

- $\mathsf{Zero} \in X$.
- For all $x \in X$, we have $\mathsf{Succ}(x) \in X$.

A *set of all natural numbers* is a minimal (and therefore least) nat-inductive set. The axiom of *Infinity* says that there is a set of all natural numbers, written $\mathbb{N}$. As this uniquely specifies a set, I prefer it to the equivalent statement "There is a nat-inductive set", which does not.

Example 1.1. $\mathsf{Succ}(\mathsf{Succ}(\mathsf{Succ}(\mathsf{Zero})))$ is a natural number.

1.4.2 Signature Infinity

Our next infinity principle uses the following notions.

For a set K and class C, a K*-tuple within* C is a function from K to C. We write it as $[a_k]_{k\in K}$ or as a column of maplets $k \mapsto a_k$. The empty tuple is written $[\,]$.

A *signature* $S = (K_i)_{i\in I}$ is a family of sets, meaning that it consists of a set I and, for each $i \in I$, a set K_i. An element $i \in I$ is called a *symbol*, the set K_i its *arity*, and an element of K_i a *position*.[1] A set X is said to be S*-inductive* when, for any $i \in I$ and K_i-tuple $[a_k]_{k\in K_i}$ within X, we have $\langle i, [a_k]_{k\in K_i}\rangle \in X$.

A *set of all* S*-terms* is a minimal (and therefore least) S-inductive set. *Signature Infinity* is the assertion that, for any signature S, there is a set of all S-terms, written $\mathsf{Term}(S)$.[2] We shall prove it below (Proposition 2.6).

Example 1.2. Let S be the signature indexed by $\{5, 6, 7, 8\}$, where symbol 5 has arity $\{0, 1, 2, 3\}$, symbols 6 and 7 have arity $\emptyset$ and symbol 8 has arity $\{0, 1, 2\}$. The following are S-terms:

$$\langle 6, [\,]\rangle$$

$$\langle 7, [\,]\rangle$$

$$\langle 5, \begin{bmatrix} 0 & \mapsto & \langle 7, [\,]\rangle \\ 1 & \mapsto & \langle 6, [\,]\rangle \\ 2 & \mapsto & \langle 7, [\,]\rangle \\ 3 & \mapsto & \langle 7, [\,]\rangle \end{bmatrix}\rangle$$

$$\langle 8, \begin{bmatrix} 0 & \mapsto & \langle 5, \begin{bmatrix} 0 & \mapsto & \langle 7, [\,]\rangle \\ 1 & \mapsto & \langle 6, [\,]\rangle \\ 2 & \mapsto & \langle 7, [\,]\rangle \\ 3 & \mapsto & \langle 7, [\,]\rangle \end{bmatrix}\rangle \\ 1 & \mapsto & \langle 7, [\,]\rangle \\ 2 & \mapsto & \langle 6, [\,]\rangle \end{bmatrix}\rangle$$

An S-term can be visualized as a well-founded tree. For example, the last S-term in Example 1.2 is visualized in Figure 1, using the vertical dimension for $\begin{bmatrix} \vdots \end{bmatrix}$ and the horizontal dimension for internal structure, with the root appearing at the left.

[1] Some authors use "container" for signature [1].

[2] This statement is called "Smallness of W-types" in [37, page 15].

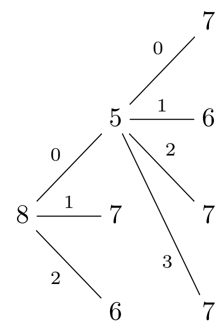

Figure 1: Visualization of an S-term

1.4.3 Reduced Broad Infinity

For our next principle, the first step is to define $\mathsf{Begin} \in \mathfrak{T}$ and $\mathsf{Make} : \mathfrak{T}^2 \to \mathfrak{T}$ in such a way that Make is injective and never yields Begin. We achieve this as follows:

$$\begin{array}{rcl} \mathsf{Begin} & \stackrel{\text{def}}{=} & \emptyset \\ \mathsf{Make}(x, y) & \stackrel{\text{def}}{=} & \{\{x\}, \{x, y\}\} \end{array}$$

A *reduced broad signature* F is a function $F : \mathfrak{T} \to \mathfrak{S}$. For any x, we call Fx the *arity* of x. A set X is said to be F*-inductive* when it satisfies the following.

- $\mathsf{Begin} \in X$.
- For any $x \in X$ and Fx-tuple $[a_k]_{k \in Fx}$ within X, we have $\mathsf{Make}(x, [a_k]_{k \in Fx}) \in X$.

A *set of all* F*-broad numbers* is a minimal (and therefore least) F-inductive set. The axiom scheme of *Reduced Broad Infinity* states that, for every reduced broad signature F, there is a set of all F-broad numbers, written $\mathsf{rBroad}(F)$.

Example 1.3. Let F be the reduced broad signature that sends $\mathsf{Make}(\mathsf{Begin}, [\,])$ to $\{0, 1\}$, and everything else to $\emptyset$. The following are F-broad numbers:

$$\begin{array}{l} \mathsf{Begin} \\ \mathsf{Make}(\mathsf{Begin}, [\,]) \\ \mathsf{Make}(\mathsf{Make}(\mathsf{Begin}, [\,]), \begin{bmatrix} 0 & \mapsto & \mathsf{Begin} \\ 1 & \mapsto & \mathsf{Make}(\mathsf{Begin}, [\,]) \end{bmatrix}) \\ \mathsf{Make}(\mathsf{Make}(\mathsf{Make}(\mathsf{Begin}, [\,]), \begin{bmatrix} 0 & \mapsto & \mathsf{Begin} \\ 1 & \mapsto & \mathsf{Make}(\mathsf{Begin}, [\,]) \end{bmatrix}), [\,]) \end{array}$$

An F-broad number can be visualized as a well-founded three-dimensional tree, using the vertical dimension for $\begin{bmatrix} \vdots \end{bmatrix}$, the horizontal dimension for Make, and the depth dimension for internal structure. The root appears at the front, and the Begin-marked leaves at the rear.

As will be apparent from Proposition 9.2 and our other results, ZFC + Reduced Broad Infinity implies the consistency of ZFC. Therefore, by Gödel's second theorem, ZFC does not prove Reduced Broad Infinity, assuming the consistency of ZFC.

Philosophical remarks. Since there is no hope of proving Reduced Broad Infinity from accepted axioms, one may ask whether it is intuitively justified.[3] Proponents may argue that, at each occurrence of $\mathsf{Make}(x, [a_k]_{k \in Fx})$ inside an F-broad number, the size of the tuple is determined by applying F to the left component x, which "has already been constructed". This seems to provide a clearly specified construction process, by contrast with (say) the process of constructing an ordinal, where one takes *any* transitive set of already-constructed ordinals. Nonetheless the interplay of two forms of justification—from the left and from the rear—may cause some anxiety.

1.4.4 Broad Infinity

For the last of our infinity principles, the first step is to define $\mathsf{Start} \in \mathfrak{T}$ and $\mathsf{Build} : \mathfrak{T}^3 \to \mathfrak{T}$ in such a way that Build is injective and never yields Start. We achieve this as follows:

$$\begin{aligned} \mathsf{Start} &\stackrel{\text{def}}{=} \emptyset \\ \mathsf{Build}(x, y, z) &\stackrel{\text{def}}{=} \{\{x\}, \{x, \{\{y\}, \{y, z\}\}\}\} \end{aligned}$$

A *broad signature* G is a function $\mathfrak{T} \to \mathsf{Sig}$, where Sig is the class of all signatures. A set X is said to be *G-inductive* when the following conditions hold.

- $\mathsf{Start} \in X$.
- For any $x \in X$ with $Gx = (K_i)_{i \in I}$, and any $i \in I$ and K_i-tuple $[a_k]_{k \in K_i}$ within X, we have $\mathsf{Build}(x, i, [a_k]_{k \in K_i}) \in X$.

A *set of all G-broad numbers* is a minimal (and therefore least) G-inductive set. The axiom scheme of *Broad Infinity* states that, for every broad signature G, there is a set of all G-broad numbers, written $\mathsf{Broad}(G)$.

Example 1.4. Let G be the broad signature that

- sends $\mathsf{Build}(\mathsf{Start}, 6, [\,])$ to the signature indexed by $\{7, 8, 9\}$ in which 7 and 8 have arity $\{0, 1\}$ and 9 has arity $\emptyset$
- sends everything else to the signature indexed by $\{4, 5, 6\}$ in which 4 has arity $\{0, 1\}$ and 5 and 6 have arity $\emptyset$.

[3]The justification of set-theoretic axioms is discussed, for example, in [21].

The following are G-broad numbers:

$$\mathsf{Start}$$
$$\mathsf{Build}(\mathsf{Start}, 5, [\,])$$
$$\mathsf{Build}(\mathsf{Start}, 6, [\,])$$
$$\mathsf{Build}(\mathsf{Build}(\mathsf{Start}, 6, [\,]), 8, \begin{bmatrix} 0 & \mapsto & \mathsf{Start} \\ 1 & \mapsto & \mathsf{Build}(\mathsf{Start}, 5, [\,]) \end{bmatrix})$$
$$\mathsf{Build}(\mathsf{Build}(\mathsf{Build}(\mathsf{Start}, 6, [\,]), 8, \begin{bmatrix} 0 & \mapsto & \mathsf{Start} \\ 1 & \mapsto & \mathsf{Build}(\mathsf{Start}, 5, [\,]) \end{bmatrix}), 6, [\,])$$

As before, a G-broad number can be visualized as a well-founded three-dimensional tree, using the vertical dimension for $\begin{bmatrix} \vdots \end{bmatrix}$, the horizontal dimension for Build, and the depth dimension for internal structure. The root appears at the front, and the Start-marked leaves at the rear.

We shall see that Broad Infinity and Reduced Broad Infinity are equivalent. But this relies on the use of classical logic, which includes the law of Excluded Middle. In intuitionistic set theory. where Excluded Middle is not accepted, the ($\Rightarrow$) proof is valid but it may be that Reduced Broad Infinity is strictly weaker. In my opinion, although Broad Infinity is more complicated, it is no less intuitively justified, so it would be strange to accept Reduced Broad Infinity but not Broad Infinity.

We give the name *Broad ZF* to ZF with Infinity replaced by Broad Infinity (or Reduced Broad Infinity). Of course, Infinity follows. Likewise Broad ZFA, Broad ZFC, etc.

1.5 Generation of sets

We now give principles for generating a set from a suitable collection of rules, called a "rubric" or "broad rubric".

1.5.1 Every rubric generates a set

The basic notions are as follows.

Definition 1.5. Let C be a class.

1. A *family within* C consists of a set J, and a function from J to C. It is written as $(b_j)_{j \in J}$. The empty family is written $()$.

2. The class of subsets of C is written $\mathcal{P}_{\mathsf{s}}C$, and the class of families within C is written $\mathsf{Fam}(C)$. In particular, $\mathfrak{S} = \mathcal{P}_{\mathsf{s}}(\mathfrak{T})$, and $\mathcal{P}_{\mathsf{s}}(\mathfrak{S})$ is the class of all sets of sets, , and $\mathsf{Fam}(\mathfrak{T})$ is the class of all families, and $\mathsf{Sig} = \mathsf{Fam}(\mathfrak{S})$.

3. A *rule* $\langle K, R \rangle$ on C consists of a set K (the *arity*) and a function $R : C^K \to \mathsf{Fam}(C)$.[4]

[4]For the purposes of generating a set, we could just as well say function $C^K \to \mathcal{P}_{\mathsf{s}}C$. But for the purposes of generating a family (Section 1.6), it is more suitable to say function $C^K \to \mathsf{Fam}(C)$.

4. A *rubric* $\mathcal{R} = (\langle K_i, R_i \rangle)_{i \in I}$ on C is a family of rules, meaning that it consists of a set I and, for each $i \in I$, a rule $\langle K_i, R_i \rangle$ on C.

We give an example.

Example 1.6. Here is a rubric on $\mathbb{N}$, indexed by $\{0, 1\}$.

- Rule 0 has arity $\{0, 1\}$ and sends $\begin{bmatrix} 0 & \mapsto & m_0 \\ 1 & \mapsto & m_1 \end{bmatrix} \mapsto (m_0 + m_1 + p)_{p \geqslant 2m_0}$.
- Rule 1 has arity $\emptyset$ and sends $[\,] \mapsto (2p)_{p \geqslant 50}$.

These rules prescribe when an element of $\mathbb{N}$ is acceptable. (As we have not defined "acceptable", this is just informal motivation.) Rule 0 says that, if m_0 and m_1 are acceptable, then $m_0 + m_1 + p$ is acceptable for all $p \geqslant 2m_0$. Rule 1 says that $2p$ is acceptable for all $p \geqslant 50$. So 100, 102 and 402 are acceptable, and by induction every acceptable number is $\geqslant 100$.

Informally, the "set generated by $\mathcal{R}$" is the set of all acceptable elements. Here is a precise formulation.

Definition 1.7. Let C be a class.

1. Let $\langle K, R \rangle$ be a rule on C. A subset X of C is said to be $\langle K, R \rangle$-inductive when, for every K-tuple $[a_k]_{k \in K}$ within X with $R_i[a_k]_{k \in K} = (y_p)_{p \in P}$, and every $p \in P$, we have $y_p \in X$.
2. Let $\mathcal{R} = (\langle K_i, R_i \rangle)_{i \in I}$ be a rubric on C. A subset X of C is said to $\mathcal{R}$*-inductive* when, for every $i \in I$, it is $\langle K_i, R_i \rangle$-inductive.

A *set generated by* $\mathcal{R}$ is a minimal (and therefore least) $\mathcal{R}$-inductive subset of C. The *Set Generation* scheme says that says that any rubric $\mathcal{R}$ on $\mathfrak{T}$ generates a set, written $\mathsf{Gen}(\mathcal{R})$. As we shall see (Corollary 3.10), this implies that any rubric on any class does so.

Assuming AC, we shall see that Set Generation holds. As explained in Section 10, a result of Gitik [15] implies that this cannot be shown without AC, assuming the consistency of the existence of arbitrarily large strongly compact cardinals.

1.5.2 Every broad rubric generates a set

We turn to the broad version of the set generation story.

Definition 1.8. Let C be a class. A *broad rubric* $\mathcal{B}$ on C consists of $\mathcal{B}_0 \in \mathbf{Rub}(C)$ and a function $\mathcal{B}_1 : C \to \mathbf{Rub}(C)$, where $\mathbf{Rub}(C)$ denotes the collection of all rubrics on C. (This will be made precise in Section 3.2.) We call $\mathcal{B}_0$ the *basic* rubric. For $x \in C$, we call $\mathcal{B}_1(x)$ the rubric *triggered* by x.

Example 1.9. Here is a broad rubric on $\mathbb{N}$. The basic rubric is as follows, indexed by $\{0, 1\}$.

- Rule 0 has arity $\{0, 1\}$ and sends $\begin{bmatrix} 0 & \mapsto & m_0 \\ 1 & \mapsto & m_1 \end{bmatrix} \mapsto (m_0 + m_1 + p)_{p \geqslant 2m_0}$.

- Rule 1 has arity $\emptyset$ and sends $[\,] \mapsto (2p)_{p\geqslant 50}$.

7 triggers the following rubric, indexed by $\{0\}$.

- Rule 0 has arity $\{0,1\}$ and sends $\begin{bmatrix} 0 & \mapsto & m_0 \\ 1 & \mapsto & m_1 \end{bmatrix} \mapsto (m_0 + m_1 + 500p)_{p\geqslant 9}$.

100 triggers the following rubric, indexed by $\{0,1,2\}$.

- Rule 0 has arity $\{0,1,2\}$ and sends $\begin{bmatrix} 0 & \mapsto & m_0 \\ 1 & \mapsto & m_1 \\ 2 & \mapsto & m_2 \end{bmatrix} \mapsto (m_0 + m_1 m_2 + p)_{p\geqslant 17}$.
- Rule 1 has arity $\emptyset$ and sends $[\,] \mapsto (p)_{p\geqslant 1000}$.
- Rule 2 has arity $\{0,1\}$ and sends $\begin{bmatrix} 0 & \mapsto & m_0 \\ 1 & \mapsto & m_1 \end{bmatrix} \mapsto (m_1 + p)_{p\geqslant 4}$.

Every other natural number triggers the empty rubric.

These rules prescribe when an element of $\mathbb{N}$ is acceptable. (As we have not defined "acceptable", this is just informal motivation.) For example, rule 0 of $\mathcal{B}_1(100)$ says that if 100 is acceptable and m_0, m_1, m_2 are too, then so is $m_0 + m_1 m_2 + p$ for all $p \geqslant 17$. So 100, 102, 402 and 107 are acceptable, and by induction every acceptable number is $\geqslant 100$.

Informally, the "set generated by $\mathcal{R}$" is the set of all acceptable elements. Here is a precise formulation.

Definition 1.10. Let $\mathcal{B}$ be a broad rubric on a class C. A subset X of C is said to be *$\mathcal{B}$-inductive* when it is $\mathcal{B}_0$-inductive and, for every $x \in X$, it is $\mathcal{B}_1(x)$-inductive.

A *set generated by* $\mathcal{B}$ is a minimal (and therefore least) $\mathcal{B}$-inductive subset of C. The *Broad Set Generation* scheme says that every broad rubric $\mathcal{B}$ on $\mathfrak{T}$ generates a set, written $\mathsf{Gen}(\mathcal{B})$. For an illustration of how this is applied, see Section 9.

Assuming AC, we shall see that Broad Infinity implies Broad Set Generation. I do not know whether this can be shown without AC; see the discussion in Section 10.

1.6 Generation of families

Next we give principles for generating a family from a rubric or broad rubric. As we shall see in Section 6.1, they can be proved without assuming AC.

For a family $x = (x_m)_{m\in M}$ and subset $N \subseteq M$, we define the family $x \upharpoonright_N \stackrel{\text{def}}{=} (x_m)_{m\in N}$. For a class C, we partially order the class of families on C by writing $y \leqslant x$ when $x = (x_m)_{m\in M}$ and $y = (y_n)_{n\in N}$ and $N \subseteq M$ and $y = x \upharpoonright_M$. We say that y is *included* in x. The meet of two families $(x_m)_{m\in M}$ and $(y_n)_{n\in N}$ is $(x_m)_{m\in L}$, where $L \stackrel{\text{def}}{=} \{m \in M \cap N \mid x_m = y_m\}$.

1.6.1 Every rubric generates a family

Given a rubric on a class, a *derivation* is a way of showing that an element is acceptable. (As we have not defined "derivation", this is just informal motivation.) Here are some examples of derivations.

Example 1.11. For the rubric in Example 1.6:

- $\langle 1, [\,], 50\rangle$ derives 100.
- $\langle 1, [\,], 51\rangle$ derives 102.
- $\langle 0, \begin{bmatrix} 0 & \mapsto & \langle 1, [\,], 50\rangle \\ 1 & \mapsto & \langle 1, [\,], 50\rangle \end{bmatrix}, 202\rangle$ and $\langle 0, \begin{bmatrix} 0 & \mapsto & \langle 1, [\,], 50\rangle \\ 1 & \mapsto & \langle 1, [\,], 51\rangle \end{bmatrix}, 200\rangle$ derive 402.

Note that each derivation is a triple consisting of a rule index, a tuple of derivations, and an index that yields the result. Informally, the "family generated by $\mathcal{R}$" is the family $(x_m)_{m\in M}$, where M is the set of derivations and $m \in M$ derives x_m. Here is a precise formulation.

Definition 1.12. Let $\mathcal{R}$ be a rubric on a class C. Let $x = (x_m)_{m\in M}$ be a family within C.

1. We say that x is *$\mathcal{R}$-inductive* when the following condition holds. Writing $\mathcal{R} = (\langle K_i, R_i\rangle)_{i\in I}$, for every $i \in I$ and $g : K_i \to M$ with $R_i[x_{g(k)}]_{k\in K_i} = (y_p)_{p\in P}$, and every $p \in P$, we have $\langle i, g, p\rangle \in M$ and $x_{\langle i,g,p\rangle} = y_p$.
2. If x is $\mathcal{R}$-inductive, then a subset N of M is said to be *relatively inductive* when $x \upharpoonright_N$ is $\mathcal{R}$-inductive. This reduces to the following condition: For any $i \in I$ and $g : K_i \to N$ with $R_i[x_{g(k)}]_{k\in K_i} = (y_p)_{p\in P}$ and any $p \in P$, we have $\langle i, g, p\rangle \in N$.

A *family generated by* $\mathcal{R}$ is a minimal (and therefore least) $\mathcal{R}$-inductive family $(x_m)_{m\in M}$ within C. Minimality can be expressed as follows: every relatively inductive subset of M is equal to M. The *Family Generation* scheme says that every rubric $\mathcal{R}$ on $\mathfrak{T}$ generates a family, written $\mathsf{GenFam}(\mathcal{R})$. We shall prove this below (Proposition 6.2).

1.6.2 Every broad rubric generates a family

For our next principle, the first step is to define $\mathsf{Basic} : \mathfrak{T}^3 \to \mathfrak{T}$ and $\mathsf{Trigger} : \mathfrak{T}^4 \to \mathfrak{T}$ in such a way that they are injective and have disjoint range. We achieve this as follows:

$$\begin{aligned} \mathsf{Basic}(x, y, z) &\stackrel{\text{def}}{=} \langle 0, \langle x, y, z\rangle\rangle \\ \mathsf{Trigger}(x, y, z, w) &\stackrel{\text{def}}{=} \langle 1, \langle x, y, z, w\rangle\rangle \end{aligned}$$

Given a broad rubric on a class, a *derivation* is a way of showing that an element is acceptable. (As we have not defined "derivation", this is just informal motivation.) Here are some examples of derivations.

Example 1.13. For the broad rubric in Example 1.9:

- $\mathsf{Basic}(1, [\,], 50)$ is a derivation of 100.
- $\mathsf{Basic}(1, [\,], 51)$ is a derivation of 102.
- $\mathsf{Trigger}(\mathsf{Basic}(1, [\,], 50), 2, \begin{bmatrix} 0 & \mapsto & \mathsf{Basic}(1, [\,], 70) \\ 1 & \mapsto & \mathsf{Basic}(1, [\,], 51) \end{bmatrix}, 5)$ is a derivation of 107.

Informally, the "family generated by $\mathcal{B}$" is the family $(x_m)_{m \in M}$, where M is the set of derivations and $m \in M$ derives x_m. Here is a precise formulation.

Definition 1.14. Let $\mathcal{B}$ be a broad rubric on a class C. Let $x = (x_m)_{m \in M}$ be a family within C.

1. We say that x is *$\mathcal{B}$-inductive* when the following conditions hold.
   - Writing $\mathcal{B}_0 = (\langle K_i, R_i \rangle)_{i \in I}$, for every $i \in I$ and $g : K_i \to M$ with $R_i[x_{g(k)}]_{k \in K_i} = (y_p)_{p \in P}$, and every $p \in P$, we have $\mathsf{Basic}(i, g, p) \in M$ and $x_{\mathsf{Basic}(i,g,p)} = y_j$.
   - For every $m \in M$ with $\mathcal{B}_1(x_m) = (\langle K_i, R_i \rangle)_{i \in I}$, and every $i \in I$ and $g : K_i \to M$ with $R_i[x_{g(k)}]_{k \in K_i} = (y_p)_{p \in P}$, and every $p \in P$, we have $\mathsf{Trigger}(m, i, g, p) \in M$ and $x_{\mathsf{Trigger}(m,i,g,p)} = y_p$.
2. If x is $\mathcal{B}$-inductive, then a subset N of M is said to be *relatively inductive* when $x \upharpoonright_N$ is $\mathcal{B}$-inductive. This reduces to the following conditions:
   - Writing $\mathcal{B}_0 = (\langle K_i, R_i \rangle)_{i \in I}$, for any $i \in I$ and $g : K_i \to N$ with $R_i[x_{g(k)}]_{k \in K_i} = (y_p)_{p \in P}$, and any $p \in P$, we have $\mathsf{Basic}(i, g, p) \in N$.
   - For any $m \in N$ with $\mathcal{B}_1(x_m) = (\langle K_i, R_i \rangle)_{i \in I}$, and any $i \in I$ and $g : K_i \to N$ with $R_i[x_{g(k)}]_{k \in K_j} = (y_p)_{p \in P}$, and any $p \in P$, we have $\mathsf{Trigger}(m, i, g, p) \in N$.

A *family generated by $\mathcal{B}$* is a minimal (and therefore least) $\mathcal{B}$-inductive family $(x_m)_{m \in M}$ within C. Minimality can be expressed as follows: every relatively inductive subset of M is equal to M. The *Broad Family Generation* scheme says that every broad rubric $\mathcal{B}$ on $\mathfrak{T}$ generates a family, written $\mathsf{GenFam}(\mathcal{B})$. For an illustration of how this is applied, see Section 9. Broad Infinity implies Broad Family Generation, as we shall see in Proposition 6.3.

1.7 Generation of regular limits

We now come to principles that have appeared in the literature. We write Ord for the class of ordinals, and $\mathrm{S}(\alpha)$ for the successor of an ordinal α, and $\bigvee_{i \in I} \alpha_i$ for the supremum of a family of ordinals $(\alpha_i)_{i \in I}$. Recall that an ordinal is a *limit* when it is neither 0 nor a successor, and *regular* when it is equal to its cofinality. Thus a regular ordinal is either 0, 1 or a regular limit.

For an ordinal α, a *regular limit generated by α* is a minimal (and therefore least) regular limit $\geqslant \alpha$. The *Blass Generation* principle [6] says that every ordinal α generates a regular limit, written $\mathsf{Gen}(\alpha)$.

Let J be an *ordinal function*, meaning a (not necessarily monotone) function $\mathsf{Ord} \to \mathsf{Ord}$. An ordinal λ is said to be $\geqslant J$ when $\lambda \geqslant J\beta$ for all $\beta < \lambda$. A *regular limit generated by* J is a minimal (and therefore least) regular limit $\geqslant J$.

The *Jorgensen Generation* scheme [17, 24] says that every ordinal function J generates a regular limit, written $\mathsf{Gen}(J)$. Note that this gives us arbitrarily large regular limits $\geqslant J$. To see this, let J_α be the ordinal function sending β to $J\beta \vee \alpha$. Then, for any ordinal $\lambda > 0$, it is $\geqslant J_\alpha$ iff it is both $\geqslant J$ and $\geqslant \alpha$. For an illustration of how Jorgensen generation is applied, see Section 9.

We shall see that Blass Generation is equivalent to Set Generation, and Jorgensen Generation to Broad Set Generation. But this relies on our adoption of classical logic. In the intuitionistic setting, the story is more subtle, beginning with appropriate definitions of "limit" and "regular limit". All this is presented in Section 7–8.

1.8 Classes of ordinals

In the literature, where Excluded Middle is assumed, the following notions are often used. A class C of ordinals is *unbounded* when for any ordinal α there is $\beta \in B$ such that $\beta > \alpha$. It is *closed* when, for any limit λ, if $\lambda = \mathsf{sup}(\lambda \cap C)$, then $\lambda \in C$. These notions give rise to the following principles.

- *Blass's axiom* [6]: The class of regular ordinals is unbounded.
- The *Ord-is-Mahlo* scheme [20, 39, 16]: Every closed unbounded class of ordinals contains a regular ordinal.

These principles are connected to this paper as follows.

Proposition 1.15.

1. *Blass's axiom is equivalent to Blass Generation.*
2. *[17] Ord-is-Mahlo is equivalent to Jorgensen Generation.*

Proof.

1. Obvious.
2. For ($\Rightarrow$), given an ordinal function J, let C be the class of limits $\geqslant J$. For any subset X of C, we have $\sup X \in C$, so we need only prove unboundedness. Given an ordinal $\alpha > 1$, let the sequence $(\beta_n)_{n\in\mathbb{N}}$ be defined by $\beta_0 \stackrel{\text{def}}{=} \alpha$ and $\beta_{n+1} \stackrel{\text{def}}{=} \bigvee_{\gamma<\beta_n} J\gamma$. Then $\bigvee_{n\in\mathbb{N}} \beta_n$ is the least $\beta \geqslant \alpha$ such that $\beta \in C$.
   For ($\Leftarrow$), let C be a closed unbounded class. For each ordinal β, let $G\beta$ be the least ordinal $> \beta$ that is in B, and let $J\beta \stackrel{\text{def}}{=} \mathrm{S}((G\beta))$. Let λ be the regular limit generated by J. We show that $\lambda = \bigvee(\lambda \cap C)$, giving $\lambda \in C$. Suppose $\beta < \lambda$. Then $J\beta \leqslant \lambda$, hence $G\beta < \lambda$, hence $G\beta \in \lambda \cap B$, and $\beta < G\beta$, so $\beta \in \bigvee(\lambda \cap B)$. $\square$

Various principles equivalent to Ord-is-Mahlo have been studied [20, 26, 10], and similar principles have been given for type theory [28, 34] and Explicit Mathematics [18]. Other principles have been shown to be equiconsistent with Ord-is-Mahlo [16, 23].

2 The Base Theory

2.1 Motivation

The primary goal of this paper is to study extensions of ZFC. But I also have secondary goals:

1. To track the use of the Axiom of Choice (AC) and Excluded Middle.
2. To make clear that the main results still hold if urelements and/or non-well-founded sets are admitted.

For the sake of these secondary goals, the paper adopts a base theory that does not assume AC or Excluded Middle, and allows urelements and non-well-founded sets.

2.2 The Base Theory

To begin, say that a *logical signature* Σ consists of a set of *predicate symbols* and a set of *function symbols*, where each symbol is equipped with a natural number, called its *arity*. The *Base Theory* on Σ is an intuitionistic first-order theory with equality. It uses the predicate symbols isSet and $\in$ and all the symbols in Σ. The following syntactic notion helps us to formulate axiom and theorem schemes.

Definition 2.1. Let $\overrightarrow{u}$ be a list of variables. A *formula on* $\overrightarrow{u}$ is a formula whose free variables all appear in $\overrightarrow{u}$. More generally, for $n \in \mathbb{N}$, an *n-ary abstracted formula on* $\overrightarrow{u}$ is a formula whose free variables all appear in $\overrightarrow{u}, \overrightarrow{x}$, where $\overrightarrow{x}$ is a list of n special variables disjoint from $\overrightarrow{u}$.

The theory is axiomatized in two parts. The first part is as follows.

- Axiom of *Extensionality*: Any two sets with the same elements are equal.
- Axiom of *Inhabitation*: Anything that has an element is a set.
- Axiom of *Empty Set*: There is a set with no elements.
- Axiom of *Pairing*: For any a and b, there is a set whose elements are a and b.
- Axiom of *Union Set*: For any set of sets $\mathcal{A}$, there is a set of all elements of elements of $\mathcal{A}$.
- Axiom scheme of *Replacement*: For any relation R (given by a binary abstracted formula) and set A, if every $a \in A$ has a unique R-image (i.e., b such that $a \, R \, b$), then there is a set of all R-images of elements of A.
- Axiom scheme of *Truth Value Separation*: Write $* \stackrel{\text{def}}{=} \emptyset$. For every proposition ψ (given by a formula), there is a set of all x such that $x = *$ and ψ holds. It is called the *truth value* of ψ and written 1_ψ.

Before continuing, we note that the *Separation* scheme is already provable: For every set A and predicate P, there is a set of all $x \in A$ such that $P(x)$. To see this, put

$$\{x \in A \mid P(x)\} \quad \stackrel{\text{def}}{=} \quad \bigcup_{x \in A} \bigcup_{y \in 1_{P(x)}} \{x\}$$

So an intersection of two sets is a set, and therefore any minimal nat-inductive set is a least one. The second part of the axiomatization is as follows.

- Axiom of *Infinity*: There is a set of all natural numbers.
- Axiom of *Exponentiation*: For any sets A and B, there is a set B^A of all functions from A to B.

Henceforth we fix a logical signature Σ, and assume the Base Theory on Σ.

A *truth value* is a subset of $1 \stackrel{\text{def}}{=} \{*\}$, e.g. 0 or 1.

Proposition 2.2. *The following are equivalent.*

Axiom of Truth Value Set*: There is a set Ω of truth values.*

Axiom of Powerset*: For any set A, there is a set $\mathcal{P}A$ of subsets of A.*

Proof. Via $\mathcal{P}A \stackrel{\text{def}}{=} \{\mathsf{Range}(f) \mid f \in \Omega^A\}$ and $\Omega \stackrel{\text{def}}{=} \mathcal{P}1$. □

As usual in intuitionistic mathematics, for a proposition ψ, its *negation* $\neg\psi$ is defined to be $\psi \Rightarrow \mathsf{False}$.

Proposition 2.3. *The following are equivalent.*

- *Axiom of* Boolean Truth*: Every truth value is either 0 or 1.*
- *Law of* Excluded Middle*: For every proposition ψ, either ψ or $\neg\psi$.*
- *Axiom of* Decidable Equality *[4]: For all a, b, either $a = b$ or $a \neq b$.*

Proof.

Decidable Equality $\Rightarrow$ Boolean Truth: for any truth value t, we have either $t = 1$ or $t \neq 1$.

Boolean Truth $\Rightarrow$ Excluded Middle: for any ψ, the truth value 1_ψ is either 0 or 1.

Excluded Middle $\Rightarrow$ Decidable Equality: obvious. □

Remark on related work.

There are two major schools of set theory that do not accept Boolean Truth. One is the IZF school, which accepts both Truth Value Separation and Truth Value Set. The other is the CZF school, which restricts the former and does not accept the latter. These are explained in [8, 13, 12, 3, 37, 33, 4].

The Base Theory we have adopted follows an intermediate policy: acceptance of Truth Value Separation but not Truth Value Set. See Appendix A for a version of the paper using a weaker base theory that meets the requirements of the CZF school.

In this paper, I generally refer to Truth Value Separation rather than Separation, to Truth Value Set rather than Powerset, and to Boolean Truth rather than Excluded Middle. This is to emphasize the role of truth values, and in the last case also to be compatible with Appendix A, where Excluded Middle is stronger than Boolean Truth.

2.3 Descendants

To analyze the nature of $\mathfrak{T}$ and $\mathfrak{S}$, we first mention the following hypotheses.

- Axiom scheme of $\in$-*induction*: Any predicate that is $\in$-*inductive*—i.e., satisfied by everything whose elements all satisfy it—is satisfied by everything.
- Axiom of *Purity*: Everything is a set.
- Axiom of *Decidable Sethood* [4]: Everything is either a set or not a set.

Note that Decidable Sethood follows from Boolean Truth, and also from Purity. When reading the next result, bear in mind that Decidable Sethood is not assumed.

Proposition 2.4.

1. *Every thing* e *has an* element set*, meaning a set with the same elements as* e*, written* $\mathcal{E}(e)$.[5]
2. *Every thing* e *has a* descendant set*, meaning a minimal (and therefore least) transitive set* $\ni e$*, written* $\mathcal{E}^*(e)$.

Proof.

1. Put $\mathcal{E}(e) \stackrel{\text{def}}{=} \bigcup_{x \in 1_{\mathsf{IsSet}(e)}} e$.
2. Put $\mathcal{E}^*(e) \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{N}} \mathcal{E}^n(e)$, where the sequence of sets $(\mathcal{E}^n(e))_{n \in \mathbb{N}}$ is defined recursively by $\mathcal{E}^0(e) \stackrel{\text{def}}{=} \{e\}$ and $\mathcal{E}^{n+1}(e) \stackrel{\text{def}}{=} \bigcup_{x \in \mathcal{E}^n(e)} \mathcal{E}(x)$. $\square$

Definition 2.5.

1. For any set A, its *transitive closure* is $\bigcup_{x \in A} \mathcal{E}^*(x)$.
2. For any thing e, its *strict descendant set*, written $\mathcal{E}^+(e)$, is the transitive closure of $\mathcal{E}(e)$.

2.4 Establishing Signature Infinity

The following is a key property of the Base Theory.

Proposition 2.6. *Signature Infinity holds.*

Proof. Let $S = (K_i)_{i \in I}$ be a signature. For a set X, let $\Gamma_S X \stackrel{\text{def}}{=} \sum_{i \in I} X^{K_i}$. For a function f on X, let $\Gamma_S f$ be the function on $\Gamma_S X$ sending $\langle i, [a_k]_{k \in K_i} \rangle$ to $\langle i, [f a_k]_{k \in K} \rangle$. For $f : X \to Y$ we have $\Gamma_S f : \Gamma_S X \to \Gamma_S Y$, and Γ_S preserves identities and composition. In brief, Γ_S is an endofunctor on the category of sets. Let r be the unique map from $\Gamma_S 1$ to 1. Following [5] we form the ω^{op}-chain

$$1 \xleftarrow{r} \Gamma_S 1 \xleftarrow{\Gamma_S r} \Gamma_S^2 1 \xleftarrow{\Gamma_S^2 r} \cdots$$

[5]This is a special case of Proposition 3.6(1) below, as $e \mapsto \mathcal{E}(e)$ is the unique function $\mathfrak{T} \to \mathfrak{S}$ that extends the identity on $\mathfrak{S}$ and is supported on $\mathfrak{S}$.

known as the "final-coalgebra chain". (Intuitively $\Gamma_S^n 1$ is the set of S-trees with stumps at level n.) Let M be the limit, i.e. the set of sequences $[x_n]_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \Gamma_S^n 1$ such that, for all $n \in \mathbb{N}$, we have $(\Gamma_S^n r)(x_{n+1}) = x_n$. (Intuitively M is the set of S-trees.) For each $n \in \mathbb{N}$, let $\pi_n : M \to \Gamma_S^n 1$ be the projection. By the limit property, there is a unique map $\theta : \Gamma_S M \to M$ such that the triangle

$$\begin{tikzcd} M \arrow[d, "\pi_{n+1}"'] & \Gamma_S M \arrow[l, "\theta"'] \arrow[dl, "\Gamma_S \pi_n"] \\ \Gamma_S^{n+1} 1 & \end{tikzcd}$$

commutes for all $n \in \mathbb{N}$. (Intuitively θ analyzes an S-tree into its root symbol and components.) Moreover, θ is bijective, since the functor Γ_S preserves limits of connected diagrams up to isomorphism. Let N be the least subset of M closed under θ. (Intuitively N is the set of well-founded S-trees.) By well-founded recursion, there is a unique function p on N such that $p\theta\langle i, [a_k]_{k\in K_i}\rangle = \langle i, [pa_k]_{k\in K}\rangle$. By induction, it is injective. Its range is a set of all S-terms. $\square$

An S-term gives rise to a map from branches to results. For example, in the S-term shown in Figure 1, the empty branch $()$ has result 8 and the branch $(0, 3)$ has result 7. To be precise, let $S = (K_i)_{i\in I}$ be a signature. For an S-term t, a *branch* is a sequence $(k_0, \ldots, k_{n-1})$ within $\bigcup_{i\in I} K_i$ such that there is a (necessarily unique) sequence $t = s_0, \ldots, s_n$ of S-terms such that, for all $m < n$, writing $s_m = \langle i, [r_k]_{k\in K_i}\rangle$, we have $k_m \in K_i$ and $s_{m+1} = r_{k_m}$. Finally, writing $s_n = \langle i, [r_k]_{k\in K_i}\rangle$, the *result* of the branch is i. The set of branches of t is written $\mathsf{Branches}(t)$. The following implies that an S-term is determined by the map from branches to results.

Proposition 2.7. *Let S be a signature. For any $s, t \in \mathsf{Term}(S)$, if every $b \in \mathsf{Branches}(s) \cap \mathsf{Branches}(t)$ has the same result in s and t, then $s = t$.*

Proof. Induction on s. $\square$

3 Working with classes

3.1 Basic operations

This section sets out various useful ways of working with classes. We begin with the following conventions.

- A *class* is given by a unary abstracted formula.
- Let I be a class. A *class dependent on* $i \in I$ is given by a binary abstracted formula.
- Let C be a class. A *function* F on C is a relation, given by a binary abstracted formula, subject to the hypothesis that every $x \in C$ has a unique image, written $F(x)$. A *partial function* on C is a subclass B together with a function on B.

Henceforth we speak about classes in the usual relaxed way. The following operations are standard.

Definition 3.1.

1. Let C and D be classes. Then $C+D$ is the class $\{\mathsf{inl}\ x \mid x \in C\} \cup \{\mathsf{inr}\ y \mid y \in D\}$, writing $\mathsf{inl}\ x \stackrel{\text{def}}{=} \langle 0, x\rangle$ and $\mathsf{inr}\ y \stackrel{\text{def}}{=} \langle 1, y\rangle$.
2. Let I be a class, and C_i a class dependent on $i \in I$. Then $\sum_{i\in I} C_i$ is the class of pairs $\langle i, x\rangle$ with $i \in I$ and $x \in C_i$.
3. Let I be a set, and C_i be a class dependent on $i \in I$. Then $\prod_{i\in I} C_i$ is the class of functions f such that for all $i \in I$ we have $f(i) \in C_i$.

Note that for a family of sets $(A_i)_{i\in I}$, both $\sum_{i\in I} A_i$ and $\prod_{i\in I} A_i$ are sets, the latter since it is $\{f \in (\bigcup_{i\in I} A_i)^I \mid \forall i \in I.\, f(i) \in A_i\}$.

Definition 3.2.

1. The class of truth values is denoted Ω.
2. For each truth value t, we write $R(t)$ for the corresponding subset of 1. (Our chosen representation of truth values gives $R(t) = t$.)
3. For a class D, we define $D_\perp \stackrel{\text{def}}{=} \sum_{t\in\Omega} D^{R(t)}$. An element of $D_\perp$ is called a *partial element* of D.

Note that the partial elements of D correspond to the subsets of D that are *subsingleton*, i.e. have the property that any two elements are equal.

Definition 3.3. Let C be a class, and D_x a class dependent on $x \in C$.

1. A *function* $G : (x \in C) \to D_x$ is a function G on C such that for all $x \in C$ we have $G(x) \in D_x$.
2. A *partial function* $\langle B, F\rangle : (x \in C) \rightharpoonup D_x$ consists of a subclass B of C (the *domain*) and a function $F : (x \in B) \to D_x$.

Note that a partial function $(x \in C) \rightharpoonup D_x$ corresponds to a function $C \to (D_x)_\perp$.

The usual case is where D_x is the same class D for all $x \in C$. We then write simply $G : C \to D$ or $\langle B, F\rangle : C \rightharpoonup D$.

3.2 Functions That Yield Families

We look next at a special kind of function: one that yields a family or a rubric. For example, the definition of broad rubric on a class C involves a function $C \to \mathbf{Rub}(C)$, where $\mathbf{Rub}(C)$ denotes the "colletion" of rubrics on C. We explain such functions as follows.

Let C be a class and D_x a class depending on $x \in C$. Then a function $(x \in C) \to \mathsf{Fam}(D_x)$ can be represented by the following data.

- A function $F\colon C \to \mathfrak{S}$, given by a binary abstracted formula.
- A function G sending each $x \in C$ and $i \in G(y)$ to an element of D_x, given by a 3-ary abstracted formula.

The data (F, G) represents the function sending $x \in C$ to the signature $(G(x,i))_{i \in F(x)}$. In the same way, a function $(x \in C) \to \mathbf{Rub}(D_x)$ is represented by the following data.

- A function $F\colon C \to \mathfrak{S}$, given by a binary abstracted formula.
- A function G sending each $x \in C$ and $i \in F(y)$ to a set, given by a 3-ary abstracted formula.
- A function H sending each $x \in C$ and $i \in F(y)$ and $G(x,i)$-tuple $[a_k]_{k \in G(x,i)}$ within D_x to an element of D_x, given by a 4-ary abstracted formula.

The data (F, G, H) represents the function sending $x \in C$ to the rubric $(\langle G(x,i), R_i \rangle)_{i \in F(x)}$ on D_x, where, for $i \in F(x)$, the function R_i sends a $G(x,i)$-tuple $[a_k]_{k \in G(x,i)}$ within D_x to $h(x, i, [a_k]_{k \in G(x,i)})$.

3.3 Extending Functions

We often want to extend a function defined on a class B to a larger class C. We now give some methods for doing this, to be used (in particular) in the proof of Proposition 6.3. They involve the following notions.

Definition 3.4.

1. A set X is *inhabited* when it has an element.
2. A family $(x_i)_{i \in I}$ is *inhabited* when I has an element.
3. A rubric $(\langle K_i, R_i \rangle)_{i \in I}$ on a class C is *inhabited* when I has an element.

Note that "inhabited" implies not empty, and conversely if Boolean Truth is assumed.

Definition 3.5. Let C be a class and B a subclass. Let D_x be a class depending on $x \in C$.

1. A function $G : (x \in C) \to \mathcal{P}_{\mathsf{s}} D_x$ is said to be *supported on* B when, for all $x \in C$, if $G(x)$ is inhabited, then $x \in B$.
2. Likewise for a function $G : (x \in C) \to \mathsf{Fam}(D_x)$.
3. Likewise for a function $G : (x \in C) \to \mathbf{Rub}(D_x)$.

Note that "supported on B" implies that every $x \in C \setminus B$ is sent to the empty set (or the empty family, or the empty rubric) and conversely if Boolean Truth is assumed.

Proposition 3.6. *Let C be a class and B a subclass. Let D_x be a class depending on $x \in C$.*

1. *Let D_x be a class depending on $x \in C$. Any function $(x \in B) \to \mathcal{P}_{\mathsf{s}} D_x$ extends uniquely to a function $(x \in C) \to \mathcal{P}_{\mathsf{s}} D_x$ that is supported on B.*

2. *Likewise for a function $(x \in B) \to \mathsf{Fam}(D_x)$.*

3. *Likewise for a function $(x \in B) \to \mathbf{Rub}(D_x)$.*

Proof.

1. For a function $F : (x \min B) \to \mathcal{P}_{\mathsf{s}} D_x$, the extension sends $x \in C$ to $\bigcup_{y \in 1_{x \in B}} F(x)$.

2. As explained in Section 3.2, a function $B \to \mathsf{Fam}(D_x)$ is described by a pair of functions (F, G), and the required function $B \to \mathsf{Fam}(D_x)$ will be described by (F', G'). We define F' to be the unique extension of $F : B \to \mathfrak{S}$ to a function $C \to \mathfrak{S}$ that is supported on B. For $x \in C$ and $i \in F'(x)$, we have $x \in B$ and define $F'(x, i) \stackrel{\text{def}}{=} G(x, i)$.

3. Similar.

4. Similar. □

Corollary 3.7. *Let B be a class.*

1. *Every function $B \to \mathfrak{S}$ extends uniquely to a reduced broad signature that is supported on B.*

2. *Every function $B \to \mathsf{Sig}$ extends uniquely to a broad signature that is supported on B.*

Definition 3.8. Let C be a class and B a subclass.

1. Let $\mathcal{S} = (\langle K_i, S_i \rangle)_{i \in I}$ be a rubric on C.

   - It is said to *extend* a rubric $\mathcal{R} = (\langle K_i, R_i \rangle)_{i \in I}$ on B when, for all $i \in I$, the function $S_i : C^{K_i} \to \mathsf{Fam}(C)$ extends R_i.
   - It is said to be *supported on* B when, for all $i \in I$, the function S_i is supported on B.

2. Let $\mathcal{C}$ be a broad rubric on C.

   - It is said to *extend* a broad rubric $\mathcal{B}$ on B when the rubric $\mathcal{C}_0$ extends $\mathcal{B}_0$ and, for all $x \in B$, the rubric $\mathcal{C}_1(x)$ extends $\mathcal{B}_1(x)$.
   - It is said to be *supported on* B when the rubric $\mathcal{C}_0$ is supported on B, and the function $\mathcal{C}_1 : C \to \mathbf{Rub}(C)$ is supported on B, and, for all $x \in B$ (and hence for all $x \in C$), the rubric $\mathcal{C}_1(x)$ is supported on B.

Proposition 3.9. *Let C be a class and B a subclass.*

1. *Every rubric on B extends uniquely to a rubric on C that is supported on B.*

2. *Every broad rubric on B extends uniquely to a broad rubric on C that is supported on B.*

Proof. Follows from Proposition 3.6(2)–(3). □

Corollary 3.10. *For any class C, the following entailments between schemes hold.*

1. *If every rubric on C generates a set, then so does every rubric on a subclass of C.*
2. *If every rubric on C generates a family, then so does every rubric on a subclass of C.*
3. *If every broad rubric on C generates a set, then so does every broad rubric on a subclass of C.*
4. *If every broad rubric on C generates a family, then so does every broad rubric on a subclass of C.*

Strictly speaking, in results of this kind, C must be defined by a closed formula. This formula may, of course, contain symbols from the logical signature Σ.

Henceforth, we write "Set Generation (C)" for the scheme saying that every rubric on C generates a set, and likewise "Broad Set Generation (C)".

4 Spections and Introspections

For a class C, say that a *large family* within C consists of a class M and function $M \to C$. It is written $(x_m)_{m \in M}$. As with families, we write $\leqslant$ for the inclusion relation on large families.[6]

This section provides methods for generating classes and large families. They are used in proving Propositions 6.3–6.6.

4.1 Spective Generation of Classes

Given a broad signature G, before asking whether there is a set of all G-broad numbers, a preliminary question is whether there is a *class* of all G-broad numbers, meaning a minimal (and therefore least) G-inductive class. We shall now show that the answer is yes, using the following notions.

Definition 4.1.

1. A *spection* consists of a class M and, for each $e \in M$, a set $J(e)$. We say that $e \in M$ is *suitable*, and then $d \in J(e)$ is a *child* of e.
2. An *introspection* is a spection $(J(e))_{e \in M}$ such that for all $e \in M$ we have $J(e) \subseteq \mathcal{E}^+(e)$.

All the spections used in this paper are introspections.

Proposition 4.2. *Let $\mathcal{M} = (J(e))_{e \in M}$ be a spection. Say that a set X is $\mathcal{M}$-*transitive *when, for all $x \in X \cap M$, we have $J(x) \subseteq M$.*

[6] The phrase "large family" has a different meaning in [14].

1. *Every e has an $\mathcal{M}$-*descendant set*, meaning a minimal (and therefore least) $\mathcal{M}$-transitive set $\ni e$, written $J^*(e)$.*

2. *Moreover, if $\mathcal{M}$ is an introspection, then $J^*(e) \subseteq \mathcal{E}^*(e)$.*

Proof.

1. Put $J^*(e) \stackrel{\text{def}}{=} \bigcup_{n\in\mathbb{N}} J^n(e)$, where the sequence of sets $(J^n(e))_{n\in\mathbb{N}}$ is defined recursively by $J^0(e) \stackrel{\text{def}}{=} \{e\}$ and $J^{n+1}(e) \stackrel{\text{def}}{=} \bigcup_{x\in J^n(e)\cap M} J(x)$.

2. Since $\mathcal{E}^*(e)$ is an $\mathcal{M}$-transitive set $\ni e$. □

Definition 4.3. Let $\mathcal{M} = (J(e))_{e\in M}$ be a spection. Then the monotone operator $\Gamma_\mathcal{M}$ on classes sends A to to the class of all $e \in M$ such that $J(e) \subseteq A$.

Definition 4.4. Let $\mathcal{M} = (J(e))_{e\in M}$ be a spection. Let A be a class.

- We say that A is $\mathcal{M}$-*inductive* when $\Gamma_\mathcal{M}(A) \subseteq A$. Spelling this out: For all $e \in M$, if $J(e) \subseteq A$, then $e \in A$.

- We say that A is $\mathcal{M}$-*coinductive* when $A \subseteq \Gamma_\mathcal{M}(A)$. Spelling this out: For all $e \in A$, we have $e \in M$ and $J(e) \subseteq A$.

- We say that A is *generated by* $\mathcal{M}$ when it is a minimal (and therefore least) $\mathcal{M}$-inductive class.

- We say that A is *cogenerated by* $\mathcal{M}$ when it is a maximal (and therefore greatest) $\mathcal{M}$-coinductive class.

- We say that A is *bigenerated by* $\mathcal{M}$ when it is both generated and cogenerated by $\mathcal{M}$.

Note that any bigenerated class is a unique fixpoint of $\Gamma_\mathcal{M}$.

Proposition 4.5.

1. *Any spection $\mathcal{M}$ generates a class, written* $\mathsf{Gen}(\mathcal{M})$*, and cogenerates a class, written* $\mathsf{Cogen}(\mathcal{M})$.

2. (Assuming $\in$-induction.) *Any introspection bigenerates a class.*

Proof.

1. Let $\mathcal{M} = (J(e))_{e\in M}$ be a spection. Define $\mathsf{Cogen}(\mathcal{M})$ to consist of all e such that $J^*(e) \subseteq M$. For $\mathsf{Gen}(\mathcal{M})$, we give two constructions. (The latter is used in Section A.3.)

   - Say that a subset $X \subseteq J^*(e)$ is e-*inductive* when, for all $x \in X \cap M$, if $J(x) \subseteq X$ then $x \in X$. Define $\mathsf{Gen}(\mathcal{M})$ to consist of every e that belongs to every e-inductive subset of $J^*(e)$.

- For any e, define the signature $S_e \stackrel{\text{def}}{=} (J(e))_{e \in J^*(e) \cap M}$. By induction, for all $d \in J^*(e)$, we have $\mathsf{Term}(S_d) \subseteq \mathsf{Term}(S_e)$. A *derivation* for e is $t \in \mathsf{Term}(S_e)$ such that the result of $\overrightarrow{b} \in \mathsf{Branches}(t)$ is e if $\overrightarrow{b}$ is empty, and the final entry in $\overrightarrow{b}$ otherwise. By Proposition 2.7, it is unique. Note that, for any $\langle a, [s_b]_{b \in J(a)} \rangle \in \mathsf{Term}(S_e)$, it derives e iff both $a = e$ and, for all $b \in J(e)$, the term s_b derives b. Define $\mathsf{Gen}(\mathcal{M})$ to consist of every e that some (unique) $t \in \mathsf{Term}(S_e)$ is a derivation of e.

2. Let $\mathcal{M} = (J(e))_{e \in M}$ be an introspection. For any $\mathcal{M}$-coinductive class A and $\mathcal{M}$-inductive class B, we prove by $\in$-induction on x that $\mathcal{E}^*(x) \cap A \subseteq \mathcal{E}^*(x) \cap B$, and deduce $A \subseteq B$. Hence $\mathsf{Cogen}(\mathcal{M})$ is generated by $\mathcal{M}$. □

Here are some examples of introspectively generated classes.

- The class V_{impure} of *vonniads*—the word alludes to "von Neumann iteration"—is the least $\in$-inductive class. It exists because it is generated by the introspection $(\mathcal{E}(e))_{e \in \mathfrak{T}}$.

- The class V_{pure} of *pure vonniads* is the least class X such that every subset of X is in X. It exists because it is generated by the introspection $(e)_{e \in \mathcal{P}_{\mathsf{s}}(\mathfrak{T})}$.

- The class Ord of *ordinals* is the least class X such that any transitive subset of X is in X. It exists because it is generated by the introspection $(e)_{e \in M}$ where M is the class of all transitive sets. We write $\alpha < \beta$ for $\alpha \in \beta$.

- The set $\mathbb{N}$ is the least nat-inductive class. It is generated by the following introspection. A suitable thing e is either Zero, in which case $J(e) = \emptyset$, or of the form $\mathsf{Succ}(x)$, in which case $J(e) \stackrel{\text{def}}{=} \emptyset$.

- For any signature $S = (K_i)_{i \in I}$, the set $\mathsf{Term}(S)$ is the least S-inductive class. It is generated by the following introspection. A suitable thing e is of the form $\langle i, [a_k]_{k \in K_i} \rangle$, where $i \in I$, with $J(e) \stackrel{\text{def}}{=} \{a_k \mid k \in K_i\}$.

- For any broad signature G, the *class of all G-broad numbers*, written $\mathsf{Broad}(G)$, is the least G-inductive class. It exists because is generated by the following introspection. A suitable thing e is either Start, in which case $J(e) \stackrel{\text{def}}{=} \emptyset$, or of the form $\mathsf{Build}(x, i, [a_k]_{k \in K_i})$ with $Gx = (K_i)_{i \in I}$ and $i \in I$, in which case $J(e) \stackrel{\text{def}}{=} \{x\} \cup \{a_k \mid k \in K\}$. Likewise for a reduced broad signature.

The following are facts that make use of the classes we have just defined.

- The $\in$-induction scheme can be stated as the axiom: $V_{\mathsf{impure}} = \mathfrak{T}$.

- The combination of Purity and $\in$-induction can be stated as the axiom: $V_{\mathsf{pure}} = \mathfrak{T}$.

- Broad Infinity can be stated as follows: For every broad signature G, the class $\mathsf{Broad}(G)$ is a set. Likewise for Reduced Broad Infinity.

- For any broad signatures G and G' that have the same restriction to $\mathsf{Broad}(G) \cap \mathsf{Broad}(G')$, we have $\mathsf{Broad}(G) = \mathsf{Broad}(G')$. Thus the only part of a broad signature G that matters is its restriction to $\mathsf{Broad}(G)$. Likewise for reduced broad signatures.

4.2 Recursion over a Class

The following recursion principle is often useful, especially in the usual case where C_x is the same class C for all $x \in E$.

Proposition 4.6. *Let $\mathcal{M} = (J(e))_{e \in M}$ be a spection that generates the class E. Let C_x be a class depending on $x \in E$. For each $x \in E$, let $H_x : (\prod_{y \in J(x)} C_y) \to C_x$ be a function. Then there is a unique function $F : (x \in E) \to C_x$ such that for all $x \in E$ we have $F(x) = H_x(F \restriction_{J(e)})$.*

Proof. For $e \in E$, say that an *attempt* for e is a function $g : (x \in J^*(e)) \to C_x$ such that for all $x \in J^*(e)$ we have $g(x) = H_x(g \restriction_{J(x)})$. By induction, every $e \in E$ has a unique attempt g, and we define $F(e) \stackrel{\text{def}}{=} g(e)$. Then F has the required property. □

4.3 Spective generation of large families

Let $\mathcal{R}$ be a rubric or broad rubric on a class C. Before asking whether it generates a set, a preliminary question is whether it generates a *class*, that is, whether there is a least $\mathcal{R}$-inductive class. I do not know the answer in our setting; cf. [3, Theorem 12.1.1].

A similar issue arises for generation of families. Before asking whether $\mathcal{R}$ generates a family, a preliminary question is whether it generates a large family. In this case the answer is yes, as we shall now show. The following notions are used.

Definition 4.7. Let C be a class.

1. A *fam-spection* on C consists of a class M and, for each $e \in M$, a set $J(e)$ and partial function $\langle W_e, L_e \rangle : C^{J(e)} \rightharpoonup C$.

2. A *fam-introspection* on C is a fam-spection $(\langle J(e), W_e, L_e \rangle)_{e \in M}$ such that for all $e \in M$ we have $J(e) \subseteq \mathcal{E}^+(e)$.

All the fam-spections used in this paper are fam-introspections.

Definition 4.8. Let $\mathcal{S} = (\langle J(e), W_e, L_e \rangle)_{e \in M}$ be a fam-spection on a class C. Then the monotone operator $\Gamma_{\mathcal{S}}$ on large families within C sends $u = (u_a)_{a \in A}$ to $(L_e(u \restriction_{J(e)}))_{e \in B}$, where B is the class of all $e \in M$ such that $J(e) \subseteq A$ and $u \restriction_{J(e)} \in W_e$.

Definition 4.9. Let $\mathcal{S} = (\langle J(e), W_e, L_e \rangle)_{e \in M}$ be a fam-spection on a class C. Let $u = (u_a)_{a \in A}$ be a large family within C.

- We say that u is *$\mathcal{S}$-inductive* when $\Gamma_{\mathcal{S}}(u) \leqslant u$. Spelling this out: For all $e \in M$, if $J(e) \subseteq A$ and $u \restriction_{J(e)} \in W_e$ and $L_e : u \restriction_{J(e)} \mapsto y$, then $e \in A$ and $u_e = y$.

- We say that u is *$\mathcal{S}$-coinductive* when $u \leqslant \Gamma_{\mathcal{S}}(u)$. Spelling this out: For all $e \in A$, we have $e \in M$ and $J(e) \subseteq A$ and $u \restriction_{J(e)} \in W_e$ and $L_e : u \restriction_{J(e)} \mapsto u_e$.

- If u is $\mathcal{S}$-inductive, a subclass B of A is *relatively inductive* when $u \upharpoonright_B$ is $\mathcal{S}$-inductive. This reduces to the following condition: For all $e \in M$ such that $J(e) \subseteq B$ and $u \upharpoonright_{J(e)} \in W_e$, we have $e \in B$.

- We say that u is *generated by* $\mathcal{S}$ when it is a minimal (and therefore least) $\mathcal{S}$-inductive large family within C. Minimality can be expressed as follows: every relatively inductive subclass of A is equal to A.

- We say that u is *bigenerated by* $\mathcal{S}$ when it is generated by $\mathcal{S}$ and also a greatest $\mathcal{S}$-coinductive large family within C.

Note that any bigenerated large family is a unique fixpoint of $\Gamma_{\mathcal{S}}$.

Proposition 4.10. *Let C be a class.*

1. *Any fam-spection $\mathcal{S}$ on C generates a large family, written* $\mathsf{GenFam}(\mathcal{S})$.

2. (Assuming $\in$-induction.) *Any fam-introspection on C bigenerates a large family.*

Proof.

1. Let $\mathcal{S} = (\langle J(e), W_e, L_e\rangle)_{e\in M}$ be a fam-spection. The spection $(J(e))_{e\in M}$ generates a class D. By Proposition 4.6, there is a unique function $F : D \to C_\perp$ that sends $e \in D$ to $\{y \mid g \in \prod_{d\in J(e)} Fd,\ L_e : g \mapsto y\}$. This corresponds to a partial function $(E, e \mapsto x_e)$ from D to C. Put $\mathsf{GenFam}(\mathcal{S}) \stackrel{\text{def}}{=} (x_e)_{e\in E}$, which is clearly a fixpoint of $\Gamma_{\mathcal{S}}$. For any inductive subclass B of E, induction on $e \in D$ shows that $e \in E$ implies $e \in B$.

2. Let $\mathcal{S}$ be a fam-introspection. Let $(u_a)_{a\in A}$ be an $\mathcal{S}$-coinductive and $(v_b)_{b\in B}$ an $\mathcal{S}$-inductive large family. Then $\in$-induction on e shows that, for all $a \in \mathcal{E}^*(e) \cap A$, we have $a \in B$ and $u_a = v_a$. Thus $(u_a)_{a\in A} \leqslant (v_b)_{b\in B}$. $\square$

Here is an example an of introspectively generated large family. Let $\mathcal{B}$ be a broad rubric on a class C. The *large family generated by* $\mathcal{B}$, written $\mathsf{GenFam}(\mathcal{B})$, is the minimal (and therefore least) $\mathcal{B}$-inductive large family $(x_m)_{m\in M}$. Minimality can be expressed as follows: every relatively inductive subclass of M is equal to M. This large family exists because it is generated by the following fam-introspection $\mathcal{S} = (\langle J(e), W_e, L_e\rangle)_{e\in M}$ on C. A thing e is in M when one of the following conditions hold.

- $e = \mathsf{Basic}(i, g, p)$, where i and p are anything and g is a function. In this case, we put $J(e) \stackrel{\text{def}}{=} \mathsf{Range}(g)$. For $h \in C^{J(e)}$, we say $h \in W_e$ when, writing $\mathcal{B}_0 = (\langle K_i, R_i\rangle)_{i\in I}$, we have $i \in I$ and $\mathsf{dom}(g) = K_i$ and $R_i[h(gk)]_{k\in K_i} = (y_p)_{p\in P}$ and $p \in P$. We then define $L_e(h)$ to be y_p (which is in C).

- $e = \mathsf{Trigger}(m, i, g, p)$, where m, i, p are anything and g is a function. In this case, we put $J(e) \stackrel{\text{def}}{=} \{m\} \cup \mathsf{Range}(g)$. For $h \in C^{J(e)}$, we say $h \in W_e$ when, writing $\mathcal{B}_1(h(m)) = (\langle K_i, R_i\rangle)_{i\in I}$, we have $i \in I$ and $\mathsf{dom}(g) = K_i$ and $R_i[h(gk)]_{k\in K_i} = (y_p)_{p\in P}$ and $p \in P$. We then define $L_e(h)$ to be y_p (which is in C).

Thus Broad Family Generation scheme can be stated as follows: For every broad rubric $\mathcal{B}$ on $\mathfrak{T}$, the large family $\mathsf{GenFam}(\mathcal{B})$ is a family.

The following, in combination with Proposition 4.6, allows recursion over the domain of a large family.

Proposition 4.11. *Let $\mathcal{S} = (\langle J(e), L_e \rangle)_{e \in M}$ be a fam-spection on a class C, and let $\mathsf{GenFam}(\mathcal{S}) = (x_e)_{e \in E}$. Then $E \subseteq M$, and the spection $(E, (J(e))_{e \in E})$ generates E.*

Proof. The fact that $(x_e)_{e \in E}$ is $\mathcal{S}$-coinductive tells us that $E \subseteq M$ and, for any subclass B of E, that B is relatively inductive iff it is $(E, (J(e))_{e \in E})$-inductive. $\square$

5 Arranging the Story

5.1 Weak Forms of AC

This section looks at several principles related to AC.

Proposition 5.1. *(Diaconescu's Theorem.) AC implies Boolean Truth.*

Proof. See [9, 40]. $\square$

Although it may not be possible to entirely dispense with AC for the purpose of set generation, we shall see that a weak form of AC suffices.

Definition 5.2. Let K be a set.

1. A K-*cover* δ is a K-tuple $[A_k]_{k \in K}$ of inhabited sets.
2. The *unit* K-*cover* is $1_K \stackrel{\text{def}}{=} [1]_{k \in K}$.
3. A *map* from a K-cover $[A_k]_{k \in K}$ to a K-cover $[B_k]_{k \in K}$, is a K-tuple of functions $[f_k : A_k \to B_k]_{k \in K}$.
4. A set $\mathcal{A}$ of K-covers is *weakly initial* when, for any K-cover $[B_k]_{k \in K}$, there is $[A_k]_{k \in K} \in \mathcal{A}$ and a map $[f_k : A_k \to B_k]_{k \in K}$. We say that $\mathcal{A}$ is a *WISC* (weakly initial set of covers) for K.

Note that, if AC is assumed, then $\{1_K\}$ is a WISC for K.

Definition 5.3.

1. Let C be a class of sets. A *WISC function* on C sends each $K \in C$ to a WISC.
2. A *global WISC function* is a WISC function on $\mathfrak{S}$.
3. Axiom of *Local WISC*: On every set of sets, there is a WISC function.

Thus, if AC is assumed, then $K \mapsto \{1_K\}$ is a global WISC function.

We shall see also that a weak form of Boolean Truth is useful. It is formulated as follows.

$$
\begin{tikzcd}
\textbf{Base} \arrow[r] \arrow[dd] & \text{Local OWISC} \arrow[r] \arrow[d] & \text{Local WISC} \arrow[d] \\
 & \text{Global OWISC function} \arrow[r] \arrow[d] & \text{Global WISC function} \arrow[d] \\
\text{Truth Value Set} \arrow[r] & \text{Boolean Truth} \arrow[r] & \text{AC}
\end{tikzcd}
$$

Figure 2: Diagram of subsystems: truth values and choice

Definition 5.4. Let K be a set.

1. An *ordinal K-cover* is a K-tuple $[A_k]_{k\in K}$ of inhabited sets of ordinals.

2. A set $\mathcal{A}$ of K-covers is *ordinal-weakly initial* when for any ordinal K-cover $[B_k]_{k\in K}$ there is $[A_k]_{k\in K} \in \mathcal{A}$ and a map $[f_k : A_k \to B_k]_{k\in K}$. We say that $\mathcal{A}$ is an *OWISC* (ordinal-weakly initial set of covers) for K.

If Boolean Truth is assumed, then every inhabited set of ordinals has a least element, so $\{1_K\}$ is an OWISC for K.

Definition 5.5.

1. Let C be a class of sets. An *OWISC function* on C sends each $K \in C$ to an OWISC.

2. A *global OWISC function* is an OWISC function on $\mathfrak{S}$.

3. Axiom of *Local OWISC*: On every set of sets, there is an OWISC function.

Thus, if Boolean Truth is assumed, then $K \mapsto \{1_K\}$ is a global OWISC function.

Figure 2 summarizes the situation. The arrows indicate inclusion of theories. That is to say, reverse implication.

Remarks on related work

If the axiom scheme of Collection is assumed, then Local WISC reduces to the following statement: Every set has a WISC [36, 38]. This statement is just called WISC. It was shown in [19] that WISC is unprovable in ZF, assuming ZF is consistent. Another proof was given in [32], using topos theory.

A related notion is the following [31]. A *collection family* is a set of sets $\mathcal{D}$ such that, for every $K \in \mathcal{D}$ and K-cover $[B_k]_{k\in K}$, there is a set $Y \in \mathcal{D}$ and a surjection $p : Y \twoheadrightarrow K$ and a map $[g_k : p^{-1}\{k\} \to B_k]_{k\in K}$. For a set K, a *collective WISC* is a collection family $\mathcal{D}$ such that $K \in \mathcal{D}$.[7] For a class of sets C, a *collective WISC function* on C sends each $K \in C$ to a collective WISC. Thus, if AC is assumed, then

[7]In [25, 31], the "Axiom of Multiple Choice" is the statement that every set has a collective WISC.

$K \mapsto \{K\}$ is a global collective WISC function. Note that a collective WISC on K gives rise to a WISC on K as follows:

$$\mathcal{D} \quad \mapsto \quad \{[p^{-}\{k\}]_{k \in K} \mid Y \in \mathcal{D},\ p : Y \twoheadrightarrow K\}$$

Conversely, a global WISC function d gives rise to a global collective WISC function that sends a set K to $\bigcup_{n \in \mathbb{N}} \mathcal{D}_n$, where the sequence of sets of sets $(\mathcal{D}_n)_{n \in \mathbb{N}}$ is recursively defined as follows:

$$\begin{array}{rcl} \mathcal{D}_0 & \stackrel{\text{def}}{=} & \{K\} \\ \mathcal{D}_{n+1} & \stackrel{\text{def}}{=} & \{\sum_{l \in L} A_l \mid L \in \mathcal{D}_n,\ [A_l]_{l \in L} \in d(L)\} \end{array}$$

5.2 Arranging the generation principles

To help the reader follow the results, the main relationships between the different principles are displayed in Figures 3–4. Again, the arrows indicate inclusion of theories. That is to say, reverse implication.

6 Sets and Families

6.1 Main Results

We are now ready to study our generation principles in detail, beginning with the following straightforward implications.

Proposition 6.1.

1. *Let C be a class. Then Broad Set Generation (C) implies Set Generation (C).*

2. *Broad Set Generation implies Broad Infinity and Reduced Broad Infinity.*

3. *Broad Family Generation implies Broad Infinity and Reduced Broad Infinity.*

Proof.

1. For a rubric $\mathcal{R}$ on C, let $\hat{\mathcal{R}}$ be the following broad rubric on C: the basic rubric is $\mathcal{R}$ and each $x \in C$ triggers the empty rubric. A set is $\hat{\mathcal{R}}$-inductive iff it is $\mathcal{R}$-inductive, so a set generated by $\hat{\mathcal{R}}$ is generated by $\mathcal{R}$.

2. For a broad signature G, let $[G]$ be the following broad rubric on $\mathfrak{T}$: the basic rubric is $(\langle \emptyset, [\,] \mapsto (\mathsf{Start}) \rangle)$, and a thing x with $Gx = (K_i)_{i \in I}$ triggers the rubric

$$(\langle K_i, [a_k]_{k \in K_i} \mapsto (\mathsf{Build}(x, i, (a_k)_{k \in K_i})) \rangle)_{i \in I}$$

   A set is $[G]$-inductive iff it is G-inductive, so the set generated by $[G]$ is a set of all G-broad numbers. Likewise for a reduced broad signature.

$$
\begin{tikzcd}
\begin{array}{l}\textbf{Base}\\ \text{\textsc{Signature Infinity}}\\ \text{Family Generation}\end{array} \arrow[r] \arrow[d] & \text{\textsc{Reduced Broad Infinity}} \arrow[r] & \begin{array}{l}\text{\textsc{Broad Infinity}}\\ \text{Broad Family Generation}\end{array} \arrow[dd] \\
\text{Blass Generation} \arrow[r] \arrow[d] & \text{Jorgensen Generation} \arrow[dr] & \\
\text{Ordinal Generation} \arrow[rr] \arrow[d] & & \text{Broad Ordinal Generation} \arrow[d] \\
\text{Set Generation } (V_{\mathsf{pure}}) \arrow[rr] \arrow[d] & & \text{Broad Set Generation } (V_{\mathsf{pure}}) \arrow[d] \\
\text{Set Generation} \arrow[rr] & & \text{Broad Set Generation}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\begin{array}{l}\textbf{Base} + \textbf{Truth Value Set}\\ \text{\textsc{Signature Infinity}}\\ \text{Family Generation}\end{array} \arrow[r] \arrow[d] & \text{\textsc{Reduced Broad Infinity}} \arrow[r] & \begin{array}{l}\text{\textsc{Broad Infinity}}\\ \text{Broad Family Generation}\end{array} \arrow[dd] \\
\begin{array}{l}\text{Blass Generation}\\ \text{Ordinal Generation}\end{array} \arrow[d] & & \\
\begin{array}{l}\text{Extended Blass Generation}\\ \text{Extended Ordinal Generation}\\ \text{Set Generation } (V_{\mathsf{pure}})\end{array} \arrow[rr] \arrow[d] & & \begin{array}{l}\text{Jorgensen Generation}\\ \text{Broad Ordinal Generation}\\ \text{Broad Set Generation } (V_{\mathsf{pure}})\end{array} \arrow[d] \\
\text{Set Generation} \arrow[rr] & & \text{Broad Set Generation}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\begin{array}{l}\textbf{Base} + \textbf{Boolean Truth} = \textbf{ZFAN}\\ \text{\textsc{Signature Infinity}}\\ \text{Family Generation}\end{array} \arrow[r] \arrow[d] & \begin{array}{l}\text{\textsc{Reduced Broad Infinity}}\\ \text{\textsc{Broad Infinity}}\\ \text{Broad Family Generation}\end{array} \arrow[d] \\
\begin{array}{l}\text{Blass Generation}\\ \text{Ordinal Generation}\\ \text{Set Generation}\\ \text{\textsc{Blass's Axiom}}\end{array} \arrow[r] & \begin{array}{l}\text{Jorgensen Generation}\\ \text{Broad Ordinal Generation}\\ \text{Broad Set Generation}\\ \text{\textsc{Ord-is-Mahlo}}\end{array}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\begin{array}{l}\textbf{Base} + \textbf{AC} = \textbf{ZFCAN}\\ \text{\textsc{Signature Infinity}}\\ \text{Family Generation}\\ \text{Blass Generation}\\ \text{Ordinal Generation}\\ \text{Set Generation}\\ \text{\textsc{Blass's Axiom}}\end{array} \arrow[r] & \begin{array}{l}\text{\textsc{Reduced Broad Infinity}}\\ \text{\textsc{Broad Infinity}}\\ \text{Broad Family Generation}\\ \text{Jorgensen Generation}\\ \text{Broad Ordinal Generation}\\ \text{Broad Set Generation}\\ \text{\textsc{Ord-is-Mahlo}}\end{array}
\end{tikzcd}
$$

Figure 3: Diagrams of subsystems, without assuming WISC

$$
\begin{tikzcd}[column sep=small]
\begin{array}{l} \textbf{Base} + d \\ \textsc{Signature Infinity} \\ \text{Family Generation} \\ \text{Blass Generation} \\ \text{Ordinal Generation} \\ \text{Set Generation} \end{array} \arrow[r] \arrow[dr] & \textsc{Reduced Broad Infinity} \arrow[r] & \begin{array}{l} \textsc{Broad Infinity} \\ \text{Broad Family Generation} \\ \text{Broad Ordinal Generation} \\ \text{Broad Set Generation} \end{array} \\
& \text{Jorgensen Generation} \arrow[ur] &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
\begin{array}{l} \textbf{Base} + \textbf{Truth Value Set} + d \\ \textsc{Signature Infinity} \\ \text{Family Generation} \\ \text{Blass Generation} \\ \text{Ordinal Generation} \\ \text{Set Generation} \end{array} \arrow[r] & \textsc{Reduced Broad Infinity} \arrow[r] & \begin{array}{l} \textsc{Broad Infinity} \\ \text{Broad Family Generation} \\ \text{Jorgensen Generation} \\ \text{Broad Ordinal Generation} \\ \text{Broad Set Generation} \end{array}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\begin{array}{l} \textbf{Base} + \textbf{Boolean Truth} + d \\ \quad = \textbf{ZFAN} + d \\ \textsc{Signature Infinity} \\ \text{Family Generation} \\ \text{Blass Generation} \\ \text{Ordinal Generation} \\ \text{Set Generation} \\ \textsc{Blass's Axiom} \end{array} \arrow[rr] & & \begin{array}{l} \textsc{Reduced Broad Infinity} \\ \textsc{Broad Infinity} \\ \text{Broad Family Generation} \\ \text{Jorgensen Generation} \\ \text{Broad Ordinal Generation} \\ \text{Broad Set Generation} \\ \textsc{Ord-is-Mahlo} \end{array}
\end{tikzcd}
$$

Figure 4: Diagrams of subsystems, assuming a global WISC function d

3. For a broad signature G, let $(x_m)_{m \in M}$ be the family generated by $[G]$. It has the property that $x_m = x_{m'}$ implies $m = m'$, by induction on m. Therefore the set $\{x_m \mid m \in M\}$ is a set of all G-broad numbers. Likewise for a reduced broad signature. $\square$

Proposition 6.2. *Family Generation holds.*

Proof. Let $\mathcal{R} = (\langle K_i, R_i \rangle)_{i \in I}$ be a rubric on a class C. Let S be the signature $(K_i)_{i \in I}$. We associate to each $t \in \mathsf{Term}(S)$ a family $X_t = (x_{t,m})_{m \in M_t}$ recursively as follows. For $t = \langle i, [t_k]_{k \in K_i} \rangle$, an element of M_t is a triple $\langle i, g, p \rangle$ where $i \in I$ and $g \in \prod_{k \in K_i} M_{t_k}$ with $R_i[x_{t_k, gk}]_{k \in K_i} = (y_p)_{p \in P}$ and $p \in P$, and we define $x_{t, \langle i,g,p \rangle} \stackrel{\text{def}}{=} y_p$. For any $t, t' \in \mathsf{Term}(S)$, if $M_t \cap M'_t$ is inhabited, then $t = t'$, by induction on t. We define $M \stackrel{\text{def}}{=} \bigcup_{t \in \mathsf{Term}(S)} M_t$, and for $m \in M$ we define $x_m \stackrel{\text{def}}{=} x_{t,m}$ where $m \in M_t$. Then $(x_m)_{m \in M}$ is a family generated by $\mathcal{R}$. $\square$

Proposition 6.3. *Broad Family Generation is equivalent to Broad Infinity.*

Proof. ($\Rightarrow$) is Proposition 6.1(3). For ($\Leftarrow$), we begin by defining

$$\begin{aligned} \mathsf{Basic}'(x, y, z) &\stackrel{\text{def}}{=} \mathsf{Build}(\mathsf{Build}(\mathsf{Start}, x, y), z, [\,]) \\ \mathsf{Trigger}'(x, y, z, w &\stackrel{\text{def}}{=} \mathsf{Build}(\mathsf{Build}(\mathsf{Build}(x, *, [\,]), y, z), w, [\,]) \end{aligned}$$

These are injective and have disjoint range.

Let $\mathcal{B}$ be a broad rubric on a class C. We define the notions of *$\mathcal{B}$-pseudoinductive* family $(y_m)_{m \in M}$, *relatively pseudoinductive* subset of M, and *family pseudogenerated by* $\mathcal{B}$, where "pseudo" means that we use Basic' instead of Basic, and $\mathsf{Trigger}'$ instead of $\mathsf{Trigger}$. As we saw in Section 4.3, $\mathcal{B}$ generates a large family $(x_m)_{m \in M}$, and by the same argument it pseudogenerates a large family $(u_l)_{l \in L}$. Using recursion (Propositions 4.11 and 4.6) in each direction, we construct a bijection $\theta : L \cong M$ such that for all $l \in L$ we have $x_{\theta l} = y_l$. Explicitly, θ replaces Basic' by Basic, and $\mathsf{Trigger}'$ by $\mathsf{Trigger}$. It suffices to prove that L is a set, since this implies that M is a set, as required.

Note the following "key facts about L".

- $\mathsf{Start} \notin L$.
- $\mathsf{Build}(\mathsf{Start}, x, y) \notin L$.
- $l \in L$ implies $\mathsf{Build}(l, *, [\,]) \notin L$, by induction on l.
- $l \in L$ implies $\mathsf{Build}(\mathsf{Build}(l, *, [\,]), i, f) \notin L$, by induction on l.

By these facts and Corollary 3.7(2), there is a unique broad signature G that sends

- Start, where $\mathcal{B}_0 = (\langle K_i, R_i \rangle)_{i \in I}$, to $(K_i)_{i \in I}$
- $\mathsf{Build}(\mathsf{Start}, i, f)$, where $\mathcal{B}_0 = (\langle K_i, R_i \rangle)_{i \in I}$ and $f : K_i \to L$ and $R_i[u_{fk}]_{k \in K_i} = (y_p)_{p \in P}$, to $(\emptyset)_{p \in P}$
- any $l \in L$ to $(\emptyset)$

- $\mathsf{Build}(l, *, [\,])$, where $l \in L$ and $\mathcal{B}_1(u_l) = (\langle K_i, R_i \rangle)_{i \in I}$, to $(K_i)_{i \in I}$
- $\mathsf{Build}(\mathsf{Build}(l, *, [\,]), i, f)$, where $l \in L$ and $\mathcal{B}_1(u_l) = (\langle K_i, R_i \rangle)_{i \in I}$ and $i \in I$ and $f : K_i \to L$ and $R_i[u_{fk}]_{k \in K_i} = (y_p)_{p \in P}$, to $(\emptyset)_{p \in P}$.

and is supported on these cases. By induction, every element of L is a G-broad number. So L is a set. □

Proposition 6.4. Assume AC. *Let C be a class.*

1. *Suppose the rubric $\mathcal{R}$ on C generates the family $(x_m)_{m \in M}$. Then it generates the set $\{x_m \mid m \in M\}$.*
2. *Likewise for a broad rubric.*

Proof.

1. Let $X \overset{\text{def}}{=} \{x_m \mid m \in M\}$. We first show that it is $\mathcal{R}$-closed. Given $i \in I$ and a K_i-tuple $[a_k]_{kinK_i}$ within X, with $\mathcal{R}[a_k]_{k \in K_i} = (y_p)_{p \in P}$, and $p \in P$, we want $y_p \in X$. For each $k \in K_i$ choose some $gk \in M$ such that $a_k = x_{gk}$. Then $y_p = x_{\langle i, g, p \rangle}$.

   To show minimality: for any $\mathcal{R}$-closed subset Y of X, we prove by induction on $m \in M$ that $x_m \in Y$.
2. Similar. □

If we merely assume a WISC function, rather than AC, we can still derive our generation principles for sets, as follows.

Proposition 6.5. *Let C be a class.*

1. (Assuming Local WISC.) *Family Generation (C) implies Set Generation (C).*
2. (Assuming a global WISC function d.) *Broad Family Generation (C) implies Broad Set Generation (C).*

Proof. We give a preliminary construction. For any rule $\langle K, R \rangle$ on C, and any K-cover $\delta = [D_k]_{k \in K}$, define a rule $\langle K, R \rangle^\delta \overset{\text{def}}{=} \langle L, S \rangle$ on C as follows. Put $L \overset{\text{def}}{=} \sum_{k \in K} D_k$. Let C_δ be the class of all L-tuples $b = [b_{\langle k,d \rangle}]_{\langle k,d \rangle \in L}$ within C such that for all $k \in K$ and $d, d' \in D_k$ we have $b_{\langle k,d \rangle} = b_{\langle k,d' \rangle}$. The map $\theta_\delta : C^K \to C_\delta$ sending $[a_k]_{k \in K}$ to $[a_k]_{\langle k,d \rangle \in L}$ is a bijection. By Proposition 3.6(2), let $S : C^L \to \mathsf{Fam}(C)$ be the function that sends $b \in C_\delta$ to $R(\theta_\delta^{-1} b)$ and is supported on C_δ. Thus, for any K-tuple $[a_k]_{k \in K}$ within C, we have $S[a_k]_{\langle k,d \rangle \in L} = R[a_k]_{k \in K}$. Now our proof begins.

1. Given a rubric $\mathcal{R} = (\langle K_i, R_i \rangle)_{i \in I}$ on C, let d be a WISC function on $\{K_i \mid i \in I\}$. We define the rubric $\mathcal{R}^d$ on C to be $(\langle K_i, R_i \rangle^\delta)_{i \in I, \delta \in d(K_i)}$. Let $(x_m)_{m \in M}$ be the family generated by $\mathcal{R}^d$. We show that the set $X \overset{\text{def}}{=} \{x_m \mid m \in M\}$ is generated by $\mathcal{R}$.

   First we show that X is $\mathcal{R}$-inductive. Given $i \in I$ and a K_i-tuple $[a_k]_{k \in K_i}$ within X, with $\mathcal{R}[a_k]_{k \in K_i} = (y_p)_{p \in P}$, and $p \in P$, we want $y_p \in X$. For each

$k \in K_i$, let $A_k \stackrel{\text{def}}{=} \{m \in M \mid x_m = a_k\}$, which is inhabited. Since $d(K_i)$ is weakly initial, there is $\delta = [D_k]_{k \in K_i} \in d(K_i)$ and a map $[f_k : D_k \to A_k]_{k \in K_i}$. We have $\langle K_i, R_i \rangle^\delta = \langle L, S \rangle$, where $L = \sum_{k \in K} D_k$. Writing $g : L \to M$ for the function sending $\langle k, d \rangle$ to $f_k(d)$, we have

$$\begin{aligned} S[x_{g(l)}]_{l \in L} &= S[x_{f_k(d)}]_{\langle k,d \rangle \in L} \\ &= S[a_k]_{\langle k,d \rangle \in L} \quad (\text{since } f_k(d) \in A_k) \\ &= R[a_k]_{k \in K} \\ &= (y_p)_{p \in P}. \end{aligned}$$

Therefore $\langle i, g, p \rangle \in M$ with $x_{\langle i,g,p \rangle} = y_p$, giving $y_p \in X$ as required.

It remains to show that, for any $\mathcal{R}$-inductive subset Y of C, we have $X \subseteq Y$. We do this by showing that, for all $m \in M$, we have $x_m \in Y$, by induction on m. For $i \in I$ and $\delta = [D_k]_{k \in K_i} \in d(K_i)$, giving $\langle K_i, R_i \rangle^\delta = \langle L, S \rangle$ with $L \stackrel{\text{def}}{=} \sum_{k \in K_i} D_i$, we must show that, for any $g : L \to M$ satisfying $\forall l \in L.\, x_{gl} \in Y$, with $S[x_{gl}]_{l \in L} = (y_p)_{p \in P}$, and any $p \in P$, we have $y_p \in Y$ (since $y_p = x_{\langle i,\delta \rangle, g, p})$. Since S is supported on C_δ and $p \in P$, we have $[x_{gl}]_{l \in L} \in C_\delta$, and we put $[a_k]_{k \in K_i} \stackrel{\text{def}}{=} \theta_\delta^{-1}[x_{gl}]_{l \in L}$. This means that, for all $k \in K_i$ and $d \in D_k$, we have $x_{g\langle k,d \rangle} = a_k$. Thus, for all $k \in K_i$ we have $a_k \in Y$ (since D_k is inhabited), and we have $R[a_k]_{k \in K} = S[x_{gl}]_{l \in L} = (y_p)_{p \in P}$. Consequently $\mathcal{R}$-inductivity of Y gives $y_p \in Y$, as required.

2. Given a broad rubric $\mathcal{B}$ on C, we define the broad rubric $\mathcal{B}^d$ on C. Its basic rubric is $(\mathcal{B}_0)^d$, and the rubric triggered by $x \in C$ is $(\mathcal{B}_1(x))^d$. Let $(x_m)_{m \in M}$ be the family generated by $\mathcal{B}^d$. As in part 1, we show that the set $X \stackrel{\text{def}}{=} \{x_m \mid m \in M\}$ is generated by $\mathcal{B}$. □

6.2 Reduced Broad Infinity

We shall now show see how to give Broad Infinity in a "reduced" form as described in Section 1.4.3.

Proposition 6.6. (Assuming Boolean Truth.) *Broad Infinity is equivalent to Reduced Broad Infinity.*

Proof.
($\Leftarrow$): Broad Infinity implies Broad Family Generation, which implies Reduced Broad Infinity.
($\Rightarrow$): We begin as follows:

- Define $\mathsf{Start}' \stackrel{\text{def}}{=} \mathsf{Make}(\mathsf{Begin}, [\,])$.

- For any w and signature $S = (K_i)_{i \in I}$ and $i \in I$ and tuple $[a_k]_{k \in K_i}$, define

$$\begin{array}{rcll} \mathsf{Build}'(w, S, i, [a_k]_{k \in K_i}) & \stackrel{\text{def}}{=} & \mathsf{Make}(\mathsf{Make}(w, [\,]), [b_j]_{j \in J}) & \\ \text{where} \quad J & \stackrel{\text{def}}{=} & I + \sum_{i \in I} K_i & \\ b_{\mathsf{inl}\ i} & \stackrel{\text{def}}{=} & \mathsf{Begin} & \\ b_{\mathsf{inl}\ i'} & \stackrel{\text{def}}{=} & \mathsf{Make}(\mathsf{Begin}, [\,]) & (\text{for } i' \in I,\ i' \neq i) \\ b_{\mathsf{inr}\ \langle i,k \rangle} & \stackrel{\text{def}}{=} & a_k & (\text{for } k \in K_i) \\ b_{\mathsf{inr}\ \langle i',k \rangle} & \stackrel{\text{def}}{=} & \mathsf{Begin} & (\text{for } i' \in I,\ i' \neq i,\ k \in K'_i) \end{array}$$

  which is well-defined by Decidable Equality (see Proposition 2.3).

- These are injective and have disjoint range.

- Let E be the minimal (and therefore least) class X with the following properties.

  - $\mathsf{Start}' \in X$.
  - For any $w \in X$ and signature $S = (K_i)_{i \in I}$ and $i \in I$ and tuple $[a_k]_{k \in K_i}$ within X, we have $\mathsf{Build}'(w, S, i, [a_k]_{k \in K_i}) \in X$.

  It exists because it is introspectively generated.

- Note the following "key facts about E".

  - $\mathsf{Begin} \notin E$.
  - $w \in E$ implies $\mathsf{Make}(w, [\,]) \notin E$, by induction on w.

- Let θ be the function on E that recursively sends Start' to Start and sends $\mathsf{Build}'(w, S, i, f)$ to $\mathsf{Build}(\theta w, i, \theta \circ f)$.

Let G be a broad signature. Define U be the minimal (and therefore least) subclass X of E with the following properties.

- $\mathsf{Start}' \in X$.

- For any $u \in X$ with $G(\theta u) = S = (K_i)_{i \in I}$, and any $i \in I$ and tuple $[a_k]_{k \in K_i}$ within X, we have $\mathsf{Build}'(u, i, S, [a_k]_{k \in K_i}) \in X$.

It exists because it is introspectively generated. Note that $\theta \restriction_U$ is a bijection from U to the class of G-broad numbers, using recursion to construct the inverse. So it suffices to show that U is a set.

By the "key facts about E" and Corollary 3.7(1), there is a unique reduced broad signature F that sends

- Begin to $\emptyset$

- any $z \in E$ to $\emptyset$

- $\mathsf{Make}(w, [\,])$, with $w \in E$ and $Gw = (K_i)_{i \in I}$, to $I + \sum_{i \in I} K_i$.

and is supported on these cases. By induction, every element of U is an F-broad number, so U is a set. $\square$

7 Ordinals

7.1 Basic Theory

So far we have not considered ordinals, and this is our next task. We begin by setting up the theory of ordinals in the intuitionistic setting. Most of this is standard.

Definition 7.1. Let A be a set. A *well-ordering* on A is a relation $\prec$ satisfying the following properties.

- *Well-foundedness*: every subset X of A that is inductive (i.e for all $a \in A$, if $\forall x \in A. (x \prec a \Rightarrow x \in X)$, then $a \in X$) is equal to A.
- *Transitivity*: for all $a, b, c \in A$, if $c \prec b$ and $b \prec a$, then $c \prec a$.
- *Extensionality*: For all $a, b \in A$, if for all $x \in A$ we have $x \prec a$ iff $x \prec b$, then $a = b$.

If Boolean Truth is assumed, then every well-ordered set $(A, \prec)$ has the *trichotomy* property: for any $a, b \in A$, either $a \prec b$ or $b \prec a$ or $a = b$. For a proof, see [27].

Proposition 7.2.

1. *Every ordinal is a well-ordered set, equipped with the relation* $\in$.
2. *For every well-ordered set* $(A, \prec)$*, there is a unique pair* $\langle \alpha, \theta \rangle$ *consisting of an ordinal* α *and isomorphism* $\theta : (A, \prec) \cong \alpha$*. Explicitly,* θ *recursively sends* a *to* $\{\theta b \mid b \in A,\ b \prec a\}$*, and* α *is its range.*

In summary, an ordinal is precisely the order-type of a well-ordered set.

The class Ord is partially ordered by writing $\alpha \leqslant \beta$ for $\alpha \subseteq \beta$. Any family of ordinals $(\alpha_i)_{i \in I}$ has a least upper bound $\bigvee_{i \in I} \alpha_i \stackrel{\text{def}}{=} \bigcup_{i \in I} \alpha_i$. In particular, the least ordinal is $0 \stackrel{\text{def}}{=} \emptyset$. Note that $\alpha < \beta$ implies $\alpha \leqslant \beta$, and $\alpha < \beta \leqslant \gamma$ implies $\alpha < \gamma$.

The *successor* of an ordinal α, written $\mathrm{S}(\alpha)$, is the least ordinal β such that $\alpha < \beta$, namely, $\alpha \cup \{\alpha\}$. Thus the successor function is injective (indeed reflects $\leqslant$) and never yields 0.

For a family of ordinals $(\alpha_k)_{k \in K}$, the *strict supremum* is the least strict upper bound, namely, $\mathsf{ssup}_{i \in I} \alpha_i \stackrel{\text{def}}{=} \bigvee_{i \in I} \mathrm{S}(\alpha_i)$.

The following notion will often be useful.

Definition 7.3. Let K be a set. An ordinal λ is K*-complete* when, for any K-tuple $[\alpha_k]_{k \in K}$ within λ, we have $\bigvee_{k \in K} \alpha_k < \lambda$.

Next we introduce the notion of limit ordinal. In the intuitionistic setting, various definitions are possible but the following seems most suitable.

Definition 7.4. A *limit* is an ordinal λ satisfying the following.

- For all $\alpha < \lambda$, we have $\mathrm{S}(\alpha) < \lambda$.
- λ is 0-complete, meaning $0 < \lambda$.

- λ is 2-complete, meaning that, for all $\alpha, \beta < \lambda$, we have $\alpha \vee \beta < \lambda$.

All three conditions will be used in the proof of Proposition 7.11.

Note that a limit is not 0 or a successor, and is n-complete for all $n \in \mathbb{N}$. If Boolean Truth is assumed, then every ordinal is either 0, a successor or a limit.

7.2 Inductive Chains

We recall the theory of monotone endomaps from Section 1.2. Here are some examples of such endomaps.

1. Let Γ_{nat} be the monotone operator on $\mathfrak{S}$ that sends X to the set
$$\{\mathsf{Zero}\} \cup \{\mathsf{Succ}(n) \mid n \in X\}$$
Thus a nat-inductive set is precisely a prefixpoint of Γ_{nat}.

2. For a signature $S = (K_i)_{i \in I}$, let Γ_S be the monotone operator on $\mathfrak{S}$ that sends X to the set
$$\{\langle i, [a_k]_{k \in K_i}\rangle \mid i \in I,\ [a_k]_{k \in K_i} \in X^{K_i}\}$$
Thus an S-inductive set is precisely a prefixpoint of Γ_S.

3. For a reduced broad signature F, let Γ_F be the monotone operator on $\mathfrak{S}$ that sends X to the set
$$\{\mathsf{Begin}\} \cup \{\mathsf{Make}(x, [a_k]_{k \in Fx}) \mid x \in X,\ [a_k]_{k \in Fx} \in X^{Fx}\}$$
Thus an F-inductive set is precisely a prefixpoint of Γ_F.

4. For a broad signature G, let Γ_G be the monotone operator on $\mathfrak{S}$ that sends X to the set
$$\{\mathsf{Start}\} \cup \{\mathsf{Build}(x, i, [a_k]_{k \in K_i}) \mid \\ x \in X,\ Gx = (K_i)_{i \in I},\ i \in I,\ [a_k]_{k \in K_i} \in X^{K_i}\}$$
Thus a G-inductive set is precisely a prefixpoint of Γ_G.

5. For a rubric $\mathcal{R}$ on a class C, let $\Gamma_{\mathcal{R}}$ be the monotone operator on $\mathcal{P}_{\mathsf{s}}C$ that sends X to the set
$$\{y_p \mid \mathcal{R} = (K_i)_{i \in I},\ i \in I,\ [a_k]_{k \in K_i} \in X^{K_i},\ R_i = (y_p)_{p \in P},\ p \in P\}$$
Thus an $\mathcal{R}$-inductive set is precisely a prefixpoint of $\Gamma_{\mathcal{R}}$.

6. For a broad rubric $\mathcal{B}$ on a class C, let $\Gamma_{\mathcal{B}}$ be the monotone operator on $\mathcal{P}_{\mathsf{s}}C$ that sends X to the set
$$\{y_p \mid \mathcal{B}_0 = (K_i)_{i \in I},\ i \in I,\ [a_k]_{k \in K_i} \in X^{K_i},\ R_i = (y_p)_{p \in P},\ p \in P\} \\ \cup \{y_p \mid x \in X,\ \mathcal{B}_1(x) = (K_i)_{i \in I},\ i \in I,\ [a_k]_{k \in K_i} \in X^{K_i}, \\ R_i = (y_p)_{p \in P},\ p \in P\}$$
Thus a $\mathcal{B}$-inductive set is precisely a prefixpoint of $\Gamma_{\mathcal{B}}$.

7. *(Assuming Truth Value Set.)* The monotone operator $\mathcal{P}$ on $\mathfrak{S}$ sends X to its powerset. It has no prefixpoint.

The above examples may be used in the following construction.

Definition 7.5. Let C be a class, and Γ a monotone endofunction on $\mathcal{P}_{\mathsf{s}}C$. The *inductive chain* of Γ is the sequence $(\mu^{\alpha}\Gamma)_{\alpha\in\mathsf{Ord}}$ within $\mathcal{P}_{\mathsf{s}}C$ defined recursively by $\mu^{\alpha}\Gamma \stackrel{\text{def}}{=} \bigcup_{\beta<\alpha}\Gamma\mu^{\beta}\Gamma$.

The inductive chain is increasing (i.e., $\alpha \mapsto \mu^{\alpha}\Gamma$ is monotone). Moreover, it has the following properties:

$$\begin{array}{rrcl} & \mu^{0}\Gamma & = & \emptyset \\ & \mu^{\mathrm{S}(\alpha)}\Gamma & = & \Gamma\mu^{\alpha}\Gamma \\ \text{For a limit } \alpha, & \mu^{\alpha}\Gamma & = & \displaystyle\bigcup_{\beta<\alpha}\mu^{\beta}\Gamma \end{array}$$

Note that $\mu^{\alpha}\Gamma$ is a postfixpoint of Γ for all α. We say that Γ *inductively stabilizes* at α when $\mu^{\alpha}\Gamma$ is a prefixpoint. Every prefixpoint is an upper bound of the inductive chain, so, if Γ inductively stabilizes at α, then $\mu^{\alpha}\Gamma$ is both the supremum of the inductive chain and the least prefixpoint of Γ. Under the assumption of Boolean Truth, we shall see a converse (Proposition 8.6): if Γ has a prefixpoint, then it inductively stabilizes.

7.3 Generation of Limits

This section introduces two principles for generating limit ordinals, and relates them to our other principles. We begin with the following properties.

Definition 7.6.

1. Let $\mathcal{D}$ be a set of sets. An ordinal λ is *$\mathcal{D}$-collectively complete* when, for all $K \in \mathcal{D}$, it is K-complete. (For example, ω is ω-collectively complete, though not ω-complete.)

2. Let H be a *broad set of sets*, meaning a function $H : \mathsf{Ord} \to \mathcal{P}_{\mathsf{s}}\mathfrak{S}$. An ordinal λ is said to be *H-collectively complete* when, for all $\beta < \lambda$, it is $H\beta$-collectively complete.

These properties can be combined: let $H_{\mathcal{D}}$ be the broad set of sets sending β to $H\beta\cup\mathcal{D}$. (This generalizes the J_{α} notation of Section 1.7.) Then, for any ordinal $\lambda > 0$, it is $H_{\mathcal{D}}$-collectively complete iff it is both H-collectively complete and $\mathcal{D}$-collectively complete.

Now we give our generation principles.

- For a set of sets $\mathcal{D}$, a *limit collectively generated by* $\mathcal{D}$ is a minimal (and therefore least) $\mathcal{D}$-collectively complete limit. The *Ordinal Generation* principle says that every set of sets $\mathcal{D}$ collectively generates a limit.

- For a broad set of sets H, a *limit collectively generated by* H is a minimal (and therefore least) H-collectively complete limit. The *Broad Ordinal Generation* scheme says that every broad set of sets collectively generates a limit.

Proposition 7.7.

1. *Broad Ordinal Generation implies Ordinal Generation.*

2. *Broad Ordinal Generation implies Broad Infinity.*

Proof.

1. Let $\mathcal{D}$ be a set of sets. The limit collectively generated by the broad set of sets $\beta \mapsto \mathcal{D}$ is also a limit collectively generated by $\mathcal{D}$.

2. Let G be a broad signature. Let $r : \mathsf{Broad}(G) \to \mathsf{Ord}$ be the function that recursively sends Start to 0, and $\mathsf{Build}(x, i, [a_k]_{k \in K_i})$ to the strict supremum of $\{r(x)\} \cup \{r(a_k) \mid k \in K_i\}$. By induction on w, we have $w \in \mu^{\mathrm{S}(r(w))}\Gamma_G$.

   Let H be the broad set of sets that sends β to

   $$\{K_i \mid x \in \mu^{\beta}\Gamma_G,\ Gx = (K_i)_{i \in I},\ i \in I\}$$

   Let λ be the limit collectively generated by H. To show that Γ_G inductively stabilizes at λ, we show that every $w \in \mu^{\mathrm{S}(\lambda)}\Gamma_G$ satisfies $r(w) < \lambda$, by induction on w. Either $w = \mathsf{Start}$, in which case $r(w) = 0 < \lambda$, or $w = \mathsf{Build}(x, i, [a_k]_{k \in K_i})$ with $Gx = (K_i)_{i \in I}$. In the latter case, $\mathrm{S}(r(x)) < \lambda$ by the inductive hypothesis, and $x \in \mu^{\mathrm{S}(r(x))}\Gamma_G$, so $\lambda \geqslant H$ tells us that λ is K_i-complete. For all $k \in K_i$, we have $r(a_k) < \lambda$ by the inductive hypothesis. So $r(w) < \lambda$. $\square$

Proposition 7.8.

1. *Set Generation (V_{pure}) implies Ordinal Generation.*

2. *Broad Set Generation (V_{pure}) implies Broad Ordinal Generation.*

Proof.

1. Let $\mathcal{D}$ be a set of sets. A $\mathcal{D}$-collectively complete limit is precisely an $\mathcal{R}$-inductive set of ordinals, where $\mathcal{R}$ is the following rubric on Ord indexed by $4 + \mathcal{D}$. Rule $\mathsf{inl}\ 0$ (for transitivity) has arity 1 and sends $[\alpha]$ to $(\beta)_{\beta \in \alpha}$. Rule $\mathsf{inl}\ 1$ has arity 0 and sends $[\,]$ to (0). Rule $\mathsf{inl}\ 2$ has arity 1 and sends $[\alpha] \to (\mathrm{S}(\alpha))$. Rule $\mathsf{inl}\ 3$ has arity 2 and sends $[\alpha_k]_{k \in 2}$ to $(\alpha_0 \vee \alpha_1)$. Rule $\mathsf{inr}\ K$, for $K \in \mathcal{D}$, has arity K and sends $[\alpha_k]_{k \in K}$ to $(\bigvee_{k \in K} \alpha_k)$. By Set Generation ($V_{\mathsf{pure}}$) and Corollary 3.10, $\mathcal{R}$ generates a set, and this is a limit collectively generated by $\mathcal{D}$.

2. Similar, using a broad rubric on Ord.

$\square$

Proposition 7.9.

1. (Assuming Local OWISC.) *Ordinal Generation implies Set Generation.*

2. (Assuming a global OWISC function d.) *Broad Ordinal Generation implies Broad Set Generation.*

Proof.

1. Let $\mathcal{R} = (\langle K_i, R_i\rangle)_{i\in i}$ be a rubric on a class C, and let d be an OWISC function for $\{K_i \mid i \in I\}$. Let λ be the limit collectively generated by the set of sets

$$\mathcal{D} \quad \stackrel{\text{def}}{=} \quad \{\sum_{k\in K_i} A_k \mid i \in I,\ [A_k]_{k\in K_i} \in d(K_i)\}$$

We show that $\Gamma_{\mathcal{R}}$ inductively stabilizes at λ. For any $i \in I$, and K_i-tuple $[a_k]_{k\in K_i}$ within $\mu^{\lambda}\Gamma_{\mathcal{R}}$, with $R_i[a_k]_{k\in K_i} = (x_p)_{p\in P}$, and any $p \in P$, we want $x_p \in \mu^{\lambda}\Gamma_{\mathcal{R}}$. For each $k \in K_i$, let B_k be the set of ordinals $\beta < \lambda$ such that $a_k \in \mu^{\beta}\Gamma_{\mathcal{R}}$, which is inhabited. So there is a cover $\delta = [A_k]_{k\in K_i} \in d(K_i)$ and map $[f_k : A_k \to B_k]_{k\in K_i}$, giving a map $f : \sum_{k\in K_i} A_k \to \lambda$ sending $\langle k, a\rangle$ to $f_k(a)$. Since λ is $\sum_{k\in K_i} A_k$-complete, the supremum σ of the range of f is $< \lambda$. For each $k \in K_i$, since there is $a \in A_k$, we have $a_k \in \mu^{f_k(a)}\Gamma_{\mathcal{R}} = \mu^{f\langle k,a\rangle}\Gamma_{\mathcal{R}} \subseteq \mu^{\sigma}\Gamma_{\mathcal{R}}$. So $x_p \in \mu^{\mathrm{S}(\sigma)}\Gamma_{\mathcal{R}} \subseteq \mu^{\lambda}\Gamma_{\mathcal{R}}$ as required.

2. Let $\mathcal{B}$ be a broad rubric on a class C. For $\mathcal{B}_0 = (\langle K_i, R_i\rangle)_{i\in i}$, define the set of sets

$$\mathcal{D} \quad \stackrel{\text{def}}{=} \quad \{\sum_{k\in K_i} A_k \mid i \in I,\ [A_k]_{k\in K_i} \in d(K_i)\}$$

and the broad set of sets H sending β to

$$\{\sum_{k\in K_i} A_k \mid x \in \mu^{\beta}H_{\mathcal{B}},\ \mathcal{B}_1(x) = (\langle K_i, R_i\rangle)_{i\in i},\ i \in I,\ [A_k]_{k\in K_i} \in d(K_i)\}$$

Let λ be the limit collectively generated by $H_{\mathcal{D}}$. We show that $\mu^{\lambda}H_{\mathcal{B}}$ is $\mathcal{B}$-inductive. $\mathcal{B}_0$-inductivity is as in part (1). For $x \in \mu^{\lambda}H_{\mathcal{B}}$, we show $\mathcal{B}_1(x)$-inductivity by taking $\beta < \lambda$ such that $x \in \mu^{\beta}H_{\mathcal{B}}$, and proceeding as in part (1). □

We now come to a key definition.

Definition 7.10. A limit λ is said to be *regular* when it is λ-collectively complete.

Proposition 7.11. *Any limit collectively generated by a set of sets, or by a broad set of sets, is regular.*

Proof. The broad case is sufficient, since a limit collectively generated by a set of sets $\mathcal{D}$ is the limit collectively generated by the broad set of sets $\beta \mapsto \mathcal{D}$.

Let λ be a limit collectively generated by a broad set of sets H; we must show it is regular. Write α for the set of ordinals $\beta < \lambda$ such that λ is $\mathrm{S}(\beta)$-collectively complete. (In particular, if $\beta \in \alpha$, then λ is β-complete.) It is clearly transitive, so it is an ordinal $\leqslant \lambda$. We show that it is a limit. Since $0 < \lambda$, it follows that λ is 0-complete and hence $\mathrm{S}(0)$-collectively complete, giving $0 < \alpha$. Next we show that $\beta < \alpha$ implies $\mathrm{S}(\beta) < \alpha$, meaning that if λ is $\mathrm{S}(\beta)$-collectively complete, then λ is $\mathrm{S}(\mathrm{S}(\beta))$-collectively complete. For $\gamma < \mathrm{S}(\mathrm{S}(\beta))$, either $\gamma < \mathrm{S}(\beta)$, in which case λ is γ-collectively complete, or $\gamma = \mathrm{S}(\beta) = \beta \cup \{\beta\}$, in which case, for any γ-tuple $[a_k]_{k\in\gamma}$ within λ, we have $\bigvee_{k\in\gamma} a_k = (\bigvee_{k\in\beta} a_k) \vee a_\beta$, which is $< \lambda$ since a limit is 2-complete.

Thus α is a limit. Next we show that it is H-collectively complete. For any $\delta < \alpha$ and $K \in H\delta$ and K-tuple $[\beta_k]_{k\in K}$ within α, put $\beta \stackrel{\text{def}}{=} \bigvee_{k\in K} \beta_k$, and we show that $\beta < \alpha$. For any $\gamma < \mathrm{S}(\beta)$, we must show that λ is γ-complete. Either $\gamma < \beta$ or $\gamma = \beta$. In the first case, there is $k \in K$ such that $\gamma < \beta_k$, so λ is γ-complete. In the second case, let $[a_i]_{i\in\beta}$ be a γ-tuple within λ. Then $\bigvee_{i\in\beta} a_i = \bigvee_{k\in K} \bigvee_{i\in\beta_k} a_i$. For each $k \in K$, we have $\bigvee_{i\in\beta_k} a_i < \lambda$ since $\beta_k < \alpha$. Since λ is K-complete, we deduce $\bigvee_{i\in\beta} a_i < \lambda$.

But λ is the minimal H-collectively complete limit, so $\lambda = \alpha$. Therefore, for any $\beta < \lambda$, we deduce that λ is β-complete. $\square$

Proposition 7.12. *Let λ be a regular limit.*

1. *Let α be an ordinal. Then λ is α-collectively complete iff $\lambda \geqslant \alpha$.*

2. *Let H be an ordinal function. Then λ is J-collectively complete iff $\lambda \geqslant J$.*

Proof.

1. ($\Leftarrow$): For $\beta < \alpha$, we have $\beta \leqslant \lambda$, so λ-collective completeness of λ gives β-collective completeness.

   ($\Rightarrow$): We show that $\beta < \alpha$ implies $\beta < \lambda$ by induction on β. Since β is the strict supremum of its elements, and they are all $< \alpha$ and therefore $< \lambda$, and λ is β-complete, we are done.

2. Follows. $\square$

Proposition 7.13.

1. *Let α be an ordinal. A regular limit generated by an α is precisely a limit collectively generated by α.*

2. *Let J be an ordinal function. A regular limit generated by J is precisely a limit collectively generated by J.*

Proof. By Propositions 7.11-7.12. $\square$

Corollary 7.14.

1. *Blass Generation can be stated as follows: Every ordinal collectively generates a limit.*

2. *Jorgensen Generation can be stated as follows: Every ordinal function collectively generates a limit.*

Hence Ordinal Generation implies Blass Generation, and Broad Ordinal Generation implies Jorgensen Generation.

8 Consequences of Truth Value Set

Throughout this section, Truth Value Set is assumed.

8.1 Hartogs and Lindenbaum Numbers

This section describes cardinal relationships between sets and ordinals. Although AC implies that every set A has a *cardinality*, written $\mathsf{card}\, A$, the situation is more subtle when AC is not assumed. We begin with two preorders on $\mathfrak{S}$.

Definition 8.1. For set A and B, we write

- $A \preccurlyeq B$ when there is an injection from A to B
- $A \preccurlyeq^* B$ when there is a partial surjection from B to A.

Proposition 8.2. *Let A and B be sets.*

1. $A \preccurlyeq B$ *implies* $A \preccurlyeq^* B$.
2. $A \preccurlyeq^* B$ *implies* $B \preccurlyeq \mathcal{P}A$.
3. (Assuming Boolean Truth.) $A \preccurlyeq^* B$ *iff either* $A = \emptyset$ *or there is a surjection from* B *to* A.
4. (Assuming AC.) $A \preccurlyeq B$ *iff* $A \preccurlyeq^* B$ *iff* $\mathsf{card}\, A \leqslant \mathsf{card}\, B$.

Next, we would like to convert sets to ordinals. The following are two well-established ways of doing so.

Definition 8.3. Let K be a set.

1. The *Hartogs number* of K, written $\aleph(K)$, is the set of order-types of well-ordered subsets of K.
2. A *partial partition* of K is a set $\mathcal{A}$ of inhabited subsets such that, for all $X, Y \in \mathcal{A}$, if $X \cap Y$ is inhabited, then $X = Y$. The *Lindenbaum number* of K, written $\aleph^*(K)$, is the set of order-types of well-ordered partial partitions of K.

Each of these sets of ordinals is transitive (indeed lower) and thus an ordinal. Here are some basic properties.

Proposition 8.4. *Let K be a set.*

1. *For an ordinal γ, we have $\gamma < \aleph(K)$ iff $\gamma \preccurlyeq K$.*
2. *For an ordinal γ, we have $\gamma < \aleph^*(K)$ iff $\gamma \preccurlyeq^* K$.*
3. $0 < \aleph(K) \leqslant \aleph^*(K) \leqslant \aleph(\mathcal{P}K)$.
4. (Assuming AC.) $\aleph(K) = \aleph^*(K) = (\mathsf{card}\, K)^+$. *Here κ^+ denotes the successor cardinal of a cardinal κ.*

Proof.

1. Both statements are equivalent to K having a well-ordered subset with order-type γ.

2. Similar.

3. Since $\gamma \preccurlyeq K$ implies $\gamma \preccurlyeq^* K$, which in turn implies $\gamma \preccurlyeq \mathcal{P}K$.

4. Since $\gamma \preccurlyeq K$ iff $\gamma \preccurlyeq^* K$ iff $\gamma < (\mathsf{card}\, K)^+$. $\square$

Although Hartogs numbers not used in this paper, Lindenbaum numbers are used in the following ways.

Proposition 8.5. *Let K be a set.*

1. *Let λ be a regular limit such that $\aleph^*(K) \leqslant \lambda$. Then λ is K-complete.*

2. (Assuming Boolean Truth.) *Let $(X_\alpha)_{\alpha<\aleph^*(K)}$ be an increasing sequence of subsets of K. Then there is $\alpha < \aleph^*(K)$ such that $X_\alpha = X_{\mathrm{S}(\alpha)}$.*

Proof.

1. Given a K-tuple $[\gamma_k]_{k\in K}$ within λ, let β be the order-type of $\{\gamma_k \mid k \in K\}$, with isomorphism $\theta : \{\gamma_k \mid k \in K\} \cong \beta$. The map $k \mapsto \theta(\gamma_k)$ is a surjection from K to β, so $\beta < \aleph^*(K) \leqslant \lambda$. Therefore $\bigvee_{k\in K} \gamma_k = \bigvee_{\delta<\beta} \theta^{-1}(\delta) < \lambda$.

2. Since the partial map from K to $\aleph^*(K)$ that sends x to α when $x \in X_{\mathrm{S}(\alpha)} \setminus X_\alpha$ is not surjective, there is $\alpha < \aleph^*(K)$ that is not in its range. $\square$

We see next that Boolean Truth makes inductive stabilization equivalent to the existence of a prefixpoint.

Proposition 8.6. (Assuming Boolean Truth.) *Let C be a class, and Γ be a monotone endomap on $\mathcal{P}_{\mathsf{s}}C$, with prefixpoint K. Then Γ inductively stabilizes at some ordinal $< \aleph^*(K)$.*

Proof. From Proposition 8.5(2). $\square$

8.2 Relating Generation Principles for Ordinals

Now we are in a position to establish all the remaining relationships.

Proposition 8.7. *Blass Generation is equivalent to Ordinal Generation.*

Proof. We have seen ($\Leftarrow$). For ($\Rightarrow$), given a set of sets $\mathcal{D}$, let λ be the regular limit generated by $\bigvee_{K\in\mathcal{D}} \aleph^*(K)$. For all $K \in \mathcal{D}$, since $\aleph^*(K) \leqslant \lambda$, Proposition 8.5(1) tells us that λ is K-complete. $\square$

Recall that the *cumulative hierarchy* $(V_\alpha)_{\alpha\in\mathsf{Ord}}$ is the inductive chain of $\mathcal{P}$, and consists of subsets of V_{pure}. For an element $x \in V_{\mathsf{pure}}$, its *rank* $r(x)$ is recursively defined to be the strict supremum of $\{r(y) \mid y \in x\}$. Induction on x shows that $x \in V_{\mathrm{S}(r(x))}$. Thus $V_{\mathsf{pure}} = \bigcup_{\alpha\in\mathsf{Ord}} V_\alpha$. Here is an application.

Proposition 8.8. *Jorgensen Generation, Broad Ordinal Generation and Broad Set Generation (V_{pure}) are equivalent.*

Proof. We already know Broad Set Generation (V_{pure}) $\Rightarrow$ Broad Ordinal Generation $\Rightarrow$ Jorgensen Generation, so it remains to show Jorgensen Generation $\Rightarrow$ Broad Set Generation (V_{pure}). Let $\mathcal{B}$ be a broad rubric on V_{pure}. Writing $\mathcal{B}_0 = (\langle K_i, R_i\rangle)_{i\in I}$, let α be the supremum of $\{\aleph^*(K_i) \mid i \in I\}$. Let J be the ordinal function sending β to the supremum of

$$\begin{aligned}&\{\aleph^*(x) \mid x \in V_\beta\}\\ &\quad\cup \{\aleph^*(K_i) \mid x \in V_\beta,\ \mathcal{B}_1(x) = (\langle K_i, R_i\rangle)_{i\in I},\ i \in I\}\\ &\quad\cup \{\mathrm{S}(r(y)) \mid y \in H_{\mathcal{B}} V_\beta\}\end{aligned}$$

Let λ be the regular limit generated by J_α. We first show that $x \in V_\lambda$ implies $r(x) < \lambda$, by induction on x, as follows. There is $\beta < \lambda$ such that $x \in V_\beta$. As $\aleph^*(x) \leqslant \lambda$, Proposition 8.5(1) tells us that λ is x-complete. For each $y \in x$ we have $r(y) < \lambda$, so $r(x) = \mathsf{ssup}_{y\in x} r(y) < \lambda$.

We show that V_λ is $\mathcal{B}$-inductive. We give just the triggered part, as the basic part is similar. For any $x \in V_\lambda$, with $\mathcal{B}_1(x) = (\langle K_i, R_i\rangle)_{i\in I}$, and any $i \in I$ and K_i-tuple $[a_k]_{k\in K_i}$ within V_λ, with $R_i[a_k]_{k\in K_i} = (y_p)_{p\in P}$, and any $p \in P$, we must show $y_p \in V_\lambda$. The set $\{r(a_k) \mid k \in K_i\}$ is a subset of λ with order-type $< \aleph^*(K_i) \leqslant J(\mathrm{S}(r(x))) \leqslant \lambda$ (since $x \in V_{\mathrm{S}(r(x))}$ and $r(x) < \lambda$ and $\lambda \geqslant J$). So the strict supremum σ of $\{r(x)\} \cup \{r(a_k) \mid k \in K_i\}$ is $< \lambda$, and we have $x \in V_\sigma$ and $\forall k \in K_i.\, a_k \in V_\sigma$. Since $\lambda \geqslant J$ and $y_p \in H_{\mathcal{B}} V_\sigma$, we have $r(y_p) < J\sigma \leqslant \lambda$, so $y_p \in V_{\mathrm{S}(r(y_p))} \subseteq V_\lambda$. $\square$

To obtain a non-broad analogue of Proposition 8.8, the following notion is useful. For a set of sets $\mathcal{D}$ and ordinal function J, a *limit collectively generated by $\mathcal{D}$ extended by J* is a minimal (and therefore least) $\mathcal{D}$-collectively complete limit $\geqslant J$. This gives two more generation principles.

- The *Extended Ordinal Generation* principle says that any set of sets, extended by any ordinal function, collectively generates a limit.
- Recalling Corollary 7.14(1), the *Extended Blass Generation* principle says that any ordinal, extended by any ordinal function, collectively generates a limit.

Proposition 8.9. *Extended Blass Generation, Extended Ordinal Generation and Set Generation (V_{pure}) are equivalent.*

Proof. Set Generation (V_{pure}) $\Rightarrow$ Extended Ordinal Generation is similar to Proposition 7.8, and Extended Ordinal Generation $\Rightarrow$ Extended Blass Generation is obvious, so we prove Extended Blass Generation $\Rightarrow$ Set Generation (V_{pure}). Given a rubric $\mathcal{R} = (\langle K_i, R_i\rangle)_{i\in I}$ on V_{pure}, let α be the supremum of $\{\aleph^*(K_i) \mid i \in I\}$. Let J be the ordinal function sending β to the supremum of $\{\aleph^*(x) \mid x \in V_\beta\} \cup \{\mathrm{S}(r(y)) \mid y \in H_{\mathcal{B}} V_\beta\}$. Let λ be the limit collectively generated by α extended by J, and continue as in the proof of Proposition 8.8. $\square$

9 Application: Universes and Inaccessibles

To illustrate the "broad" generation principles, we show how to directly deduce the existence of universes and inaccessibles.

Definition 9.1. *(Assuming Truth Value Set.)* A *Grothendieck universe* is a transitive set $\mathfrak{U}$ with the following properties.

- $\mathbb{N} \in \mathfrak{U}$.
- For every set of sets $\mathcal{A} \in \mathfrak{U}$, we have $\bigcup \mathcal{A} \in \mathfrak{U}$.
- For every set $A \in \mathfrak{U}$, we have $\mathcal{P}A \in \mathfrak{U}$.
- For every set $K \in \mathfrak{U}$ and K-tuple $[a_k]_{k \in K}$ within $\mathfrak{U}$, we have $\{a_k \mid k \in K\} \in \mathfrak{U}$.

Proposition 9.2. (Assuming Truth Value Set.) *Broad Set Generation implies the "Axiom of Universes": For every set X, there is a least Grothendieck universe $\mathfrak{U}$ with $X \subseteq \mathfrak{U}$.*

Proof. Define the following broad rubric $\mathcal{B}$ on $\mathfrak{S}$. The basic rubric is indexed by $X+4$:

- Rule $\mathsf{inl}\ x$ (for $x \in X$) has arity 0 and sends $[\,]$ to (x).
- Rule $\mathsf{inr}\ 0$ has arity 1 and sends $[* \mapsto A]$ to $(b)_{b \in A}$ if A is a set, and is supported on this case.
- Rule $\mathsf{inr}\ 1$ has arity 0 and sends $[\,]$ to $(\mathbb{N})$.
- Rule $\mathsf{inr}\ 2$ has arity 1 and sends $[* \mapsto \mathcal{A}]$ to $(\bigcup \mathcal{A})$ if $\mathcal{A}$ is a set of sets, and is supported on this case.
- Rule $\mathsf{inr}\ 3$ has arity 1 and sends $[* \mapsto A]$ to $(\mathcal{P}A)$ if A is a set, and is supported on this case.

Each set B triggers a rubric indexed by 1, where rule $*$ has arity B and sends $[a_k]_{k \in B}$ to $(\{a_k \mid k \in B\})$, and $\mathcal{B}_1$ is supported on this case. The set $\mathsf{Gen}(\mathcal{B})$ has the required properties. □

For our second example, which comes from type theory [22], the first step is to define

$$
\begin{array}{lllll}
\text{embed} & : & \mathfrak{T} & \to & \mathfrak{T} \\
\text{zero} & \in & \mathfrak{T} & & \\
\text{two} & \in & \mathfrak{T} & & \\
\text{eq} & : & \mathfrak{T}^3 & \to & \mathfrak{T} \\
\text{sigma} & : & \mathfrak{T}^2 & \to & \mathfrak{T} \\
\text{wtype} & : & \mathfrak{T}^2 & \to & \mathfrak{T}
\end{array}
$$

in such a way that they are injective and have disjoint range. We achieve this as follows:

$$\begin{array}{rcl}
\mathrm{embed}(x) & \stackrel{\text{def}}{=} & \langle 0, \langle x \rangle \rangle \\
\mathrm{zero} & \stackrel{\text{def}}{=} & \langle 1, \langle \rangle \rangle \\
\mathrm{two} & \stackrel{\text{def}}{=} & \langle 2, \langle \rangle \rangle \\
\mathrm{eq}(x,y,z) & \stackrel{\text{def}}{=} & \langle 3, \langle x,y,z \rangle \rangle \\
\mathrm{sigma}(x,y) & \stackrel{\text{def}}{=} & \langle 4, \langle x,y \rangle \rangle \\
\mathrm{wtype}(x,y) & \stackrel{\text{def}}{=} & \langle 5, \langle x,y \rangle \rangle
\end{array}$$

Definition 9.3. Let $(B_a)_{a \in A}$ be a family of sets. A *Tarski-style universe* extending it is a family of sets $(D_m)_{m \in M}$ satisfying the following conditions.

- For all $a \in A$, we have $\mathrm{embed}(a) \in M$ with $D_{\mathrm{embed}(a)} = B_a$.
- We have $\mathrm{zero} \in M$ with $D_{\mathrm{zero}} = \emptyset$.
- We have $\mathrm{two} \in M$ with $D_{\mathrm{two}} = \{0, 1\}$.
- For any $m \in M$ and $a, b \in D_m$, we have $\mathrm{eq}(m, a, b) \in M$ with $D_{\mathrm{eq}(m,a,b)} = 1_{a=b}$.
- For any $m \in M$ and function $g : D_m \to M$, we have $\mathrm{sigma}(m, g) \in M$ with $D_{\mathrm{sigma}(m,g)} = \sum_{k \in D_m} D_{g(m)}$.
- For any $m \in M$ and function $g : D_m \to M$, we have $\mathrm{wtype}(m, g) \in M$ with $D_{\mathrm{wtype}(m,g)} = \mathsf{Term}(D_{g(m)})_{k \in D_m}$.

Proposition 9.4. *Broad Family Generation implies that, for any family of sets, there is a least Tarski-style universe extending it.*

Proof. Let $(B_a)_{a \in A}$ be a family of sets. Define $\mathcal{B}$ to be the following broad rubric on $\mathfrak{S}$. The basic rubric is indexed by $A + 2$:

- Rule $\mathsf{inl}\ a$ (for $a \in A$) has arity 0 and sends $[\,]$ to (B_a).
- Rule $\mathsf{inr}\ 0$ has arity 0 and sends $[\,]$ to $(\emptyset)$.
- Rule $\mathsf{inr}\ 1$ has arity 1 and sends $[\,]$ to $(\{0, 1\})$.

A set D triggers a rubric indexed by $D^2 + 2$:

- Rule $\mathsf{inl}\ \langle d, e \rangle$ (for $d, e \in D$) has arity 0 and sends $[\,]$ to $(1_{d=e})$
- Rule $\mathsf{inr}\ 0$ has arity D and sends $[E_k]_{k \in D}$ to $(\sum_{k \in K} E_k)$.
- Rule $\mathsf{inr}\ 1$ has arity D and sends $[E_k]_{k \in D}$ to $(\mathsf{Term}(E_k)_{k \in D})$.

Let $\mathsf{GenFam}(\mathcal{B}) = (E_n)_{n \in N}$. Define the function θ on N that recursively sends

$$\begin{array}{rcl} \mathsf{Basic}(\mathsf{inl}\ a, [\,], *) & \mapsto & \mathrm{embed}(a) \\ \mathsf{Basic}(\mathsf{inr}\ 0, [\,], *) & \mapsto & \mathrm{zero} \\ \mathsf{Trigger}(n, \mathsf{inl}\ \langle d, e \rangle, [\,], *) & \mapsto & \mathrm{eq}(\theta n, d, e) \\ \mathsf{Trigger}(n, \mathsf{inr}\ 0, g, *) & \mapsto & \mathrm{sigma}(\theta n, \theta \circ g) \\ \mathsf{Trigger}(n, \mathsf{inr}\ 1, g, *) & \mapsto & \mathrm{wtype}(\theta n, \theta \circ g) \end{array}$$

By induction, θ is injective. Let M be its range. Then $(E_{\theta^{-1}(m)})_{m \in M}$ is the desired family. $\square$

Our last example concerns the notion of inaccessible cardinal. AC is assumed, cf. [7].

Definition 9.5. *(Assuming AC.)* An *inaccessible* is a regular limit $\kappa > \omega$ such that, for any cardinal $\lambda < \kappa$, we have $2^\lambda < \kappa$.

Proposition 9.6. (Assuming AC.) *Jorgensen Generation implies that there are arbitrarily large inaccessibles.*

Proof. Let J be the ordinal function that sends λ to $\mathrm{S}(2^\lambda)$, if λ is an cardinal, and is supported on this case. For any $\alpha > \omega$, the regular limit generated by J_α is the least inaccessible $\geqslant \alpha$. $\square$

10 Conclusions and Further Work

We have introduced the new principle of Broad Infinity, and have seen that, assuming AC, it is equivalent to Jorgensen Generation and hence to Ord-is-Mahlo. We assumed a global WISC function to prove ($\Rightarrow$) and Truth Value Set to prove ($\Leftarrow$), so the equivalence might not hold in weaker systems.

One question in particular remains: does Broad ZF prove that there is an uncountable regular ordinal? Gitik [15] showed that ZF does not, assuming the consistency of the existence of arbitrarily large strongly compact cardinals. A similar result for Broad ZF would clarify (subject to a consistency hypothesis) the relationship between Broad Infinity and AC.

Another topic to investigate is the relationship between Broad Family Generation and the *induction-recursion* principles [11, 14] used in type theory and the proof assistant Agda. These allow the formation of Tarski-style universes, as in Proposition 9.4, and are modelled in [11] using a Mahlo cardinal.

In a different direction, one may consider models of IZF and CZF, such as those appearing in [4, 13, 37, 35]. While it is true (on the assumption that AC holds in reality) that many such models validate Local WISC [37], I do not know whether they provide a global WISC function. And while they validate Signature Infinity, I do not know whether they validate Broad Infinity. The latter question may relate to the work of Rathjen [30] giving type theoretic semantics of CZF with a Mahlo universe.

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A Restricted Separation

A.1 Introduction

As mentioned in Section 2.2, the CZF school does not accept unrestricted Truth Value Separation. In this appendix, we modify some aspects of the paper in order to conform to that viewpoint. Section A.2 presents a suitable base theory, and Section A.3 lists the changes that we make.

A.2 The Restricted Separation Base Theory

Our first task is to modify the Base Theory. Let us say that a proposition ψ is *separable*, written $\bigcirc\,\psi$, when it has a truth value. Formally:

$$\bigcirc\,\psi \quad \overset{\text{def}}{\iff} \quad \exists X.\, \forall y.\, (y \in X \iff (y = * \wedge \psi))$$

Likewise a predicate P is *separable* on a class A when $P(x)$ is separable for all $x \in A$.

On a logical signature Σ, the *Restricted Separation Base Theory* is a subsystem of the Base Theory, defined in two parts. The first part is the same as in Section 2.2, except that Truth Value Separation (which asserts that every proposition is separable) is replaced by the following.

- Axiom of *Equality Separation*: For any a and b, we have $\bigcirc\, a = b$.
- Axiom of *Sethood Separation*: For any a, we have $\bigcirc\, \mathsf{IsSet}(a)$.

Before continuing, note that we already have the following result, which is adapted from [3, Theorem 9.5.6 and Proposition 9.6.2] and [4, Lemma 2.2].

Proposition A.1.

1. *Let P and Q be n-ary predicates, for some $n \in \mathbb{N}$. If $\exists! \overrightarrow{y}.\, P(\overrightarrow{y})$ and $\forall \overrightarrow{y}.\, P(\overrightarrow{y}) \Rightarrow \bigcirc\, Q(\overrightarrow{y})$, then $\bigcirc\,\psi$, where ψ is $\exists \overrightarrow{y}.\, P(\overrightarrow{y}) \wedge Q(\overrightarrow{y})$ or equivalently $\forall \overrightarrow{y}.\, P(\overrightarrow{y}) \Rightarrow Q(\overrightarrow{y})$.*
2. *For any set A of truth values, the following classes are sets:*

$$\bigvee A \quad \overset{\text{def}}{=} \quad \{x \in 1 \mid \exists y \in A.\, x \in y\}$$
$$\bigwedge A \quad \overset{\text{def}}{=} \quad \{x \in 1 \mid \forall y \in A.\, x \in y\}$$

3. *If $\bigcirc\,\phi$ and $\phi \Rightarrow \bigcirc\,\psi$, then $\bigcirc\,(\phi \wedge \psi)$ and $\bigcirc\,(\phi \Rightarrow \psi)$.*
4. *For any a and b, we have $\bigcirc\, a \in b$.*
5. *If $\bigcirc\,\phi$ and $\bigcirc\,\psi$, then $\bigcirc\,(\phi \vee \psi)$ and $\bigcirc\,(\phi \wedge \psi)$ and $\bigcirc\,(\phi \Rightarrow \psi)$.*
6. *For any separable predicate P over a set A, we have $\bigcirc\,\exists x \in A.\, P(x)$ and $\bigcirc\,\forall x \in A.\, P(x)$.*
7. *If $\psi \vee \neg\psi$, then $\bigcirc\,\psi$.*
8. *Let $P(a)$ be any of the following predicates:*
   - *a is an ordered pair.*
   - *a is of the form* $\mathsf{Succ}(x)$.
   - *a is of the form* $\mathsf{Make}(x, y)$.
   - *a is of the form* $\mathsf{Build}(x, y, z)$.
   - *a is of the form* $\mathsf{Basic}(x, y, z)$.

- *a is of the form* $\mathsf{Trigger}(x, y, z, w)$.

For all a, we have $\bigcirc\, P(a)$.

9. *Let A be a set, and P a predicate. Then $\{x \in A \mid P(x)\}$ is a set iff P is separable over A.*

10. *The intersection of two sets is a set.*

Proof.

1. Let $\overrightarrow{a}$ be the unique $\overrightarrow{y}$ such that $P(\overrightarrow{y})$. Then $\bigcirc\, Q(\overrightarrow{a})$, and ψ is equivalent to $Q(\overrightarrow{a})$.

2. The class $\bigvee A$ is $\bigcup A$, and therefore a set. For $\bigwedge A$, first note that $B \stackrel{\text{def}}{=} \bigcup_{x \in A} \bigcup_{y \in 1_{x=1}} \{x\}$ is the set of all $x \in A$ such that $x = 1$. So $\forall x \in A.\, x = 1$ is equivalent to $B = A$, and therefore $\bigwedge A = 1_{B=A}$.

3. Since $1_{\phi \wedge \psi} = \bigvee_{x \in 1_\phi} 1_\psi$ and $1_{\phi \Rightarrow \psi} = \bigwedge_{x \in 1_\phi} 1_\psi$.

4. If $\mathsf{IsSet}(b)$, then $\bigcirc\, a \in b$ since $a \in b \iff b = b \cup \{a\}$. Part 3 gives $\bigcirc\, (\mathsf{IsSet}(b) \wedge a \in b)$ and therefore $\bigcirc\, a \in b$.

5. From parts 2–3.

6. From part 2.

7. By case analysis.

8. The statement "a is an ordered pair" is equivalent to the statement "a is a set of sets and there is $x, y \in \bigcup a$ such that $a = \{x, y\}$". Likewise for the others.

9. ($\Rightarrow$): for any $y \in A$, we have $P(y)$ iff $y \in \{x \in A \mid P(x)\}$, which is separable by part (4). For ($\Leftarrow$), use $\{x \in A \mid P(x)\} = \bigcup_{x \in A} \bigcup_{u \in 1_{P(x)}} \{x\}$.

10. By part (9), using $A \cap B = \{x \in A \mid x \in B\}$. $\square$

Proposition A.1 gives us several tools to verify $\bigcirc\, \psi$. In particular, parts 4–6 give the case where ψ has no unbounded quantifiers and no predicate symbols from Σ (other than ones that are hypothesized to be separable). Furthermore, parts 1 and 3 allow ψ to use class functions, even ones that are defined using unbounded quantifiers.

The second part of the theory is given as follows:

- Axiom of *Signature Infinity Specification*: For any signature S, there is a minimal (and therefore least) S-inductive set.

- Axiom scheme of *Signature Infinity Induction*: For any signature S, any minimal S-inductive set is a minimal (and therefore least) S-inductive class.

For a signature S, we now define *set of all S-terms* to mean a set that is a minimal (and therefore least) S-inductive class. The two parts of Signature Infinity together assert the existence of such a set.

The theory's name emphasizes that only Separation is restricted, not induction or Replacement.

We consider the theory to be acceptable to the CZF school, because Signature Infinity Specification—though omitted from the original CZF formulation [2]—is often assumed in recent literature [37]. All the other axioms and axiom schemes are provable in CZF, with Sethood Separation following from Purity, and Signature Infinity Induction from $\in$-induction, cf. [29].

We henceforth assume this theory, and have the following.

Proposition A.2.

1. Infinity*: There is a* set of all natural numbers*, i.e. a set that is a minimal (and therefore least) nat-inductive class.*

2. Exponentiation*: For any sets A and B, there is a set B^A of all functions from A to B.*

Proof.

1. Let S be the signature indexed by $\{0, 1\}$, where 0 has arity $\emptyset$ and 1 has arity 1. Let θ be the function on $\mathsf{Term}(S)$ recursively defined to send $\langle 0, [\,]\rangle$ to Zero and $\langle 1, [* \mapsto a]\rangle$ to $\mathsf{Succ}(\theta a)$. By induction, θ is injective, and its range is a set of all natural numbers.

2. Let S be the signature indexed by $1 + B$, where $\mathsf{inl}\ *$ has arity A and $\mathsf{inr}\ b$ has arity 1. Then B^A is the class of all $x \in \mathsf{Term}(S)$ that have the form $\langle \mathsf{inl}\ *, [\langle b_a, [\,]\rangle]_{a \in A}\rangle$, and this predicate on $\mathsf{Term}(S)$ is separable. □

A.3 List of Alterations

We now describe all the changes that we make to the paper as a result of adopting the weaker base theory.

Excluded Middle In Proposition 2.3, we replace Excluded Middle by *Separable Excluded Middle*: For every separable proposition ψ, either ψ or $\neg\psi$. The full law of Excluded Middle is equivalent to the combination of Boolean Truth and Truth Value Separation.

Diagrams of Subsystems In Figures 3–4, we remove the references to ZFAN and ZF-CAN, and the ones to Blass's Axiom and Ord-is-Mahlo.

Induction Just like Signature Infinity, each infinity principle and generation principle has two parts: Specification and Induction. For example, given a rubric or broad rubric $\mathcal{R}$ on a class C, a *set generated by* $\mathcal{R}$ is a set that is a minimal (and therefore least) $\mathcal{R}$-inductive class. So the Set Generation principle has a Specification part, saying that there is a minimal $\mathcal{R}$-inductive set, and an Induction part saying that every minimal $\mathcal{R}$-inductive set is a minimal $\mathcal{R}$-inductive class. Likewise a *family generated by* $\mathcal{R}$ is a family that is a minimal (and therefore least) $\mathcal{R}$-inductive large family $(x_m)_{m\in M}$. As usual, minimality can be expressed by saying that every relative inductive subclass of M is equal to M.

We likewise modify the definition of *set of all G-broad numbers* (for a broad signature or reduced broad signature G) and *descendant set* and *$\mathcal{M}$-descendant set* (for a spection $\mathcal{M}$).

For the generation principles involving ordinals, we use the notion of a *large ordinal*, i.e. transitive class of ordinals. (Excluded Middle is equivalent to the scheme saying that every large ordinal is either an ordinal or Ord.) For example, given a set of sets $\mathcal{D}$, we define *limit collectively generated by* $\mathcal{D}$ to mean an ordinal that is a minimal (and therefore least) collectively $\mathcal{D}$-complete large limit.

Signature Infinity Proposition 2.6 is removed, since Signature Infinity is part of the axiomatization.

Separability We have to distinguish between separable and general classes, as we now explain.

Definition A.3.

1. A class C is *separable* when for all x we have $\bigcirc x \in C$.

2. For a separable class I, a class C_i depending on $i\in I$ is *separable* when for all $i\in I$ and all x we have $\bigcirc x \in C_i$.

3. On a separable class C, a partial function $\langle B, F\rangle$ is *separable* when B is separable.

4. A large family $(x_m)_{m\in M}$ is *separable* when M is separable.

The sentence following Definition 3.3 is modified as follows: for a separable class C, a separable partial function $(x\in C) \rightharpoonup D_x$ corresponds to a function $C \to (D_x)_\perp$.

In the definition of *spection* $\mathcal{M} = (J(e))_{e\in M}$ we require M to be separable; this is used in the construction of $J^*(e)$. Likewise, in defining *fam-spection* $\mathcal{S} = (\langle J(e), W_e, L_e\rangle)_{e\in M}$ on a class C, we assume that C is separable, and require separability of M and, for all $e \in M$, of W_e. Propositions 4.5(1) and 4.10(1) are modified as follows.

Proposition A.4.

1. *Any spection $\mathcal{M}$ generates a class, written* $\mathsf{Gen}(\mathcal{M})$*, and cogenerates a class, written* $\mathsf{Cogen}(\mathcal{M})$*. Moreover, both are separable.*

2. *Let C be a separable class. Any fam-spection $\mathcal{S}$ on C generates a large family, written* $\mathsf{GenFam}(\mathcal{S})$. *Moreover, it is separable.*

Proof.

1. The same as Proposition 4.5(1), using the second construction of $\mathsf{Gen}(\mathcal{M})$ to show separability.

2. The same as Proposition 4.10(1). □

Throughout Section 3.3 until Proposition 3.9 it is assumed that B and C are separable. In Corollary 3.10 it is assumed that C is separable and "subclass" is replaced by "separable subclass". In the rest of the paper, whenever we consider a rubric or broad rubric on a class C, it is assumed that C is separable.

Proofs Other than as indicated above, the proofs in the paper are unchanged. Proposition A.1 and A.4 are used to verify separability, such as when forming a spection or fam-spection. The class L in the proof of Proposition 6.3 is a set because it is a subclass of a set and also separable (being introspectively generated). Likewise U in the proof of Proposition 6.6.

We must pay attention to the following statement and others like it: for a rubric $\mathcal{R}$, if an $\mathcal{R}$-inductive set exists, then $\mathcal{R}$ generates a set. Such statements hold if we assume either Truth Value Separation or Truth Value Set. Happily, they are used only in the proof of Propositions 8.7–8.9, which assume Truth Value Set.