Thunkable implies central
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It is established in [1, Proposition 2.20] that, for a strong monad on a cartesian category \( C \), any Kleisli map that is thunkable is also central. This note shows that (as expected) this generalizes to the setting where \( C \) is merely monoidal.

Firstly let \( C \) be a monoidal category with a monad \( T \) and left strength \( t_{A,B} : TA \otimes B \to T(A \otimes B) \).

For a \( T \)-algebra \( (P, \theta) \) and map \( h : A \otimes \Delta \to P \), we write \( h^{\♯\theta} \) for the left Kleisli extension, i.e. the following composite:

\[
TA \times \Delta \xrightarrow{t_{A,\Delta}} T(A \times \Delta) \xrightarrow{Th} TP \xrightarrow{\theta} P
\]

**Proposition 1.** For a map \( f : \Gamma \to TA \), the following are equivalent.

(a) The map \( f \) is thunkable, i.e. the diagram

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{f} & TA \\
\downarrow{f} & & \downarrow{\eta_T A} \\
TA & \xrightarrow{T\eta_A} & T^2A
\end{array}
\]

commutes.

(b) For any object \( \Delta \) and \( T \)-algebra \( (P, \theta) \) and map \( h : \Delta \otimes TA \to P \), the diagram

\[
\begin{array}{ccc}
\Gamma \otimes \Delta & \xrightarrow{f \otimes \Delta} & TA \otimes \Delta \\
\downarrow{f \otimes \Delta} & & \downarrow{h} \\
TA \otimes \Delta & \xrightarrow{(\eta_T A \otimes h)^{\theta}} & P
\end{array}
\]

commutes.

**Proof.** For (a)\( \Rightarrow \) (b), we take

Surprisingly, the converse is also true in the case of a continuation monad [4, Remark 3.5]. But in general a central map need not be thunkable, even if it is an isomorphism. For example, the writer monad \( \mathbb{Z}_2 \times - \) on \( \text{Set} \) is commutative, so every Kleisli map is central, and in particular the Kleisli map \( 1 \to 1 \) sending \( * \mapsto (1,*) \) is a central involution that is not thunkable, cf. [3, Section 5.2].

\[^1\text{Surprisingly, the converse is also true in the case of a continuation monad [4, Remark 3.5]. But in general a central map need not be thunkable, even if it is an isomorphism. For example, the writer monad } \mathbb{Z}_2 \times - \text{ on } \text{Set} \text{ is commutative, so every Kleisli map is central, and in particular the Kleisli map } 1 \to 1 \text{ sending } *\mapsto (1,*) \text{ is a central involution that is not thunkable, cf. [3, Section 5.2].}\]
For (b)⇒(a), we take Δ to be 1 and ignore − ⊗ 1, and we take (P, θ) to be the free algebra on TA. Then we have

![Diagram]

Corollary 1. For any algebra (P, θ) and maps h, k: TA ⊗ Δ → P, the following are equivalent.

(a) The diagram $A ⊗ Δ \xrightarrow{\eta_A \otimes Δ} TA \otimes Δ \xrightarrow{f} P$ commutes.

(b) For any object Γ and thunkable $f: Γ → TA$, the diagram

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(1)
Γ \otimes Δ \xrightarrow{f \otimes Δ} TA \otimes Δ \xrightarrow{k} P
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commutes.

Proof. The implication (a)⇒(b) follows from Proposition 1(a)⇒(b). For (b)⇒(a), put $Γ = A$ and $f = η_A$. □

Now suppose that $T$ has bistrength consisting of $t_{A,B}: TA \otimes B → T(A \otimes B)$ and $t'_{A,B}: A \otimes TB → T(A \otimes B)$. (Recall from [2] that “bistrength” means that the two maps $(A \otimes TB) \otimes C → T((A \otimes B) \otimes C)$ are always equal. While this condition is not used in our argument, it is needed to ensure that the Kleisli category is premonoidal, specifically that the associator $A \otimes (B \otimes C) ≃ (A \otimes B) \otimes C$ is natural in $B$. In the case of a symmetric monoidal category, it follows from the condition that $t$ and $t'$ correspond across the symmetry. I do not know whether there are interesting examples of bistrong monads other than these.)

For maps $f: A → TB$ and $g: C → TD$, the condition that $f$ commutes with $g$ is equivalent to the instance of (b) where $(P, θ)$ is the free algebra on $A \otimes B$ and $h$ is the composite

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TA \otimes Δ \xrightarrow{T \otimes g} TA \otimes TB \xrightarrow{t_{A,B}} T(A \otimes TB) \xrightarrow{T T_{A,B}} T^2(A \otimes B) \xrightarrow{\mu_{A \otimes B}} T(A \otimes B)
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and $k$ is the composite

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TA \otimes Δ \xrightarrow{T \otimes g} TA \otimes TB \xrightarrow{t'_{A,B}} T(TA \otimes B) \xrightarrow{T T_{A,B}} T^2(A \otimes B) \xrightarrow{\mu_{A \otimes B}} T(A \otimes B)
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So if $f$ is thunkable then it commutes with $g$. So thunkability implies left centrality, and likewise it implies right centrality.

References

