Reducibility proofs in the λ-calculus

Fairouz Kamareddine, Vincent Rahli and J. B. Wells*

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Abstract

Reducibility has been used to prove a number of properties in the λ-calculus and is well known to offer on one hand very general proofs which can be applied to a number of instantiations, and on the other hand, to be quite mysterious and inflexible. In this paper, we look at two related but different results in λ-calculi with intersection types. We show that one such result (which aims at giving reducibility proofs of Church-Rosser, standardisation and weak normalisation for the untyped λ-calculus) faces serious problems which break the reducibility method and then we provide a proposal to partially repair the method. Then, we consider a second result whose purpose is to use reducibility for typed terms to show Church-Rosser of β-developments for untyped terms (without needing to use strong normalisation), from which Church-Rosser of β-reduction easily follows. We extend the second result to encompass both βI- and $\beta\eta$-reduction rather than simply β-reduction.

1 Introduction

Based on realisability semantics [Kle45], the reducibility method has been developed by Tait [Tai67] in order to prove normalisation of some functional theories. The idea is to interpret types by sets of λ-terms closed under some properties. Krivine [Kri90] uses reducibility to prove the strong normalisation of system D. Koletsos [Kol85] proves that the set of simply typed λ-terms has the Church-Rosser property. Gallier [Gal97, Gal03] uses some aspects of Koletsos's method to prove a number of results such as the strong normalisation of the λ-terms that are typable in systems like D or $D\Omega$ [Kri90]. In particular, Gallier states some conditions a property needs to satisfy in order to be enjoyed by some typable terms under some restrictions. Similarly, Ghilezan and Likavec [GL02] state some conditions a property on λ-terms has to satisfy in order to be held by all λ-terms that are typable under some restriction on types in a type system which is close to $D\Omega$. Additionally Ghilezan and Likavec state a condition that a property needs to satisfy in order to step from "a λ-term typable under

*School of Mathematical and Computer Sciences, Heriot-Watt Univ., Riccarton, Edinburgh EH14 4AS, Scotland

some restrictions on types holds the property" to "a λ-term of the untyped λ-calculus holds the property". If successful, the method designed by Ghilezan and Likavec would provide an attractive method for establishing properties like Church-Rosser for all the untyped λ-terms, simply by showing easier conditions on typed terms. However, we show in this paper that Ghilezan and Likavec's method fails for the typed terms, and that also the step of passing from typed to untyped terms fails. We show why we also fail to entirely repair the first result and how far we succeeded to get when trying to repair it (we reach a result similar to one already obtained by Ghilezan and Likavec). The second result seems unrepairable. Ghilezan and Likavec also present a weaker version of their method for a type system similar to system D, which allows using reducibility to prove properties of the term typable by this system, namely the strongly normalisable terms. As far as we know, this portion of their result is correct. (They do not actually apply this weaker method to any sets of terms.)

In addition to the method proposed by Ghilezan and Likavec (which does not actually work for the full untyped λ-calculus), other steps of establishing properties like Church-Rosser (also called confluence) for typed λ-terms and concluding the properties for all the untyped λ-terms have been successfully exploited in the literature. Koletsos and Stavrinos [KS08] use reducibility to state that λ-terms that are typable in system D hold the Church-Rosser property. Using this result together with a method based on β-developments [Klo80, Kri90], they show that β-developments are Church-Rosser and this in turn will imply the confluence of the untyped λ-calculus. Although Klop proves the confluence of β-developments [BBKV76], his proof is based on strong normalisation whereas the Koletsos and Stavrinos's proof only uses an embedding of β-developments in the reduction of typable λ-terms. In this paper, we apply Koletsos and Stavrinos's method to βI-reduction and then generalise it to $\beta\eta$-reduction.

In section 2 we introduce the formal machinery and establish the basic needed lemmas. In section 3 we present the reducibility method used by Ghilezan and Likavec and show that it fails at a number of important propositions which makes it inapplicable to the full untyped λ-calculus, although a version of their method works for the strongly normalisable terms. We give counterexamples which show that all the conditions stated in Ghilezan and Likavec's paper are satisfied, yet the claimed property does not hold. In section 4 we give some indications on the limits of the method. We show how these limits affect the salvation of the method, we partially salvage it and we show that this can now be correctly used to establish confluence, standardisation and weak head normal forms but only for restricted sets of lambda terms and types (that we believe to be equal to the set of strongly normalisable terms). We also point out some links between the work done by Ghilezan and Likavec and the work done by Gallier. In section 5 we adapt the Church-Rosser proof of Koletsos and Stavrinos [KS08] to βI-reduction. In section 6 we non-trivially generalise Koletsos and Stavrinos's method to handle $\beta\eta$-reduction. We conclude in section 7.

2 The Formal Machinery

In this section we provide some known formal machinery and introduce new definitions and lemmas that are necessary for the paper. Let n, m be metavariables which range over the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. We take as convention that if a metavariable v ranges over a set s then the metavariables v_i such that $i \geq 0$ and the metavariables $v', v'', etc.$ also range over s.

A binary relation is a set of pairs. Let rel range over binary relations. Let $\mathrm{dom}(rel) = \{x \mid \langle x, y\rangle \in rel\}$ and $\mathrm{ran}(rel) = \{y \mid \langle x, y\rangle \in rel\}$. A function is a binary relation fun such that if $\{\langle x, y\rangle, \langle x, z\rangle\} \subseteq fun$ then $y = z$. Let fun range over functions. Let $s \to s' = \{fun \mid \mathrm{dom}(fun) \subseteq s \wedge \mathrm{ran}(fun) \subseteq s'\}$.

Given n sets $s_1, \ldots, s_n$, where $n \geq 2$, $s_1 \times \ldots \times s_n$ stands for the set of all the tuples built on the sets $s_1, \ldots, s_n$. If $x \in s_1 \times \ldots \times s_n$, then $x = \langle x_1, \ldots, x_n\rangle$ such that $x_i \in s_i$ for all $i \in \{1, \ldots, n\}$.

2.1 Familiar background on λ-calculus

This section consists of one long definition of some familiar (mostly standard) concepts of the λ-calculus and one lemma which deals with the shape of reductions.

Definition 2.1.

1. let $x, y, z, etc.$ range over $\mathcal{V}$, a countable infinite set of λ-term variables. The set of terms of the λ-calculus is defined as follows:

   $$M \in \Lambda ::= x \mid (\lambda x.M) \mid (M_1 M_2)$$

   We let $M, N, P, Q, etc.$ range over Λ. We assume the usual definition of subterms: we write $N \subseteq M$ if N is a subterm of M. We also assume the usual convention for parenthesis and omit these when no confusion arises. In particular, we write $M\ N_1 ... N_n$ instead of $(...(M\ N_1)\ N_2 ... N_{n-1})\ N_n$.

   We take terms modulo α-conversion and use the Barendregt convention (BC) where the names of bound variables differ from the free ones. When two terms M and N are equal (modulo α), we write $M = N$. We write $\mathrm{fv}(M)$ for the set of the free variables of term M.

2. Let $n \geq 0$. We define $M^n(N)$, by induction on n, as follows: $M^0(N) = N$ and $M^{n+1}(N) = M(M^n(N))$.

3. The set of paths is defined as follows:

   $$p \in \mathsf{Path} ::= 0 \mid 1.p \mid 2.p$$

   We define $M|_p$ as follows: $M|_0 = M$, $(\lambda x.M)|_{1.p} = M|_p$, $(MN)|_{1.p} = M|_p$ and $(MN)|_{2.p} = N|_p$. We define $2^n.p$ by induction on $n \geq 0$: $2^0.p = p$ and $2^{n+1}.p = 2^n.2.p$.

4. The set $\Lambda\mathrm{I} \subset \Lambda$, of terms of the λI-calculus is defined by the following rules:

   (a) If $x \in \mathcal{V}$ then $x \in \Lambda\mathrm{I}$.

   (b) If $x \in \mathrm{fv}(M)$ and $M \in \Lambda\mathrm{I}$ then $\lambda x.M \in \Lambda\mathrm{I}$.

   (c) If $M, N \in \Lambda\mathrm{I}$ then $MN \in \Lambda\mathrm{I}$.

5. We define as usual the substitution $M[x := N]$ of N for all free occurrences of x in M. We let $M[x_i := N_i, \ldots, x_n := N_n]$ be the simultaneous substitution of N_i for all free occurrences of x_i in M for $1 \leq i \leq n$.

6. Let define the four common following relations:

   - $\mathsf{Beta} ::= \langle (\lambda x.M)N, M[x := N] \rangle$.
   - $\mathsf{BetaI} ::= \langle (\lambda x.M)N, M[x := N] \rangle$, where $x \in \mathrm{fv}(M)$.
   - $\mathsf{Eta} ::= \langle \lambda x.Mx, M \rangle$, where $x \notin \mathrm{fv}(M)$.
   - $\mathsf{BetaEta} = \mathsf{Beta} \cup \mathsf{Eta}$.

   Let $\langle r, s \rangle \in \{\langle \mathsf{Beta}, \beta \rangle, \langle \mathsf{BetaI}, \beta I \rangle, \langle \mathsf{Eta}, \eta \rangle, \langle \mathsf{BetaEta}, \beta\eta \rangle\}$. We define $\mathcal{R}^s$ to be $\{L \mid \langle L, R \rangle \in r\}$. If $\langle L, R \rangle \in r$ then we call L a s-redex and R the s-contractum of L (or the L s-contractum). We define the ternary relation $\rightarrow_s$ as follows:

   - $M \xrightarrow{0}_s M'$ if $\langle M, M' \rangle \in r$.
   - $\lambda x.M \xrightarrow{1.p}_s \lambda x.M'$ if $M \xrightarrow{p}_s M'$.
   - $MN \xrightarrow{1.p}_s M'N$ if $M \xrightarrow{p}_s M'$.
   - $NM \xrightarrow{2.p}_s NM'$ if $M \xrightarrow{p}_s M'$.

   We define the binary relation $\rightarrow_s$ (we use the same name as for the just defined ternary relation $\rightarrow_s$ to simplify the notations) as follows: $M \rightarrow_s M'$ if there exists p such that $M \xrightarrow{p}_s M'$. We define $\mathcal{R}^s_M = \{p \mid M|_p \in \mathcal{R}^s\}$.

7. Let $M \in \Lambda$ and $\mathcal{F} \subseteq \Lambda$. $\mathcal{F} \upharpoonright M = \{N \mid N \in \mathcal{F} \land N \subseteq M\}$.

8. $\rightarrow_{h\beta} ::= \langle \lambda x_1. \ldots x_n.(\lambda x.M_0)M_1 \ldots M_m, \lambda x_1. \ldots x_n.M_0[x := M_1]M_2 \ldots M_m \rangle$, where $n \geq 0$ and $m \geq 1$.

   If $\langle L, R \rangle \in \rightarrow_{h\beta}$ then $L = \lambda x_1. \ldots x_n.(\lambda x.M_0)M_1 \ldots M_m$ where $n \geq 0$ and $m \geq 1$ and $(\lambda x.M_0)M_1$ is called the β-head redex of L.

   We define the binary relation $\rightarrow_{i\beta}$ as $\rightarrow_\beta \setminus \rightarrow_{h\beta}$.

9. Let $r \in \{\rightarrow_\beta, \rightarrow_\eta, \rightarrow_{\beta\eta}, \rightarrow_{\beta I}, \rightarrow_{h\beta}, \rightarrow_{i\beta}\}$. We use $\rightarrow_r^*$ to denote the reflexive transitive closure of $\rightarrow_r$. We let $\simeq_r$ denote the equivalence relation induced by $\rightarrow_r$. If the r-reduction from M to N is in k steps, we write $M \rightarrow_r^k N$.

10. Let $r \in \{\beta I, \beta\eta\}$ and $n \geq 0$. A term $(\lambda x.M')N'_0N'_1 \dots N'_n$ is a direct r-reduct of $(\lambda x.M)N_0N_1 \dots N_n$ iff $M \rightarrow^*_r M'$ and $\forall i \in \{0, \dots, n\}$. $N_i \rightarrow^*_r N'_i$.

11. $\mathsf{NF}_\beta = \{\lambda x_1. \dots \lambda x_n.x_0N_1 \dots N_m \mid n, m \geq 0, N_1, \dots, N_m \in \mathsf{NF}_\beta\}$.

12. $\mathsf{WN}_\beta = \{M \in \Lambda \mid \exists N \in \mathsf{NF}_\beta, M \rightarrow^*_\beta N\}$.

13. Let $r \in \{\beta, \beta I, \beta\eta\}$.

    - We say that M has the Church-Rosser property for r (has r-CR) if whenever $M \rightarrow^*_r M_1$ and $M \rightarrow^*_r M_2$ then there is an M_3 such that $M_1 \rightarrow^*_r M_3$ and $M_2 \rightarrow^*_r M_3$.
    - $\mathsf{CR}^r = \{M \mid M \text{ has } r\text{-}\mathsf{CR}\}$.
    - $\mathsf{CR}^r_0 = \{xM_1 \dots M_n \mid n \geq 0 \wedge x \in \mathcal{V} \wedge (\forall i \in \{1, \dots, n\}, M_i \in \mathsf{CR}^r)\}$.
    - We use CR to denote CR^β and CR_0 to denote CR^β_0.
    - A term is a weak head normal form if it is a λ-abstraction (a term of the form $\lambda x.M$) or if it starts with a variable (a term of the form $xM_1 \cdots M_n$). A term is weakly head normalising if it reduces to a weak head normal form. Let $\mathsf{W}^r = \{M \in \Lambda \mid \exists n \geq 0, \exists x \in \mathcal{V}, \exists P, P_1, \dots, P_n \in \Lambda, M \rightarrow^*_r \lambda x.P \text{ or } M \rightarrow^*_r xP_1 \dots P_n\}$. We use W to denote W^β.

14. We say that M has the standardisation property if whenever $M \rightarrow^*_\beta N$ then there is an M' such that $M \rightarrow^*_h M'$ and $M' \rightarrow^*_i N$. Let $\mathsf{S} = \{M \in \Lambda \mid M \text{ has the standardisation property}\}$. □

The next lemma deals with the shape of reductions.

Lemma 2.2.

1. *$M \xrightarrow{p}_{\beta\eta} M'$ iff ($M \xrightarrow{p}_\beta M'$ or $M \xrightarrow{p}_\eta M'$).*
2. *If $x \in \mathrm{fv}(M_1)$ then $\mathrm{fv}((\lambda x.M_1)M_2) = \mathrm{fv}(M_1[x := M_2])$ and if $(\lambda x.M_1)M_2 \in \Lambda I$ then $M_1[x := M_2] \in \Lambda I$.*
3. *If $M \rightarrow^*_{\beta\eta} M'$ then $\mathrm{fv}(M') \subseteq \mathrm{fv}(M)$.*
4. *If $M \rightarrow^*_{\beta I} M'$ then $\mathrm{fv}(M) = \mathrm{fv}(M')$ and if $M \in \Lambda I$ then $M' \in \Lambda I$.*
5. *$\lambda x.M \xrightarrow{p}_{\beta\eta} P$ iff either ($p = 1.p'$, $P = \lambda x.M'$ and $M \xrightarrow{p'}_{\beta\eta} M'$) or ($p = 0$, $M = Px$ and $x \notin \mathrm{fv}(P)$).*
6. *Let $r \in \{\beta I, \beta\eta\}$, $n \geq 0$, P is not a direct r-reduct of $(\lambda x.M)N_0 \dots N_n$ and $(\lambda x.M)N_0 \dots N_n \rightarrow^k_r P$. Then the following holds:*
   (a) *$k \geq 1$, and if $k = 1$ then $P = M[x := N_0]N_1 \dots N_n$.*
   (b) *There exists a direct r-reduct $(\lambda x.M')N'_0N'_1 \dots N'_n$ of $(\lambda x.M)N_0 \dots N_n$ such that $M'[x := N'_0]N'_1 \dots N'_n \rightarrow^*_r P$.*

7. *Let $r \in \{\beta I, \beta\eta\}$, $n \geq 0$ and $(\lambda x.M)N_0N_1 \ldots N_n \to_r^* P$. There exists P' such that $P \to_r^* P'$ and if ($r = \beta I$ and $x \in \mathrm{fv}(M)$) or $r = \beta\eta$ then $M[x := N_0]N_1 \ldots N_n \to_r^* P'$.*

8. *There exists M' such that $M \stackrel{p}{\to}_r M'$ iff $p \in \mathcal{R}_M^r$.*

9. *If $M \stackrel{p}{\to}_r M_1$ and $M \stackrel{p}{\to}_r M_2$ then $M_1 = M_2$* □

2.2 Formalising the background on developments

In this section we go through some needed background from [Kri90] on developments and we precisely formalise and establish all the necessary properties. In order not to clutter the paper, we have put all the proofs of this section in an appendix. Throughout the paper, we take c to be a metavariable ranging over $\mathcal{V}$. As far as we know, this is the first precise formalisation of developments.

The next definition adapts Λ_c of [Kri90] to deal with βI- and $\beta\eta$-reduction. Basically, $\Lambda\mathrm{I}_c$ is Λ_c where in the abstraction construction rule (R1).2, we restrict abstraction to ΛI. In $\Lambda\eta_c$ we introduce the new rule (R4) and replace the abstraction rule of Λ_c by (R1).3 and (R1).4.

Definition 2.3 ($\Lambda\eta_c$, $\Lambda\mathrm{I}_c$)**.**

1. We let $\mathcal{M}_c$ range over $\Lambda\eta_c, \Lambda\mathrm{I}_c$ defined as follows (note that $\Lambda\mathrm{I}_c \subset \Lambda\mathrm{I}$):

   (R1) If x is a variable distinct from c then

   1. $x \in \mathcal{M}_c$.
   2. If $M \in \Lambda\mathrm{I}_c$ and $x \in \mathrm{fv}(M)$ then $\lambda x.M \in \Lambda\mathrm{I}_c$.
   3. If $M \in \Lambda\eta_c$ then $\lambda x.M[x := c(cx)] \in \Lambda\eta_c$.
   4. If $Nx \in \Lambda\eta_c$ such that $x \notin \mathrm{fv}(N)$ and $N \neq c$ then $\lambda x.Nx \in \Lambda\eta_c$.

   (R2) If $M, N \in \mathcal{M}_c$ then $cMN \in \mathcal{M}_c$.

   (R3) If $M, N \in \mathcal{M}_c$ and M is a λ-abstraction then $MN \in \mathcal{M}_c$.

   (R4) If $M \in \Lambda\eta_c$ then $cM \in \Lambda\eta_c$.

□

Here is a lemma related to terms of $\mathcal{M}_c$.

Lemma 2.4 (Generation)**.**

1. *$M[x := c(cx)] \neq x$ and for any N, $M[x := c(cx)] \neq Nx$.*

2. *Let $x \notin \mathrm{fv}(M)$. Then, $M[y := c(cx)] \neq x$ and for any N, $M[y := c(cx)] \neq Nx$.*

3. *If $M \in \mathcal{M}_c$ then $M \neq c$.*

4. *If $M, N \in \mathcal{M}_c$ then $M[x := N] \neq c$.*

5. *Let $MN \in \mathcal{M}_c$. Then $N \in \mathcal{M}_c$ and either*

- $M = cM'$ where $M' \in \mathcal{M}_c$ or
- $M = c$ and $\mathcal{M}_c = \Lambda\eta_c$ or
- $M = \lambda x.P$ is in $\mathcal{M}_c$

6. If $c^n(M) \in \mathcal{M}_c$ then $M \in \mathcal{M}_c$.

7. If $\lambda x.P \in \Lambda\eta_c$ then $x \neq c$ and either

   - $P = Nx$ where $N, Nx \in \Lambda\eta_c$ where $x \notin \mathrm{fv}(N)$ and $N \neq c$ or
   - $P = N[x := c(cx))]$ where $N \in \Lambda\eta_c$

8. If $\lambda x.P \in \Lambda I_c$ then $x \neq c$, $x \in \mathrm{fv}(P)$ and $P \in \Lambda I_c$.

9. If $M, N \in \mathcal{M}_c$ and $x \neq c$ then $M[x := N] \in \mathcal{M}_c$.

10. Let $y \notin \{x, c\}$. Then:

    - if $M[x := c(cx)] = y$ then $M = y$,
    - if $M[x := c(cx)] = Py$ then $M = Ny$ and $P = N[x := c(cx)]$,
    - if $M[x := c(cx)] = \lambda y.P$ then $M = \lambda y.N$ and $P = N[x := c(cx)]$.
    - if $M[x := c(cx)] = PQ$ then either $M = x$, $P = c$ and $Q = cx$ or $M = P'Q'$ and $P = P'[x := c(cx)]$ and $Q = Q'[x := c(cx)]$.
    - if $M[x := c(cx)] = (\lambda y.P)Q$ then $M = (\lambda y.P')Q'$ and $P = P'[x := c(cx)]$ and $Q = Q'[x := c(cx)]$.

11. Let $M \in \Lambda\eta_c$.

    (a) If $M = \lambda x.P$ then $P \in \Lambda\eta_c$.

    (b) If $M = \lambda x.Px$ then $Px, P \in \Lambda\eta_c$, $x \notin \mathrm{fv}(P) \cup \{c\}$ and $P \neq c$.

12. (a) Let $x \neq c$. $M[x := c(cx)] \xrightarrow{p}_{\beta\eta} M'$ iff $M' = N[x := c(cx)]$ and $M \xrightarrow{p}_{\beta\eta} N$.

    (b) Let $n \geq 0$. If $c^n(M) \xrightarrow{p}_{\beta\eta} M'$ then $p = 2^n.p'$ and there exists $N \in \Lambda\eta_c$ such that $M' = c^n(N)$ and $M \xrightarrow{p'}_{\beta\eta} N$.

$\square$

Here is a lemma about the paths of redexes in a term:

Lemma 2.5. *Let $r \in \{\beta I, \beta\eta\}$.*

- If $M \in \mathcal{V}$ then $\mathcal{R}^r_M = \varnothing$.
- If $M = \lambda x.N$ then:
  - if $M \in \mathcal{R}^r$ then $\mathcal{R}^r_M = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_N\}$.
  - else, $\mathcal{R}^r_M = \{1.p \mid p \in \mathcal{R}^r_N\}$.

- *If $M = PQ$ then:*
  - *if $M \in \mathcal{R}^r$ then $\mathcal{R}^r_M = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_P\} \cup \{2.p \mid p \in \mathcal{R}^r_Q\}$.*
  - *else, $\mathcal{R}^r_M = \{1.p \mid p \in \mathcal{R}^r_P\} \cup \{2.p \mid p \in \mathcal{R}^r_Q\}$.*

$\square$

Here is a lemma about the set of redexes in a term:

Lemma 2.6. *Let $r \in \{\beta I, \beta\eta\}$ and $\mathcal{F} \subseteq \mathcal{R}^r_M$.*

- *If $M \in \mathcal{V}$ then $\mathcal{F} = \varnothing$.*
- *If $M = \lambda x.N$ then $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^r_N$ and:*
  - *if $M \in \mathcal{R}^r$ then $\mathcal{F} \setminus \{0\} = \{1.p \mid p \in \mathcal{F}'\}$.*
  - *else, $\mathcal{F} = \{1.p \mid p \in \mathcal{F}'\}$.*
- *If $M = PQ$ then $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^r_P$, $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}^r_Q$ and:*
  - *if $M \in \mathcal{R}^r$ then $\mathcal{F} \setminus \{0\} = \{1.p \mid p \in \mathcal{F}_1\} \cup \{2.p \mid p \in \mathcal{F}_2\}$.*
  - *else, $\mathcal{F} = \{1.p \mid p \in \mathcal{F}_1\} \cup \{2.p \mid p \in \mathcal{F}_2\}$.*

$\square$

The next lemma shows the role on redexes of substitutions involving c.

Lemma 2.7. *Let $r \in \{\beta\eta, \beta I\}$. and $x \neq c$.*

1. *$M \in \mathcal{R}^{\beta\eta}$ iff $M[x := c(cx)] \in \mathcal{R}^{\beta\eta}$.*
2. *If $p \in \mathcal{R}^{\beta\eta}_M$ then $M[x := c(cx)]|_p = M|_p[x := c(cx)]$.*
3. *$p \in \mathcal{R}^{\beta\eta}_{\lambda x.M[x:=c(cx)]}$ iff $p = 1.p'$ and $p' \in \mathcal{R}^{\beta\eta}_{M[x:=c(cx)]}$.*
4. *$\mathcal{R}^{\beta\eta}_{M[x:=c(cx)]} = \mathcal{R}^{\beta\eta}_M$.*
5. *$\mathcal{R}^{\beta\eta}_{c^n(M)} = \{2^n.p \mid p \in \mathcal{R}^{\beta\eta}_M\}$.* $\square$

The next lemma shows that any element $(\lambda x.P)Q$ of $\Lambda\mathrm{I}_c$ (resp. $\Lambda\eta_c$) is a βI- (resp. $\beta\eta$-) redex.

Lemma 2.8. *Let $(\mathcal{M}_c, r) \in \{(\Lambda I_c, \beta I), (\Lambda\eta_c, \beta\eta)\}$ and $M \in \mathcal{M}_c$. If $M = (\lambda x.P)Q$ then $M \in \mathcal{R}^r$.* $\square$

The next lemma shows that $\Lambda\mathrm{I}_c$ (resp. $\Lambda\eta_c$) contains all the βI-redexes (resp. $\beta\eta$-redexes) of all its terms.

Lemma 2.9. *Let $(\mathcal{M}_c, r) \in \{(\Lambda\mathrm{I}_c, \beta I), (\Lambda\eta_c, \beta\eta)\}$ and $M \in \mathcal{M}_c$. If $p \in \mathcal{R}^r_M$ then $M|_p \in \mathcal{M}_c$.* $\square$

In order to deal with βI- and $\beta\eta$-reduction, the next lemma generalises a lemma given in [Kri90] (and used in [KS08]). It states that $\Lambda\eta_c$ and $\Lambda\mathrm{I}_c$ are closed under $\to_{\beta\eta}$- resp. $\to_{\beta I}$-reduction.

Lemma 2.10.

1. *If $M \in \Lambda\eta_c$ and $M \to_{\beta\eta} M'$ then $M' \in \Lambda\eta_c$.*
2. *If $M \in \Lambda I_c$ and $M \to_{\beta I} M'$ then $M' \in \Lambda I_c$.* $\square$

The next definition again taken from [Kri90], erases all the c's from a $\mathcal{M}_c$-term. We extend it to paths.

Definition 2.11 ($|-|^c$)**.** We define $|M|^c$ and $|\langle M, p\rangle|^c$ inductively as follows:

- $|x|^c = x$
- $|cP|^c = |P|^c$
- $|\langle M, 0\rangle|^c = 0$
- $|\langle cM, 2.p\rangle|^c = |\langle M, p\rangle|^c$
- $|\langle MN, 1.p\rangle|^c = 1.|\langle M, p\rangle|^c$
- $|\lambda x.N|^c = \lambda x.|N|^c$, if $x \neq c$
- $|NP|^c = |N|^c|P|^c$ if $N \neq c$
- $|\langle \lambda x.M, 1.p\rangle|^c = 1.|\langle M, p\rangle|^c$, if $x \neq c$
- $|\langle NM, 2.p\rangle|^c = 2, |\langle M, p\rangle|^c$ if $N \neq c$

Let $\mathcal{F} \subseteq \mathsf{Path}$ then we define $|\langle M, \mathcal{F}\rangle|^c = \{|\langle M, p\rangle|^c \mid p \in \mathcal{F}\}$. $\square$

Now, c^n is indeed erased from $|c^n(M)|^c$.

Lemma 2.12. *Let $n \geq 0$ then $|c^n(M)|^c = |M|^c$.* $\square$

Lemma 2.13. $|\langle c^n(M), \mathcal{R}^{\beta\eta}_{c^n(M)}\rangle|^c = |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c$. $\square$

Lemma 2.14. $|\langle c^n(M), 2^n.p\rangle|^c = |\langle M, p\rangle|^c$. $\square$

Also, c^n is erased from $|c^n(N)|^c$ for any $c^n(N)$ subterm of M.

Lemma 2.15. *Let $|M|^c = P$.*

- *If $P \in \mathcal{V}$ then $\exists n \geq 0$ such that $M = c^n(P)$.*
- *If $P = \lambda x.Q$ then $\exists n \geq 0$ such that $M = c^n(\lambda x.N)$ and $|N|^c = Q$.*
- *If $P = P_1P_2$ then $\exists n \geq 0$ such that $M = c^n(M_1M_2)$, $M_1 \neq c$, $|M_1|^c = P_1$ and $|M_2|^c = P_2$.* $\square$

If the c-erasure of two paths of M are equal, then these paths are also equal:

Lemma 2.16. *Let $r \in \{\beta I, \beta\eta\}$. If $p, p' \in \mathcal{R}^r_M$ and $|\langle M, p\rangle|^c = |\langle M, p'\rangle|^c$ then $p = p'$.* $\square$

Inside a term, substituting x by $c(cx)$ is undone by c-erasure.

Lemma 2.17. *Let $x \neq c$. Then, $|M[x := c(cx)]|^c = |M|^c$.* $\square$

Lemma 2.18. *Let $x \neq c$ and $p \in \mathcal{R}^{\beta\eta}_M$. Then, $|\langle M[x := c(cx)], p\rangle|^c = |\langle M, p\rangle|^c$.* $\square$

The next lemma shows that c is definitely erased from the free variables of $|M|^c$.

Lemma 2.19. *If* $M \in \mathcal{M}_c$ *then* $\mathrm{fv}(M) \setminus \{c\} = \mathrm{fv}(|M|^c)$. □

Erasure propagates through substitutions.

Lemma 2.20. *If* $M, N \in \mathcal{M}_c$ *and* $x \neq c$ *then* $|M[x := N]|^c = |M|^c[x := |N|^c]$. □

Now, c-erasing an $\Lambda\mathrm{I}_c$-term returns an ΛI-term.

Lemma 2.21. *If* $M \in \Lambda I_c$ *then* $|M|^c \in \Lambda I$. □

Lemma 2.22. *Let* $(\mathcal{M}_c, r) \in \{(\Lambda I_c, \beta I), (\Lambda\eta_c, \beta\eta)\}$ *and* $M \in \mathcal{M}_c$. *If* $p \in \mathcal{R}^r_M$ *and* $M \xrightarrow{p}_r M'$ *then* $|M|^c \xrightarrow{p'}_r |M'|^c$ *such that* $p' = |\langle M, p\rangle|^c$. □

Lemma 2.23. *Let* $(\mathcal{M}_c, r) \in \{(\Lambda I_c, \beta I), (\Lambda\eta_c, \beta\eta)\}$, $M_1, N_1, M_2, N_2 \in \mathcal{M}_c$, $x \neq c$, $|\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$, $|\langle N_1, \mathcal{R}^r_{N_1}\rangle|^c \subseteq |\langle N_2, \mathcal{R}^r_{N_2}\rangle|^c$, $|M_1|^c = |M_2|^c$ *and* $|N_1|^c = |N_2|^c$. *Then,* $|\langle M_1[x := N_1], \mathcal{R}^r_{M_1[x:=N_1]}\rangle|^c \subseteq |\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c$. □

Lemma 2.24. *Let* $(\mathcal{M}_c, r) \in \{(\Lambda I_c, \beta I), (\Lambda\eta_c, \beta\eta)\}$, $M_1, M_2 \in \mathcal{M}_c$ *such that* $|\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$ *and* $|M_1|^c = |M_2|^c$. *If* $M_1 \xrightarrow{p_1}_r M'_1$, $M_2 \xrightarrow{p_2}_r M'_2$ *such that* $|\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c$ *then* $|\langle M'_1, \mathcal{R}^r_{M'_1}\rangle|^c \subseteq |\langle M'_2, \mathcal{R}^r_{M'_2}\rangle|^c$. □

2.3 Background on Types and Type Systems

In this section we give the background necessary for the type systems used in this paper.

Definition 2.25. Let $i \in \{1, 2\}$.

1. Let $\mathcal{A}$ be a denumerably infinite set of type variables, let α range over $\mathcal{A}$ and let $\Omega \notin \mathcal{A}$ be a constant type. The sets of types $\mathsf{Type}^1 \subset \mathsf{Type}^2$ are defined as follows:

$$\sigma \in \mathsf{Type}^1 ::= \alpha \mid \sigma_1 \to \sigma_2 \mid \sigma_1 \cap \sigma_2$$

$$\tau \in \mathsf{Type}^2 ::= \alpha \mid \tau_1 \to \tau_2 \mid \tau_1 \cap \tau_2 \mid \Omega$$

2. We let $\Gamma \in \mathcal{B}^1 = \{\{x_1 : \sigma_1, \ldots, x_n : \sigma_n\} \mid \forall i, j \in \{1, \ldots, n\}.\ x_i = x_j \Rightarrow \sigma_i = \sigma_j\}$ and $\Gamma, \Delta \in \mathcal{B}^2 = \{\{x_1 : \tau_1, \ldots, x_n : \tau_n\} \mid \forall i, j \in \{1, \ldots, n\}.\ x_i = x_j \Rightarrow \tau_i = \tau_j\}$. We define $\mathrm{dom}(\Gamma) = \{x \mid x : \sigma \in \Gamma\}$. When $\mathrm{dom}(\Gamma_1) \cap \mathrm{dom}(\Gamma_2) = \varnothing$, we write Γ_1, Γ_2 for $\Gamma_1 \cup \Gamma_2$. We write $\Gamma, x : \sigma$ for $\Gamma, \{x : \sigma\}$ and $x : \sigma$ for $\{x : \sigma\}$. We denote $\Gamma = x_m : \sigma_m, \ldots, x_n : \sigma_n$ where $n \geq m \geq 0$, by $(x_i : \sigma_i)^m_n$. If $m = 1$, we simply denote Γ by $(x_i : \sigma_i)_n$.

   If $\Gamma_1 = (x_i : \tau_i)_n, (y_i : \tau''_i)_p$ and $\Gamma_2 = (x_i : \tau'_i)_n, (z_i : \tau'''_i)_q$ where $x_1, \ldots, x_n$ are the only shared variables, then $\Gamma_1 \sqcap \Gamma_2 = (x_i : \tau_i \cap \tau'_i)_n, (y_i : \tau''_i)_p, (z_i :$

$\tau_i''')_q$.
Let $X \subseteq \mathcal{V}$. We define $\Gamma \upharpoonright X = \Gamma' \subseteq \Gamma$ where $\mathrm{dom}(\Gamma') = \mathrm{dom}(\Gamma) \cap X$.
Let $\sqsubseteq$ be the reflexive transitive closure of the axioms $\tau_1 \cap \tau_2 \sqsubseteq \tau_1$ and $\tau_1 \cap \tau_2 \sqsubseteq \tau_2$. If $\Gamma = (x_i : \tau_i)_n$ and $\Gamma' = (x_i : \tau_i')_n$ then $\Gamma \sqsubseteq \Gamma'$ iff for all $i \in \{1, \ldots, n\}$, $\tau_i \sqsubseteq \tau_i'$.

3. • – Let $\nabla_1 = \{(ref), (tr), (in_L), (in_R), (\rightarrow\text{-}\cap), (mon'), (mon), (\rightarrow\text{-}\eta)\}$.
   – Let $\nabla_2 = \nabla_1 \cup \{(\Omega), (\Omega' - lazy)\}$.
   – Let $\nabla_D = \{(in_L), (in_R)\}$.
   – Let $\nabla_{D_I} = \nabla_D \cup \{(idem)\}$
   • – $\mathsf{Type}^{\nabla_1} = \mathsf{Type}^{\nabla_D} = \mathsf{Type}^{\nabla_{D_I}} = \mathsf{Type}^1$.
   – $\mathsf{Type}^{\nabla_2} = \mathsf{Type}^2$.
   • – Let ∇ be a set of axioms from Figure 1. The relation $\leq^\nabla$ is defined on types Type^∇ and axioms ∇. We use $\leq^1$ instead of $\leq^{\nabla_1}$ and $\leq^2$ instead of $\leq^{\nabla_2}$.
   – The equivalence relation is defined by: $\tau_1 \sim^\nabla \tau_2 \iff \tau_1 \leq^\nabla \tau_2 \wedge \tau_2 \leq^\nabla \tau_1$. We use $\sim^1$ instead of $\sim^{\nabla_1}$ and $\sim^2$ instead of $\sim^{\nabla_2}$.
   • – Let $\lambda\cap^1$ be the type system built on Λ, Type^1 and $\vdash^1$ such that $\vdash^1$ is the type derivability relation on $\mathcal{B}^1$, Λ and Type^1 generated using the following typing rules of Figure 2: (ax), $(\rightarrow_E)$, $(\rightarrow_I)$, $(\cap_I)$ and $(\leq^1)$).
   – Let $\lambda\cap^2$ be the type system built on Λ, Type^2 and $\vdash^2$ such that $\vdash^2$ is type derivability relation on $\mathcal{B}^2$, Λ and Type^2 generated using the following typing rules of Figure 2: (ax), $(\rightarrow_E)$, $(\rightarrow_I)$, $(\cap_I)$, $(\leq^2)$ and (Ω).
   – Let D be the type system built on Λ, Type^1 and $\vdash^{\beta\eta}$ where $\vdash^{\beta\eta}$ is the type derivability relation on $\mathcal{B}^1$, Λ and Type^1 generated using the following typing rules of Figure 2: (ax), $(\rightarrow_E)$, $(\rightarrow_I)$, $(\cap_I)$, $(\cap_{E1})$ and $(\cap_{E2})$.
   – Let D_I be the type system built on Λ, Type^1 and $\vdash^{\beta I}$ where $\vdash^{\beta I}$ is the type derivability relation on $\mathcal{B}^1$, Λ and Type^1 generated using the following typing rule of Figure 2: (ax^I), $(\rightarrow_{E^I})$, $(\rightarrow_I)$, $(\cap_I)$, $(\cap_{E1})$ and $(\cap_{E2})$. Moreover, in this type system, we assume that $\sigma \cap \sigma = \sigma$.

□

3 Problems of Ghilezan and Likavec's reducibility method [GL02]

In this section we introduce the reducibility method of [GL02] and show where exactly it fails. Throughout, we let $\circledast = \lambda x.xx$.

(ref)	$\tau \leq \tau$	(Ω)	$\tau \leq \Omega$
(tr)	$(\tau_1 \leq \tau_2 \wedge \tau_2 \leq \tau_3) \Rightarrow \tau_1 \leq \tau_3$	$(\Omega'\text{-}lazy)$	$\tau \to \Omega \leq \Omega \to \Omega$
(in_L)	$\tau_1 \cap \tau_2 \leq \tau_1$	$(idem)$	$\tau \leq \tau \cap \tau$
(in_R)	$\tau_1 \cap \tau_2 \leq \tau_2$	$(\Omega\text{-}\eta)$	$\Omega \leq \Omega \to \Omega$
$(\to\text{-}\cap)$	$(\tau_1 \to \tau_2) \cap (\tau_1 \to \tau_3) \leq \tau_1 \to (\tau_2 \cap \tau_3)$	$(\Omega\text{-}lazy)$	$\tau_1 \to \tau_2 \leq \Omega \to \Omega$
(mon')	$(\tau_1 \leq \tau_2 \wedge \tau_1 \leq \tau_3) \Rightarrow \tau_1 \leq \tau_2 \cap \tau_3$		
(mon)	$(\tau_1 \leq \tau_1' \wedge \tau_2 \leq \tau_2') \Rightarrow \tau_1 \cap \tau_2 \leq \tau_1' \cap \tau_2'$		
$(\to\text{-}\eta)$	$(\tau_1 \leq \tau_1' \wedge \tau_2' \leq \tau_2) \Rightarrow \tau_1' \to \tau_2' \leq \tau_1 \to \tau_2$		

Figure 1: Ordering axioms on types

$$\frac{}{\Gamma, x:\tau \vdash x:\tau}\ (ax) \qquad \frac{}{x:\tau \vdash x:\tau}\ (ax^I)$$

$$\frac{\Gamma \vdash M:\tau_1 \to \tau_2 \quad \Gamma \vdash N:\tau_1}{\Gamma \vdash MN:\tau_2}\ (\to_E) \qquad \frac{\Gamma_1 \vdash M:\tau_1 \to \tau_2 \quad \Gamma_2 \vdash N:\tau_1}{\Gamma_1 \sqcap \Gamma_2 \vdash MN:\tau_2}\ (\to_{E^I})$$

$$\frac{\Gamma, x:\tau_1 \vdash M:\tau_2}{\Gamma \vdash \lambda x.M:\tau_1 \to \tau_2}\ (\to_I) \qquad \frac{\Gamma \vdash M:\tau_1 \quad \Gamma \vdash M:\tau_2}{\Gamma \vdash M:\tau_1 \cap \tau_2}\ (\cap_I)$$

$$\frac{\Gamma \vdash M:\tau_1 \cap \tau_2}{\Gamma \vdash M:\tau_1}\ (\cap_{E1}) \qquad \frac{\Gamma \vdash M:\tau_1 \cap \tau_2}{\Gamma \vdash M:\tau_2}\ (\cap_{E2})$$

$$\frac{\Gamma \vdash M:\tau_1 \quad \tau_1 \leq^{\nabla} \tau_2}{\Gamma \vdash M:\tau_2}\ (\leq^{\nabla}) \qquad \frac{}{\Gamma \vdash M:\Omega}\ (\Omega)$$

Figure 2: Typing rules

Definition 3.1 (Type systems and reducibility of [GL02]). Let $i \in \{1,2\}$. Let $\mathcal{P}$ range over 2^Λ.

1. The type interpretation $[\![-]\!]^i_- \in \mathsf{Type}^i \to 2^\Lambda \to 2^\Lambda$ is defined by:
   - $[\![\alpha]\!]^i_{\mathcal{P}} = \mathcal{P}$.
   - $[\![\tau_1 \cap \tau_2]\!]^i_{\mathcal{P}} = [\![\tau_1]\!]^i_{\mathcal{P}} \cap [\![\tau_2]\!]^i_{\mathcal{P}}$.
   - $[\![\Omega]\!]^2_{\mathcal{P}} = \Lambda$.
   - $[\![\sigma_1 \to \sigma_2]\!]^1_{\mathcal{P}} = \{M \mid \forall N \in [\![\sigma_1]\!]^1_{\mathcal{P}}.MN \in [\![\sigma_2]\!]^1_{\mathcal{P}}\}$.
   - $[\![\tau_1 \to \tau_2]\!]^2_{\mathcal{P}} = \{M \in \mathcal{P} \mid \forall N \in [\![\tau_1]\!]^2_{\mathcal{P}}, MN \in [\![\tau_2]\!]^2_{\mathcal{P}}\}$.
2. A valuation of term variables in Λ is a function $\nu \in \mathcal{V} \to \Lambda$. We write $v(x := M)$ for the function v' where $v'(x) = M$ and $v'(y) = v(y)$ if $y \neq x$.
3. let ν be a valuation of term variables in Λ. Then $[\![-]\!]_\nu \in \Lambda \to \Lambda$ is defined by:
   $[\![M]\!]_\nu = M[x_1 := \nu(x_1), \ldots, x_n := \nu(x_n)]$, where $FV(M) = \{x_1, \ldots, x_n\}$.
4. - $\nu \models^i_{\mathcal{P}} M : \tau$ iff $[\![M]\!]_\nu \in [\![\tau]\!]^i_{\mathcal{P}}$
   - $\nu \models^i_{\mathcal{P}} \Gamma$ iff $\forall (x : \tau) \in \Gamma.\ \nu(x) \in [\![\tau]\!]^i_{\mathcal{P}}$
   - $\Gamma \models^i_{\mathcal{P}} M : \tau$ iff $\forall \nu \in \mathcal{V} \to \Lambda.\ \nu \models^i_{\mathcal{P}} \Gamma \Rightarrow \nu \models^i_{\mathcal{P}} M : \tau$
5. Let $\mathcal{X} \subseteq \Lambda$. Let us recall the variable, saturation, closure and invariance under abstraction predicates defined by Ghilezan and Likavec:
   - $\mathrm{VAR}^i(\mathcal{P}, \mathcal{X}) \iff \mathcal{V} \subseteq \mathcal{X}$.
   - $\mathrm{SAT}^1(\mathcal{P}, \mathcal{X}) \iff$
     $(\forall M \in \Lambda.\ \forall x \in \mathcal{V}.\ \forall N \in \mathcal{P}.\ M[x := N] \in \mathcal{X} \Rightarrow (\lambda x.M)N \in \mathcal{X})$.
   - $\mathrm{SAT}^2(\mathcal{P}, \mathcal{X}) \iff$
     $(\forall M, N \in \Lambda.\ \forall x \in \mathcal{V}.\ M[x := N] \in \mathcal{X} \Rightarrow (\lambda x.M)N \in \mathcal{X})$.
   - $\mathrm{CLO}^1(\mathcal{P}, \mathcal{X}) \iff (\forall M \in \Lambda.\ \forall x \in \mathcal{V}.\ Mx \in \mathcal{X} \Rightarrow M \in \mathcal{P})$.
   - $\mathrm{CLO}^2(\mathcal{P}, \mathcal{X}) \iff \mathrm{CLO}(\mathcal{P}, \mathcal{X}) \iff$
     $(\forall M \in \Lambda.\ \forall x \in \mathcal{V}.\ M \in \mathcal{X} \Rightarrow \lambda x.M \in \mathcal{P})$.
   - $\mathrm{VAR}(\mathcal{P}, \mathcal{X}) \iff (\forall x \in \mathcal{V}.\ \forall n \in \mathbb{N}.\ \forall N_1, \ldots, N_n \in \mathcal{P}.\ xN_1 \ldots N_n \in \mathcal{X})$.
   - $\mathrm{SAT}(\mathcal{P}, \mathcal{X}) \iff (\forall M, N \in \Lambda.\ \forall x \in \mathcal{V}.\ \forall n \in \mathbb{N}.\ \forall N_1, \ldots, N_n \in \mathcal{P}.$
     $M[x := N]N_1 \ldots N_n \in \mathcal{X} \Rightarrow (\lambda x.M)NN_1 \ldots N_n \in \mathcal{X})$.
   - $\mathrm{INV}(\mathcal{P}) \iff (\forall M \in \Lambda.\ \forall x \in \mathcal{V}.\ M \in \mathcal{P} \iff \lambda x.M \in \mathcal{P})$.

   For $\mathcal{R} \in \{\mathrm{VAR}^i, \mathrm{SAT}^i, \mathrm{CLO}^i\}$, let $\mathcal{R}(\mathcal{P}) \iff \forall \tau \in \mathsf{Type}^i.\ \mathcal{R}(\mathcal{P}, [\![\tau]\!]^i_{\mathcal{P}})$. □

Lemma 3.2 (Basic lemmas proved in [GL02]).

1. *(a)* $[\![M]\!]_{\nu(x:=N)} \equiv [\![M]\!]_{\nu(x:=x)}[x := N]$

(b) $[\![MN]\!]_\nu \equiv [\![M]\!]_\nu [\![N]\!]_\nu$

(c) $[\![\lambda x.M]\!]_\nu \equiv \lambda x.[\![M]\!]_{\nu(x:=x)}$

2. *If* $\mathrm{VAR}^1(\mathcal{P})$ *and* $\mathrm{CLO}^1(\mathcal{P})$ *then*

   (a) *for all* $\sigma \in \mathsf{Type}^1$, $[\![\sigma]\!]^1_{\mathcal{P}} \subseteq \mathcal{P}$.

   (b) *if* $\mathrm{SAT}^1(\mathcal{P})$ *and* $\Gamma \vdash^1 M : \sigma$ *then* $\Gamma \models^1_{\mathcal{P}} M : \sigma$ *and* $M \in \mathcal{P}$

3. *For all* $\tau \in \mathsf{Type}^2$, *if* $\tau \not\sim^2 \Omega$ *then* $[\![\tau]\!]^2_{\mathcal{P}} \subseteq \mathcal{P}$

4. *If* $\tau_1 \leq^2 \tau_2$ *then* $[\![\tau_1]\!]^2_{\mathcal{P}} \subseteq [\![\tau_2]\!]^2_{\mathcal{P}}$.

5. *If* $\mathrm{VAR}^2(\mathcal{P})$, $\mathrm{SAT}^2(\mathcal{P})$ *and* $\mathrm{CLO}^2(\mathcal{P})$ *then* $\Gamma \vdash^2 M : \tau$ *implies* $\Gamma \models^2_{\mathcal{P}} M : \tau$

6. *If* $\mathrm{VAR}^2(\mathcal{P})$, $\mathrm{SAT}^2(\mathcal{P})$ *and* $\mathrm{CLO}^2(\mathcal{P})$ *then for all* $\tau \in \mathsf{Type}^2$, *if* $\tau \not\sim^2 \Omega$ *and* $\Gamma \vdash^2 M : \tau$ *then* $M \in \mathcal{P}$

7. $\mathrm{CLO}(\mathcal{P}, \mathcal{P}) \Rightarrow \forall \tau \in \mathsf{Type}^2.\ \tau \not\sim^2 \Omega \Rightarrow \mathrm{CLO}^2(\mathcal{P}, [\![\tau]\!]^2_{\mathcal{P}})$. □

Proof. We only prove 5. By induction on $\Gamma \vdash^2 M : \tau$. (ax) and (Ω) are easy. $(\cap_I)$ (resp. $(\to_E)$ resp. $(\leq^2)$) is by IH (resp. IH and 1, resp. IH and 4).

$(\to_I)$ By IH, $\Gamma, x : \tau_1 \models^2_{\mathcal{P}} M : \tau_2$. Let $\nu \models^2_{\mathcal{P}} \Gamma$ and $N \in [\![\tau_1]\!]^2_{\mathcal{P}}$. Then $\nu(x := N) \models^2_{\mathcal{P}} \Gamma$ since $x \notin \mathrm{dom}(\Gamma)$ and $\nu(x := N) \models^2_{\mathcal{P}} x : \tau_1$ since $N \in [\![\tau_1]\!]^2_{\mathcal{P}}$. Therefore $\nu(x := N) \models^2_{\mathcal{P}} M : \tau_2$, i.e. $[\![M]\!]_{\nu(x:=N)} \in [\![\tau_2]\!]^2_{\mathcal{P}}$. Hence, by lemma 3.2.1, $[\![M]\!]_{\nu(x:=x)}[x := N] \in [\![\tau_2]\!]^2_{\mathcal{P}}$. By $\mathrm{SAT}^2(\mathcal{P})$, we get $(\lambda x.[\![N]\!]_{\nu(x:=x)})N \in [\![\tau_2]\!]^2_{\mathcal{P}}$. Again by lemma 3.2.1, $([\![\lambda x.M]\!]_\nu)N \in [\![\tau_2]\!]^2_{\mathcal{P}}$. Hence $[\![\lambda x.M]\!]_\nu \in \{M \mid \forall N \in [\![\tau_1]\!]^2_{\mathcal{P}}.\ MN \in [\![\tau_2]\!]^2_{\mathcal{P}}$.

By $\mathrm{VAR}^2(\mathcal{P})$, $x \in [\![\tau_1]\!]^2_{\mathcal{P}}$, hence by the same argument as above we obtain $[\![M]\!]_{\nu(x:=x)} \in [\![\tau_2]\!]^2_{\mathcal{P}}$. So by $\mathrm{CLO}^2(\mathcal{P})$, $\lambda x.[\![M]\!]_{\nu(x:=x)} \in \mathcal{P}$ and by lemma 3.2.1, $[\![\lambda x.M]\!]_\nu \in \mathcal{P}$. Hence, we conclude that $[\![\lambda x.M]\!]_\nu \in [\![\tau_1 \to \tau_2]\!]^2_{\mathcal{P}}$. □

Ghilezan and Likavec claim that if $\mathrm{CLO}^1(\mathcal{P})$, $\mathrm{VAR}^1(\mathcal{P})$ and $\mathrm{SAT}^1(\mathcal{P})$ are true then $\mathsf{SN}_\beta \subseteq \mathcal{P}$ (note that this result does not make any use of the type system $\lambda\cap^1$).

After giving the above definitions and lemmas, [GL02] states that since the predicates ($\mathrm{VAR}^i(\mathcal{P})$, $\mathrm{SAT}^i(\mathcal{P})$ and $\mathrm{CLO}^i(\mathcal{P})$ for $i \in \{1, 2\}$ have been shown to be sufficient to develop the reducibility method, and since in order to prove these predicates one needs stronger induction hypotheses which are easier to prove, the paper sets out to show that these stronger conditions when $i = 2$ are the three predicates $\mathrm{VAR}(\mathcal{P}, \mathcal{P})$, $\mathrm{SAT}(\mathcal{P}, \mathcal{P})$ and $\mathrm{CLO}(\mathcal{P}, \mathcal{P})$. However, as we show below, this attempt fails. They do not develop the necessary stronger induction hypotheses for the case when $i = 1$, and $\lambda\cap^1$ can only anyway type strongly normalisable terms, so we will not consider the case $i = 1$ further.

Commutativity, associativity and idempotence w.r.t. the preorder relation are given by the axioms (in_L), (in_R), (mon'), (tr) and (ref):

- Commutativity: by (in_R), $\tau_1 \cap \tau_2 \leq^\Omega \tau_2$ and by (in_L), $\tau_1 \cap \tau_2 \leq^\Omega \tau_1$ so by (mon'), $\tau_1 \cap \tau_2 \leq^\Omega \tau_2 \cap \tau_1$. By (in_L), $\tau_2 \cap \tau_1 \leq^\Omega \tau_2$ and by (in_R), $\tau_2 \cap \tau_1 \leq^\Omega \tau_1$ so by (mon'), $\tau_2 \cap \tau_1 \leq^\Omega \tau_1 \cap \tau_2$. Hence, $\tau_1 \cap \tau_2 \sim^2 \tau_2 \cap \tau_1$.

- Associativity: by (in_R), $(\tau_1 \cap \tau_2) \cap \tau_3 \leq^\Omega \tau_3$, by (in_L), $(\tau_1 \cap \tau_2) \cap \tau_3 \leq^\Omega \tau_1 \cap \tau_2$, by (in_R), $\tau_1 \cap \tau_2 \leq^\Omega \tau_2$, by (in_L), $\tau_1 \cap \tau_2 \leq^\Omega \tau_1$, so by (tr), $(\tau_1 \cap \tau_2) \cap \tau_3 \leq^\Omega \tau_1$ and $(\tau_1 \cap \tau_2) \cap \tau_3 \leq^\Omega \tau_2$. By (mon'), $(\tau_1 \cap \tau_2) \cap \tau_3 \leq^\Omega \tau_2 \cap \tau_3$ and again by (mon'), $(\tau_1 \cap \tau_2) \cap \tau_3 \leq^\Omega \tau_1 \cap (\tau_2 \cap \tau_3)$. By (in_L), $\tau_1 \cap (\tau_2 \cap \tau_3) \leq^\Omega \tau_1$, by (in_R), $\tau_1 \cap (\tau_2 \cap \tau_3) \leq^\Omega \tau_2 \cap \tau_3$, by (in_L), $\tau_2 \cap \tau_3 \leq^\Omega \tau_2$, by (in_R), $\tau_2 \cap \tau_3 \leq^\Omega \tau_3$, so by (tr), $\tau_1 \cap (\tau_2 \cap \tau_3) \leq^\Omega \tau_2$ and $\tau_1 \cap (\tau_2 \cap \tau_3) \leq^\Omega \tau_3$. By (mon'), $\tau_1 \cap (\tau_2 \cap \tau_3) \leq^\Omega \tau_1 \cap \tau_2$ and again by (mon'), $\tau_1 \cap (\tau_2 \cap \tau_3) \leq^\Omega (\tau_1 \cap \tau_2) \cap \tau_3$. Hence, $(\tau_1 \cap \tau_2) \cap \tau_3 \sim^2 \tau_1 \cap (\tau_2 \cap \tau_3)$.

- Idempotence: by (in_L), $\tau \cap \tau \leq^\Omega \tau$ and by (ref) and (mon'), $\tau \leq^\Omega \tau \cap \tau$, hence, $\tau \sim^2 \tau \cap \tau$.

Let $to \in \mathsf{TypeOmega} ::= \Omega \mid to_1 \cap to_2$.

Let $\text{inInter}(\tau, \tau')$ be true iff $\tau = \tau'$ or $\tau' = \tau_1 \cap \tau_2$ and $(\text{inInter}(\tau, \tau_1)$ or $\text{inInter}(\tau, \tau_2))$.

By commutativity and associativity we write $\tau_1 \cap \cdots \cap \tau_n$, where $n \geq 1$, for any type τ such that $(\text{inInter}(\tau_0, \tau)$ iff there exists $i \in \{1, \ldots, n\}$ such that $\tau_0 = \tau_i)$.

Lemma 3.3. *If $\tau_1 \leq^\Omega \tau_2$ and $\tau_1 \in \mathsf{TypeOmega}$ then $\tau_2 \in \mathsf{TypeOmega}$.* □

Proof. We prove the lemma by induction on the size derivation of $\tau_1 \leq^\Omega \tau_2$ and then by case on the last rule of the derivation.

- (ref): $\tau \leq \tau$. Then it is done since $\tau \in \mathsf{TypeOmega}$.

- (tr): $(\tau_1 \leq^\Omega \tau_2 \wedge \tau_2 \leq^\Omega \tau_3) \Rightarrow \tau_1 \leq^\Omega \tau_3$. By IH twice, $\tau_3 \in \mathsf{TypeOmega}$.

- (in_L): $\tau_1 \cap \tau_2 \leq^\Omega \tau_1$. By definition $\tau_1 \in \mathsf{TypeOmega}$.

- (in_R): $\tau_1 \cap \tau_2 \leq^\Omega \tau_2$. By definition $\tau_2 \in \mathsf{TypeOmega}$.

- $(\rightarrow\text{-}\cap)$: $(\tau_1 \rightarrow \tau_2) \cap (\tau_1 \rightarrow \tau_3) \leq^\Omega \tau_1 \rightarrow (\tau_2 \cap \tau_3)$. If $(\tau_1 \rightarrow \tau_2) \cap (\tau_1 \rightarrow \tau_3) \in \mathsf{TypeOmega}$ then by definition $\tau_1 \rightarrow \tau_2, \tau_1 \rightarrow \tau_3 \in \mathsf{TypeOmega}$ which is false.

- (mon'): $(\tau_1 \leq^\Omega \tau_2 \wedge \tau_1 \leq^\Omega \tau_3) \Rightarrow \tau_1 \leq^\Omega \tau_2 \cap \tau_3$. By IH $\tau_2, \tau_3 \in \mathsf{TypeOmega}$. Hence, $\tau_2 \cap \tau_3 \in \mathsf{TypeOmega}$.

- (mon): $(\tau_1 \leq^\Omega \tau_1' \wedge \tau_2 \leq^\Omega \tau_2') \Rightarrow \tau_1 \cap \tau_2 \leq^\Omega \tau_1' \cap \tau_2'$. By definition $\tau_1, \tau_2 \in \mathsf{TypeOmega}$. By IH, $\tau_1', \tau_2' \in \mathsf{TypeOmega}$. So $\tau_1' \cap \tau_2' \in \mathsf{TypeOmega}$.

- $(\rightarrow\text{-}\eta)$: $(\tau_1 \leq^\Omega \tau_1' \wedge \tau_2' \leq^\Omega \tau_2) \Rightarrow \tau_1' \rightarrow \tau_2' \leq^\Omega \tau_1 \rightarrow \tau_2$. It is done because $\tau_1' \rightarrow \tau_2' \notin \mathsf{TypeOmega}$.

- (Ω): $\tau \leq^\Omega \Omega$. By definition $\Omega \in \mathsf{TypeOmega}$.

- (Ω'-*lazy*): $\tau \to \Omega \leq^{\Omega} \Omega \to \Omega$. It is done since $\tau \to \Omega \notin \mathsf{TypeOmega}$. $\square$

Lemma 3.4. *If $\tau \leq^{\Omega} \tau'$ and $\tau' \not\sim^2 \Omega$ then $\tau \not\sim^2 \Omega$.* $\square$

Proof. Let $\tau \leq^{\Omega} \tau'$. Assume $\tau \sim^2 \Omega$. Then $\Omega \leq^{\Omega} \tau$ and by transitivity $\Omega \leq^{\Omega} \tau'$. Moreover, by (Ω), $\tau' \leq^{\Omega} \Omega$. So $\tau' \sim^2 \Omega$. $\square$

Lemma 3.5. *If $\tau \cap \tau' \not\sim^2 \Omega$ then $\tau \not\sim^2 \Omega$ or $\tau' \not\sim^2 \Omega$.* $\square$

Proof. By (Ω), $\tau \cap \tau' \leq^{\Omega} \Omega$. let $\tau \sim^2 \Omega$ and $\tau' \sim^2 \Omega$, so $\Omega \leq^{\Omega} \tau$ and $\Omega \leq^{\Omega} \tau'$ and by (mon'), $\Omega \leq^{\Omega} \tau \cap \tau'$. $\square$

Lemma 3.6. *If $\tau' \sim^2 \Omega$ then $\tau \leq^{\Omega} \tau \cap \tau'$* $\square$

Proof. By (Ω), $\tau \leq^{\Omega} \Omega$ and by transitivity, $\tau \leq^{\Omega} \tau'$ because $\Omega \leq^{\Omega} \tau'$. By (ref), $\tau \leq^{\Omega} \tau$ and by (mon'), $\tau \leq^{\Omega} \tau \cap \tau'$. $\square$

Lemma 3.7. *If $\tau \leq^{\Omega} \tau'$ and $\mathrm{inInter}(\tau_1 \to \tau_2, \tau')$ and $\tau_2 \not\sim^2 \Omega$ then there exist $n \geq 1$ and $\tau'_1, \tau''_1, \dots, \tau'_n, \tau''_n$ such that for all $i \in \{1, \dots, n\}$, $\mathrm{inInter}(\tau'_i \to \tau''_i, \tau)$ and $\tau''_i \not\sim^2 \Omega$ and $\tau''_1 \cap \dots \cap \tau''_n \leq^{\Omega} \tau_2$. Moreover, if $\tau_1 \sim^2 \Omega$ then for all $i \in \{1, \dots, n\}$, $\tau'_i \sim^2 \Omega$.* $\square$

Proof. We prove the lemma by induction on the size derivation of $\tau \leq^{\Omega} \tau'$ and then by case on the last rule of the derivation.

- (ref): $\tau \leq \tau$. Then it is done with $n = 1$, $\tau''_1 = \tau_2$ and $\tau'_1 = \tau_1$.

- (tr): $(\tau_1 \leq^{\Omega} \tau_2 \wedge \tau_2 \leq^{\Omega} \tau_3) \Rightarrow \tau_1 \leq^{\Omega} \tau_3$. Let τ, τ' such that $\mathrm{inInter}(\tau \to \tau', \tau_3)$ and $\tau' \not\sim^2 \Omega$. By IH there exist $n \geq 1$ and $\tau'_1, \tau''_1, \dots, \tau'_n, \tau''_n$ such that for all $i \in \{1, \dots, n\}$, $\mathrm{inInter}(\tau'_i \to \tau''_i, \tau_2)$ and $\tau''_i \not\sim^2 \Omega$ and $\tau''_1 \cap \dots \cap \tau''_n \leq^{\Omega} \tau'$. Again by IH, for all $i \in \{1, \dots, n\}$, there exist $m_i \geq 1$ and $\tau'''_{1,i}, \tau''''_{1,i}, \dots, \tau'''_{m_i,i}, \tau''''_{m_i,i} \in \mathsf{Type}^2$ such that for all $j \in \{1, \dots, m_i\}$, $\mathrm{inInter}(\tau'''_{j,i} \to \tau''''_{j,i}, \tau_1)$ and $\tau''''_{j,i} \not\sim^2 \Omega$ and $\tau''''_1 \cap \dots \cap \tau''''_m \leq^{\Omega} \tau''_i$. Using rule (mon), associativity and commutativity, $\tau''''_{1,1} \cap \dots \cap \tau''''_{m_1,1} \cap \dots \cap \tau''''_{1,n} \cap \dots \cap \tau''''_{m_n,n} \leq^{\Omega} \tau'$.

  Let $\tau \sim^2 \Omega$. Then by IH, for all $i \in \{1, \dots, n\}$, $\tau'_i \sim^2 \Omega$. Again by IH, for all $i \in \{1, \dots, n\}$, for all $j \in \{1, \dots, m_i\}$, $\tau'''_{j,i} \sim^2 \Omega$.

- (in_L): $\tau_1 \cap \tau_2 \leq^{\Omega} \tau_1$. Let τ, τ' such that $\mathrm{inInter}(\tau \to \tau', \tau_1)$ and $\tau' \not\sim^2 \Omega$ then it is done with $n = 1$, $\tau''_1 = \tau'$ and $\tau'_1 = \tau$.

- (in_R): $\tau_1 \cap \tau_2 \leq^{\Omega} \tau_2$. Let τ, τ' such that $\mathrm{inInter}(\tau \to \tau', \tau_2)$ and $\tau' \not\sim^2 \Omega$ then it is done with $n = 1$, $\tau''_1 = \tau'$ and $\tau'_1 = \tau$.

- $(\to\text{-}\cap)$: $(\tau_1 \to \tau_2) \cap (\tau_1 \to \tau_3) \leq^{\Omega} \tau_1 \to (\tau_2 \cap \tau_3)$. Let τ, τ' such that $\mathrm{inInter}(\tau \to \tau', \tau_1 \to (\tau_2 \cap \tau_3))$ and $\tau' \not\sim^2 \Omega$ then $\tau = \tau_1$ and $\tau' = \tau_2 \cap \tau_3$. $\tau_2 \not\sim^2 \Omega$ or $\tau_3 \not\sim^2 \Omega$ because $\tau' \not\sim^2 \Omega$ and using lemma 3.5. If $\tau_2 \not\sim^2 \Omega$ and $\tau_3 \not\sim^2 \Omega$ then it is done with $n = 2$, $\tau'_1 = \tau'_2 = \tau_1$ and $\tau''_1 = \tau_2$ and $\tau''_2 = \tau_3$. If $\tau_2 \not\sim^2 \Omega$ and $\tau_3 \sim^2 \Omega$ then it is done with $n = 1$, $\tau'_1 = \tau_1$ and $\tau''_1 = \tau_2$ because $\tau_2 \leq^{\Omega} \tau_2 \cap \tau_3$ by lemma 3.6. If $\tau_2 \sim^2 \Omega$ and $\tau_3 \not\sim^2 \Omega$ then it is done with $n = 1$, $\tau'_1 = \tau_1$ and $\tau''_1 = \tau_3$ because $\tau_3 \leq^{\Omega} \tau_2 \cap \tau_3$ by lemma 3.6 and commutativity.

- (mon'): $(\tau_1 \leq^\Omega \tau_2 \wedge \tau_1 \leq^\Omega \tau_3) \Rightarrow \tau_1 \leq^\Omega \tau_2 \cap \tau_3$. Let τ, τ' such that inInter$(\tau \rightarrow \tau', \tau_2 \cap \tau_3)$ and $\tau' \not\sim^2 \Omega$. Either inInter$(\tau \rightarrow \tau', \tau_2)$ and we conclude by IH. Or inInter$(\tau \rightarrow \tau', \tau_3)$ and we conclude by IH.

- (mon): $(\tau_1 \leq^\Omega \tau_1' \wedge \tau_2 \leq^\Omega \tau_2') \Rightarrow \tau_1 \cap \tau_2 \leq^\Omega \tau_1' \cap \tau_2'$. Let τ, τ' such that inInter$(\tau \rightarrow \tau', \tau_1' \cap \tau_2')$. Either inInter$(\tau \rightarrow \tau', \tau_1')$ and it is done by IH. Or inInter$(\tau \rightarrow \tau', \tau_2')$ and it is done by IH.

- ($\rightarrow$-η): $(\tau_1 \leq^\Omega \tau_1' \wedge \tau_2' \leq^\Omega \tau_2) \Rightarrow \tau_1' \rightarrow \tau_2' \leq^\Omega \tau_1 \rightarrow \tau_2$. Let τ, τ' such that inInter$(\tau \rightarrow \tau', \tau_1 \rightarrow \tau_2)$ and $\tau' \not\sim^2 \Omega$ then $\tau = \tau_1$ and $\tau' = \tau_2$ and it is done with $n = 1$ and $\tau_1'' = \tau_2'$ because $\tau_2' \not\sim^2 \Omega$ by lemma 3.4 and because if $\tau_1 \sim^2 \Omega$ then $\tau_1' \sim^2 \Omega$.

- (Ω): $\tau_0 \leq^\Omega \Omega$. There is no τ, τ' such that inInter$(\tau \rightarrow \tau', \Omega)$.

- (Ω'-$lazy$): $\tau_0 \rightarrow \Omega \leq^\Omega \Omega \rightarrow \Omega$. there is no $\tau' \not\sim^2 \Omega$ such that inInter$(\tau \rightarrow \tau', \Omega \rightarrow \Omega)$. □

Lemma 3.8. *For all $\tau, \tau' \in \mathsf{Type}^2$, $\alpha \rightarrow \Omega \rightarrow \tau' \not\sim^2 \Omega \rightarrow \tau$* □

Proof. let $\tau' \in \mathsf{Type}^2$.

First we prove that $\Omega \rightarrow \tau' \not\sim^2 \Omega$. Assume $\Omega \rightarrow \tau' \not\sim^2 \Omega$ then $\Omega \leq^\Omega \Omega \rightarrow \tau'$. By lemma 3.3, $\Omega \rightarrow \tau' \in \mathsf{TypeOmega}$ which is false.

Let $\tau \sim^2 \Omega$. Assume $\alpha \rightarrow \Omega \rightarrow \tau' \sim^2 \Omega \rightarrow \tau$ then $\Omega \rightarrow \tau \leq^\Omega \alpha \rightarrow \Omega \rightarrow \tau'$. By lemma 3.7, $\tau \leq^\Omega \Omega \rightarrow \tau'$ which is false.

Let $\tau \not\sim^2 \Omega$. Assume $\alpha \rightarrow \Omega \rightarrow \tau' \sim^2 \Omega \rightarrow \tau$ then $\alpha \rightarrow \Omega \rightarrow \tau' \leq^\Omega \Omega \rightarrow \tau$. By lemma 3.7, $\alpha \sim^2 \Omega$ because $\Omega \sim^2 \Omega$, which is false. □

Lemma 3.9 (Lemma 3.16 of [GL02] is false)**.** *Lemma 3.16 of [GL02] stated below is false:* $\mathrm{VAR}(\mathcal{P}, \mathcal{P}) \Rightarrow \forall \tau \in \mathsf{Type}^2.\ (\forall \tau' \in \mathsf{Type}^2.\ (\tau \not\sim^2 \Omega \rightarrow \tau') \Rightarrow \mathrm{VAR}(\mathcal{P}, [\![\tau]\!]^2_{\mathcal{P}}))$. □

Proof. To show that the above statement is false, we give the following counterexample. Note that $\mathrm{VAR}(\mathcal{P}, [\![\tau]\!]^2_{\mathcal{P}}) \Rightarrow \mathcal{V} \subseteq [\![\tau]\!]^2_{\mathcal{P}}$. Let $x \in \mathcal{V}$, τ be $\alpha \rightarrow \Omega \rightarrow \alpha$ and $\mathcal{P}$ be WN_β. By lemma 3.8, for all $\tau' \in \mathsf{Type}^2$, $\tau \not\sim^2 \Omega \rightarrow \tau'$ and $\mathrm{VAR}(\mathcal{P}, \mathcal{P})$ is true. Assume $\mathrm{VAR}(\mathcal{P}, [\![\tau]\!]^2_{\mathcal{P}})$, then $x \in [\![\tau]\!]^2_{\mathcal{P}}$. Then $x \in [\![\alpha \rightarrow \Omega \rightarrow \alpha]\!]^2_{\mathcal{P}} = [\![\tau]\!]^2_{\mathcal{P}}$ because $x \in \mathcal{P} = [\![\alpha]\!]^2_{\mathcal{P}}$, and $xx(\circledast\circledast) \in [\![\alpha]\!]^2_{\mathcal{P}} = \mathcal{P}$ because $\circledast\circledast \in \Lambda = [\![\Omega]\!]^2_{\mathcal{P}}$. But $xx(\circledast\circledast) \in \mathcal{P}$ is false, so $\mathrm{VAR}(\mathcal{P}, [\![\tau]\!]^2_{\mathcal{P}})$ is false. □ □

The proof for Lemma 3.18 of [GL02] does not work (because of a misused of an induction hypothesis) but we have not yet proved or disproved that lemma:

Remark 3.10 (It is not clear that Lemma 3.18 of [GL02] holds). It is not clear whether this lemma of [GL02] holds: $\mathrm{SAT}(\mathcal{P}, \mathcal{P}) \Rightarrow \forall \tau \in \mathsf{Type}^2.\ (\forall \tau' \in \mathsf{Type}^2.\ (\tau \not\sim^2 \Omega \rightarrow \tau') \Rightarrow \mathrm{SAT}(\mathcal{P}, [\![\tau]\!]^2_{\mathcal{P}}))$.

The proof given in [GL02] does not go through and we have neither been able to prove nor disprove this lemma. It remains that this lemma is not yet proved and hence cannot be used in further proofs. □

Then, Ghilezan and Likavec give a proposition (Proposition 3.21) which is the reducibility method for typable terms. However, the proof of that proposition depends on two problematic lemmas (lemma 3.16 which we showed to fail in our lemma 3.9, and lemma 3.18 which according to remark 3.10 has not been proved).

First, here is a lemma:

Lemma 3.11. $\mathrm{VAR}(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$*,* $\mathrm{CLO}(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$*,* $\mathrm{INV}(\mathsf{WN}_\beta)$ *and* $\mathrm{SAT}(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$ *hold.* □

Proof.

- $\mathrm{VAR}(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$ holds because $\forall x \in \mathcal{V}$, $\forall n \geq 0$, $\forall N_1, \dots, N_n \in \mathsf{WN}_\beta$, $xN_1 \dots N_n \in \mathsf{WN}_\beta$.
- $\mathrm{CLO}(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$ holds, because if $\exists n, m \geq 0$, $\exists x_0 \in \mathcal{V}$, $\exists N_1, \dots, N_m \in \mathsf{NF}_\beta$ such that $M \to_\beta^* \lambda x_1. \dots \lambda x_n.x_0 N_1 \dots N_m$ then $\forall y \in \mathcal{V}$, $\lambda y.M \to_\beta^* \lambda y.\lambda x_1. \dots \lambda x_n.x_0 N_1 \dots N_m \in \mathsf{NF}_\beta$.

  $\mathrm{INV}(\mathsf{WN}_\beta)$ holds, because if $\exists n, m \geq 0$, $\exists x_0 \in \mathcal{V}$, $\exists N_1, \dots, N_m \in \mathsf{NF}_\beta$ such that $\lambda x.M \to_\beta^* \lambda x_1. \dots \lambda x_n.x_0 N_1 \dots N_m$ then $x_1 = y$ and $M \to_\beta^* \lambda x_2. \dots \lambda x_n.x_0 N_1 \dots N_m$.
- $\mathrm{SAT}(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$ holds, since if $M[x := N]N_1 \dots N_n \in \mathsf{WN}_\beta$ where $n \geq 0$ and $N_1, \dots, N_n \in \mathsf{WN}_\beta$ then $\exists P \in \mathsf{NF}_\beta$ such that
  $M[x := N]N_1 \dots N_n \to_\beta^* P$.
  Hence, $(\lambda x.M)NN_1 \dots N_n \to_\beta M[x := N]N_1 \dots N_n \to_\beta^* P$. □

Lemma 3.12 (Proposition 3.21 of [GL02] fails)**.** *Assume* $\mathrm{VAR}(\mathcal{P}, \mathcal{P})$*,* $\mathrm{SAT}(\mathcal{P}, \mathcal{P})$ *and* $\mathrm{CLO}(\mathcal{P}, \mathcal{P})$*. It is* **not** *the case that:* $\forall \tau \in \mathsf{Type}^2.\ (\tau \not\sim^2 \Omega \wedge \forall \tau' \in \mathsf{Type}^2.\ (\tau \not\sim^2 \Omega \to \tau') \wedge \Gamma \vdash^2 M : \tau \Rightarrow M \in \mathcal{P})$. □

Proof. Let $\mathcal{P}$ be WN_β. Note that $\lambda y.\lambda z.\circledast\circledast \notin \mathsf{WN}_\beta$ and $\varnothing \vdash^2 \lambda y.\lambda z.\circledast\circledast : \alpha \to \Omega \to \Omega$ is derivable, where $\alpha \to \Omega \to \Omega \not\sim^2 \Omega$ and by lemma 3.8, $\alpha \to \Omega \to \Omega \not\sim^2 \Omega \to \tau'$, for all $\tau' \in \mathsf{Type}^2$. Since $\mathrm{VAR}(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$, $\mathrm{CLO}(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$ and $\mathrm{SAT}(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$ hold, we get a counterexample for Proposition 3.21 of [GL02]. □ □

Finally, also Ghilezan and Likavec's proof method for untyped terms fails.

Lemma 3.13 (Proposition 3.23 of [GL02] fails)**.** *Proposition 3.23 of [GL02] which states that "If* $\mathcal{P} \subseteq \Lambda$ *is invariant under abstraction (i.e.,* $\mathrm{INV}(\mathcal{P})$*),* $\mathrm{VAR}(\mathcal{P}, \mathcal{P})$ *and* $\mathrm{SAT}(\mathcal{P}, \mathcal{P})$ *then* $\mathcal{P} = \Lambda$*" fails.* □

Proof. The proof given in [GL02] depends on Proposition 3.21 which fails. As $\mathrm{VAR}(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$, $\mathrm{SAT}(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$ and $\mathrm{INV}(\mathsf{WN}_\beta)$, we get a counterexample for Proposition 3.23. □ □

4 How much of the reducibility method of [GL02] can we salvage ?

Because we proved that the Proposition 3.23 of [GL02] is false, we know that the given set of properties ($\mathrm{INV}(\mathcal{P})$, $\mathrm{VAR}(\mathcal{P},\mathcal{P})$ and $\mathrm{SAT}(\mathcal{P},\mathcal{P})$) that a set of terms $\mathcal{P}$ has to fulfil to be equal to the set of terms of the untyped λ-calculus is not the right one. So even if one works on the soundness result or on the type interpretation (the set of realisers), to obtain the same result as the one claimed by Ghilezan and Likavec, one should come up with a new set of properties.

Proposition 3.23 of [GL02] states a set of properties characterising the set of terms of the untyped λ-calculus. The predicate $\mathrm{VAR}(\Lambda,\Lambda)$ states that the variables (and the terms of the form $xNM_1\cdots M_n$) belong to the untyped λ-calculus. The predicate $\mathrm{INV}(\Lambda)$ states among other things that if a term is a λ-term then the abstraction of a variable over this term is a λ-term too. To get a full characterisation of the set of terms of the untyped λ-calculus, we need a predicate, let us call it $\mathrm{APP}(\mathcal{P})$, stating that $(\lambda x.M)NM_1\cdots M_n \in \mathcal{P}$ if $M, N, M_1, \ldots, M_n \in \mathcal{P}$, to be true. Is this predicate true if $\mathrm{VAR}(\mathcal{P},\mathcal{P})$, $\mathrm{SAT}(\mathcal{P},\mathcal{P})$ and $\mathrm{INV}(\mathcal{P})$ are true? No, because we saw that we can find a set of terms (WN_β) which satisfies these properties but is not equal to the λ-calculus. For example, we cannot get the non strongly normalisable terms to be in WN_β. So, these properties are not enough to characterise the λ-calculus.

The problem with these properties is that if one tries to salvage Ghilezan and Likavec's reducibility method, the properties $\mathrm{VAR}(\mathcal{P},\mathcal{P})$ and $\mathrm{CLO}(\mathcal{P},\mathcal{P})$ are going to impose a restriction on the arrow types for which the interpretation is in $\mathcal{P}$ (the realisers of arrow types), as we can see in the arrow type case of the proof of the following lemma 4.4.5 and in the arrow type case of the proof of the following lemma 4.5. As shown at the end of this section, even if the obtained result when considering these restrictions is different from (in some sens, is an improvement of) the one given by Ghilezan and Likavec using the type system $\lambda\cap^1$, we do not succeed in salvaging their method.

The use of the non-trivial types (we recall the definition below) introduced by Gallier [Gal03] are not much of a help in this case, because of the precise restriction imposed by $\mathrm{VAR}(\mathcal{P},\mathcal{P})$. One might also want to consider the sets of properties (we do not recall them in this paper for lack of space) stated in his work [Gal03], but which are unfortunately not easy to prove for CR, because a proof of $xM \in \mathsf{CR}$ for all $M \in \Lambda$ is required. Moreover, if one succeeds in proving that the variables are included in the interpretation of a defined set of types containing $\Omega \to \alpha$, where Ω is interpreted as Λ and α as $\mathcal{P}$, then one has proved that $xM \in \mathcal{P}$, so that in the case $\mathcal{P} = \mathsf{CR}$, $M \in \mathsf{CR}$.

It is worth pointing out that a part of the work done by Gallier [Gal03] would still be valid if adapted to the type system $\lambda\cap^2$. Gallier defines the non-trivial types as follows:

$$\psi \in \mathsf{NonTrivial} ::= \alpha \mid \tau \to \psi \mid \tau \cap \psi \mid \psi \cap \tau$$

Types in Type^2 are then interpreted as follows: $[\![\alpha]\!]_{\mathcal{P}} = \mathcal{P}$, $[\![\psi\cap\tau]\!]_{\mathcal{P}} = [\![\tau\cap\psi]\!]_{\mathcal{P}} =$

$[\![\tau]\!]_{\mathcal{P}} \cap [\![\psi]\!]_{\mathcal{P}}$, $[\![\tau]\!]_{\mathcal{P}} = \Lambda$ if $\tau \notin \mathsf{NonTrivial}$ and $[\![\tau \to \psi]\!]_{\mathcal{P}} = \{M \in \mathcal{P} \mid \forall N \in [\![\tau]\!]_{\mathcal{P}}.\ MN \in [\![\psi]\!]_{\mathcal{P}}\}$. We can easily prove that if $\tau_1 \leq^2 \tau_2$ then $[\![\tau_1]\!]_{\mathcal{P}} \subseteq [\![\tau_2]\!]_{\mathcal{P}}$. Hence, considering the type system $\lambda\cap^2$ instead of the type system $D\Omega$, the method of Gallier gets a set of predicates which when satisfied by a set of terms $\mathcal{P}$ implies that the set of terms typable in the system $\lambda\cap^2$ by a non-trivial type is a subset of $\mathcal{P}$. Gallier proved that the set of head-normalising λ-terms satisfies each of the given predicates.

Using a method similar to Ghilezan and Likavec's method, Gallier proved also that the set of weakly head-normalising terms (W) is equal to the set of terms typable by a weakly non-trivial types in the type system $D\Omega$. The set of weakly non-trivial types is defined as follows:

$$\psi \in \mathsf{WeaklyNonTrivial} ::= \alpha \mid \tau \to \psi \mid \Omega \to \Omega \mid \tau \cap \psi \mid \psi \cap \tau$$

As explain above, we can try and salvage Ghilezan and Likavec's method by first restricting the set of realisers when defining the interpretation of the set of types in Type^2. The different restrictions lead us to the definition of Type^3 and the following type interpretation:

Definition 4.1. $\rho \in \mathsf{Type}^3 ::= \alpha \mid \tau \to \rho \mid \rho \cap \tau \mid \tau \cap \rho$.

- $[\![\alpha]\!]^3_{\mathcal{P}} = \mathcal{P}$.
- $[\![\tau_1 \cap \tau_2]\!]^3_{\mathcal{P}} = [\![\tau_1]\!]^3_{\mathcal{P}} \cap [\![\tau_2]\!]^3_{\mathcal{P}}$, if $\tau_1 \cap \tau_2 \in \mathsf{Type}^3$.
- $[\![\tau]\!]^3_{\mathcal{P}} = \Lambda$, if $\tau \notin \mathsf{Type}^3$.
- $[\![\tau_1 \to \tau_2]\!]^3_{\mathcal{P}} = \{M \in \mathcal{P} \mid \forall N \in [\![\tau_1]\!]^3_{\mathcal{P}}.\ MN \in [\![\tau_2]\!]^3_{\mathcal{P}}\}$, if $\tau_1 \to \tau_2 \in \mathsf{Type}^3$. □

In order to prove the relation between the stronger induction hypotheses (VAR, SAT and CLO, and particularly the variable one) and the ones depending on type interpretations (VAR^2, SAT^2 and CLO^2), and in order to be able to use these stronger induction hypotheses in the soundness lemma, we have to impose other restrictions.

Definition 4.2. We let $\varphi \in \mathsf{Type}^4 ::= \alpha \mid \Omega \mid \rho \to \varphi \mid \varphi \cap \tau \mid \tau \cap \varphi$.

We let $\Gamma \in \mathcal{B}^3 = \{\{x_1 : \varphi_1, \ldots, x_n : \varphi_n\} \mid \forall i, j \in \{1, \ldots, n\}.\ x_i = x_j \Rightarrow \varphi_i = \varphi_j\}$

Let $\vdash^3$ be the relation $\vdash^2$ where (ax) is replaced by (ax') and $\mathcal{B}^2$ is replaced by $\mathcal{B}^3$. Let $\lambda\cap^3$ be the type system $\lambda\cap^2$ where (ax) is replaced by (ax') and $\mathcal{B}^2$ is replaced by $\mathcal{B}^3$. Let $\models^3_{\mathcal{P}}$ be the relation $\models^2_{\mathcal{P}}$ where $[\![\tau]\!]^2_{\mathcal{P}}$ is replaced by $[\![\tau]\!]^3_{\mathcal{P}}$. □

Due to the saturation predicates and its uses, we could have to impose some other restrictions on the type system. Another alternative is to slightly modify this predicate (in order to not have to burden ourselves with another notation for the saturation predicate, we call it as the previous one):

Definition 4.3. $\mathrm{SAT}(\mathcal{P}, \mathcal{X}) \iff (\forall M, N \in \Lambda.\, \forall x \in \mathcal{V}.\, \forall n \in \mathbb{N}.\, \forall N_1, \dots, N_n \in \Lambda.$
$M[x := N]N_1 \dots N_n \in \mathcal{X} \Rightarrow (\lambda x.M)NN_1 \dots N_n \in \mathcal{X})$. □

We can prove that if $\mathcal{P} \in \{\mathsf{CR}, \mathsf{S}, \mathsf{W}\}$ then $\mathrm{SAT}(\mathcal{P}, \mathcal{P})$ holds.

Lemma 4.4.

1. $[\![\tau_1 \cap \tau_2]\!]^3_{\mathcal{P}} = [\![\tau_1]\!]^3_{\mathcal{P}} \cap [\![\tau_2]\!]^3_{\mathcal{P}}$.
2. $[\![\rho]\!]^3_{\mathcal{P}} \subseteq \mathcal{P}$.
3. *If* $\tau_1 \leq^3 \tau_2$ *and* $\tau_2 \in \mathit{Type}^3$ *then* $\tau_1 \in \mathit{Type}^3$.
4. *If* $\tau_1 \leq^2 \tau_2$ *then* $[\![\tau_1]\!]^3_{\mathcal{P}} \subseteq [\![\tau_2]\!]^3_{\mathcal{P}}$.
5. *If* $\mathrm{VAR}(\mathcal{P}, \mathcal{P})$ *then for all* $\varphi \in \mathit{Type}^4$, $\mathrm{VAR}(\mathcal{P}, [\![\varphi]\!]^3_{\mathcal{P}})$.
6. *If* $\mathrm{SAT}(\mathcal{P}, \mathcal{P})$ *then for all* $\tau \in \mathsf{Type}^2$, $\mathrm{SAT}(\mathcal{P}, [\![\tau]\!]^3_{\mathcal{P}})$. □

Proof.

1. If $\tau_1 \cap \tau_2 \in \mathsf{Type}^3$ then it is done by definition. Otherwise $\tau_1, \tau_2 \not\in \mathsf{Type}^3$, so $[\![\tau_1 \cap \tau_2]\!]^3_{\mathcal{P}} = \Lambda = \Lambda \cap \Lambda = [\![\tau_1]\!]^3_{\mathcal{P}} \cap [\![\tau_2]\!]^3_{\mathcal{P}}$.
2. We prove this result by induction on the structure of τ.
   - Let $\rho = \alpha$ then $[\![\rho]\!]^3_{\mathcal{P}} = \mathcal{P}$.
   - Let $\rho = \tau \to \rho'$, then by definition, $[\![\rho]\!]^3_{\mathcal{P}} \subseteq \mathcal{P}$.
   - Let $\rho = \tau \cap \rho'$, then by IH, $[\![\rho']\!]^3_{\mathcal{P}} \subseteq \mathcal{P}$. So $[\![\rho]\!]^3_{\mathcal{P}} = [\![\tau]\!]^3_{\mathcal{P}} \cap [\![\rho']\!]^3_{\mathcal{P}} \subseteq \mathcal{P}$.
   - Let $\rho = \rho' \cap \tau$, then by IH, $[\![\rho']\!]^3_{\mathcal{P}} \subseteq \mathcal{P}$. So $[\![\rho]\!]^3_{\mathcal{P}} = [\![\tau]\!]^3_{\mathcal{P}} \cap [\![\rho']\!]^3_{\mathcal{P}} \subseteq \mathcal{P}$.
3. We prove this lemma by induction on the size of the derivation of $\tau_1 \leq^2 \tau_2$ and then by case on the last step.
   - (ref): $\tau \leq \tau$. This case is trivial.
   - (Ω): $\tau \leq \Omega$. This case is trivial since $\Omega \not\in \mathsf{Type}^3$.
   - (tr): $\tau_1 \leq \tau_2 \wedge \tau_2 \leq \tau_3 \Rightarrow \tau_1 \leq \tau_3$. We conclude using IH twice.
   - $(\Omega'\text{-}lazy)$: $\tau \to \Omega \leq \Omega \to \Omega$. This case is trivial since $\Omega \to \Omega \not\in \mathsf{Type}^3$.
   - (in_L): $\tau_1 \cap \tau_2 \leq \tau_1$. This case is trivial.
   - (in_R): $\tau_1 \cap \tau_2 \leq \tau_2$. This case is trivial.
   - $(\to\text{-}\cap)$: $(\tau_1 \to \tau_2) \cap (\tau_1 \to \tau_3) \leq \tau_1 \to (\tau_2 \cap \tau_3)$. if $\tau_1 \to (\tau_2 \cap \tau_3) \in \mathsf{Type}^3$ then $\tau_2 \in \mathsf{Type}^3$ or $\tau_3 \in \mathsf{Type}^3$. Hence $\tau_1 \to \tau_2 \in \mathsf{Type}^3$ or $\tau_1 \to \tau_3 \in \mathsf{Type}^3$, so $(\tau_1 \to \tau_2) \cap (\tau_1 \to \tau_3) \in \mathsf{Type}^3$.
   - (mon'): $\tau_1 \leq \tau_2 \wedge \tau_1 \leq \tau_3 \Rightarrow \tau_1 \leq \tau_2 \cap \tau_3$. If $\tau_2 \cap \tau_3 \in \mathsf{Type}^3$ then $\tau_2 \in \mathsf{Type}^3$ or $\tau_3 \in \mathsf{Type}^3$, so by IH, $\tau_1 \in \mathsf{Type}^3$.

- (mon): $\tau_1 \leq \tau_1' \wedge \tau_2 \leq \tau_2' \Rightarrow \tau_1 \cap \tau_2 \leq \tau_1' \cap \tau_2'$. If $\tau_1' \cap \tau_2' \in \mathsf{Type}^3$ then $\tau_1' \in \mathsf{Type}^3$ or $\tau_2' \in \mathsf{Type}^3$. So by IH, $\tau_1 \in \mathsf{Type}^3$ or $\tau_2 \in \mathsf{Type}^3$, hence $\tau_1 \cap \tau_2 \in \mathsf{Type}^3$.
- $(\rightarrow\text{-}\eta)$: $\tau_1 \leq \tau_1' \wedge \tau_2' \leq \tau_2 \Rightarrow \tau_1' \rightarrow \tau_2' \leq \tau_1 \rightarrow \tau_2$. If $\tau_1 \rightarrow \tau_2 \in \mathsf{Type}^3$ then $\tau_2 \in \mathsf{Type}^3$, so by IH, $\tau_2' \in \mathsf{Type}^3$, hence $\tau_1' \rightarrow \tau_2' \in \mathsf{Type}^3$.

4. We prove this lemma by induction on the size of the derivation of $\tau_1 \leq^2 \tau_2$ and then by case on the last step.

   - (ref): $\tau \leq \tau$. This case is trivial.
   - (Ω): $\tau \leq \Omega$. This case is trivial since $[\![\Omega]\!]^3_{\mathcal{P}} = \Lambda$.
   - (tr): $\tau_1 \leq \tau_2 \wedge \tau_2 \leq \tau_3 \Rightarrow \tau_1 \leq \tau_3$. By IH, $[\![\tau_1]\!]^3_{\mathcal{P}} \subseteq [\![\tau_2]\!]^3_{\mathcal{P}}$ and $[\![\tau_2]\!]^3_{\mathcal{P}} \subseteq [\![\tau_3]\!]^3_{\mathcal{P}}$, so $[\![\tau_1]\!]^3_{\mathcal{P}} \subseteq [\![\tau_3]\!]^3_{\mathcal{P}}$.
   - $(\Omega'\text{-}lazy)$: $\tau \rightarrow \Omega \leq \Omega \rightarrow \Omega$. This case is trivial since $[\![\tau \rightarrow \Omega]\!]^3_{\mathcal{P}} = [\![\Omega \rightarrow \Omega]\!]^3_{\mathcal{P}} = \Lambda$.
   - (in_L): $\tau_1 \cap \tau_2 \leq \tau_1$. By 1, $[\![\tau_1 \cap \tau_2]\!]^3_{\mathcal{P}} = [\![\tau_1]\!]^3_{\mathcal{P}} \cap [\![\tau_2]\!]^3_{\mathcal{P}} \subseteq [\![\tau_1]\!]^3_{\mathcal{P}}$.
   - (in_R): $\tau_1 \cap \tau_2 \leq \tau_2$. By 1, $[\![\tau_1 \cap \tau_2]\!]^3_{\mathcal{P}} = [\![\tau_1]\!]^3 \cap [\![\tau_2]\!]^3_{\mathcal{P}} \subseteq [\![\tau_2]\!]^3_{\mathcal{P}}$.
   - $(\rightarrow\text{-}\cap)$: $(\tau_1 \rightarrow \tau_2) \cap (\tau_1 \rightarrow \tau_3) \leq \tau_1 \rightarrow (\tau_2 \cap \tau_3)$.
     - If $\tau_1 \rightarrow \tau_2, \tau_1 \rightarrow \tau_3 \in \mathsf{Type}^3$ then $\tau_2, \tau_3, \tau_2 \cap \tau_3 \in \mathsf{Type}^3$, so $[\![(\tau_1 \rightarrow \tau_2) \cap (\tau_1 \rightarrow \tau_3)]\!]^3_{\mathcal{P}} = [\![\tau_1 \rightarrow \tau_2]\!]^3_{\mathcal{P}} \cap [\![\tau_1 \rightarrow \tau_3]\!]^3_{\mathcal{P}} = \{M \in \mathcal{P} \mid \forall N \in [\![\tau_1]\!]^3_{\mathcal{P}}.\ MN \in [\![\tau_2]\!]^3_{\mathcal{P}}\} \cap \{M \in \mathcal{P} \mid \forall N \in [\![\tau_1]\!]^3_{\mathcal{P}}.\ MN \in [\![\tau_3]\!]^3_{\mathcal{P}}\} = \{M \in \mathcal{P} \mid \forall N \in [\![\tau_1]\!]^3_{\mathcal{P}}.\ MN \in [\![\tau_2]\!]^3_{\mathcal{P}} \cap [\![\tau_3]\!]^3_{\mathcal{P}}\} = \{M \in \mathcal{P} \mid \forall N \in [\![\tau]\!]^3_{\mathcal{P}}.\ MN \in [\![\tau_2 \cap \tau_3]\!]^3_{\mathcal{P}}\} = [\![\tau_1 \rightarrow (\tau_2 \cap \tau_3)]\!]^3_{\mathcal{P}}$.
     - If $\tau_1 \rightarrow \tau_2 \in \mathsf{Type}^3$ and $\tau_1 \rightarrow \tau_3 \notin \mathsf{Type}^3$, then $\tau_2, \tau_2 \cap \tau_3 \in \mathsf{Type}^3$ and $\tau_3 \notin \mathsf{Type}^3$, so $[\![(\tau_1 \rightarrow \tau_2) \cap (\tau_1 \rightarrow \tau_3)]\!]^3_{\mathcal{P}} = [\![\tau_1 \rightarrow \tau_2]\!]^3_{\mathcal{P}} \cap [\![\tau_1 \rightarrow \tau_3]\!]^3_{\mathcal{P}} = \{M \in \mathcal{P} \mid \forall N \in [\![\tau_1]\!]^3_{\mathcal{P}}.\ MN \in [\![\tau_2]\!]^3_{\mathcal{P}}\} = \{M \in \mathcal{P} \mid \forall N \in [\![\tau_1]\!]^3_{\mathcal{P}}.\ MN \in [\![\tau_2 \cap \tau_3]\!]^3_{\mathcal{P}}\} = [\![\tau_1 \rightarrow (\tau_2 \cap \tau_3)]\!]^3_{\mathcal{P}}$.
     - If $\tau_1 \rightarrow \tau_2 \notin \mathsf{Type}^3$ and $\tau_1 \rightarrow \tau_3 \in \mathsf{Type}^3$, then $\tau_3, \tau_2 \cap \tau_3 \in \mathsf{Type}^3$ and $\tau_2 \notin \mathsf{Type}^3$, so $[\![(\tau_1 \rightarrow \tau_2) \cap (\tau_1 \rightarrow \tau_3)]\!]^3_{\mathcal{P}} = [\![\tau_1 \rightarrow \tau_2]\!]^3_{\mathcal{P}} \cap [\![\tau_1 \rightarrow \tau_3]\!]^3_{\mathcal{P}} = \{M \in \mathcal{P} \mid \forall N \in [\![\tau_1]\!]^3_{\mathcal{P}}.\ MN \in [\![\tau_3]\!]^3_{\mathcal{P}}\} = \{M \in \mathcal{P} \mid \forall N \in [\![\tau_1]\!]^3_{\mathcal{P}}.\ MN \in [\![\tau_2 \cap \tau_3]\!]^3_{\mathcal{P}}\} = [\![\tau_1 \rightarrow (\tau_2 \cap \tau_3)]\!]^3_{\mathcal{P}}$.
     - If $\tau_1 \rightarrow \tau_2, \tau_1 \rightarrow \tau_3 \notin \mathsf{Type}^3$, then $\tau_2, \tau_3, \tau_2 \cap \tau_3 \notin \mathsf{Type}^3$, so $[\![(\tau_1 \rightarrow \tau_2) \cap (\tau_1 \rightarrow \tau_3)]\!]^3_{\mathcal{P}} = [\![\tau_1 \rightarrow (\tau_2 \cap \tau_3)]\!]^3_{\mathcal{P}} = \Lambda$.
   - (mon'): $\tau_1 \leq \tau_2 \wedge \tau_1 \leq \tau_3 \Rightarrow \tau_1 \leq \tau_2 \cap \tau_3$. By IH, $[\![\tau_1]\!]^3_{\mathcal{P}} \subseteq [\![\tau_2]\!]^3_{\mathcal{P}}$ and $[\![\tau_1]\!]^3_{\mathcal{P}} \subseteq [\![\tau_3]\!]^3_{\mathcal{P}}$. So by 1, $[\![\tau_1]\!]^3_{\mathcal{P}} \subseteq [\![\tau_2]\!]^3_{\mathcal{P}} \cap [\![\tau_3]\!]^3_{\mathcal{P}} = [\![\tau_2 \cap \tau_3]\!]^3_{\mathcal{P}}$.
   - (mon): $\tau_1 \leq \tau_1' \wedge \tau_2 \leq \tau_2' \Rightarrow \tau_1 \cap \tau_2 \leq \tau_1' \cap \tau_2'$. By IH, $[\![\tau_1]\!]^3_{\mathcal{P}} \subseteq [\![\tau_1']\!]^3_{\mathcal{P}}$ and $[\![\tau_2]\!]^3_{\mathcal{P}} \subseteq [\![\tau_2']\!]^3_{\mathcal{P}}$. So by 1, $[\![\tau_1 \cap \tau_2]\!]^3_{\mathcal{P}} = [\![\tau_1]\!]^3_{\mathcal{P}} \cap [\![\tau_2]\!]^3_{\mathcal{P}} \subseteq [\![\tau_1']\!]^3_{\mathcal{P}} \cap [\![\tau_2']\!]^3_{\mathcal{P}} = [\![\tau_1' \cap \tau_2']\!]^3_{\mathcal{P}}$.
   - $(\rightarrow\text{-}\eta)$: $\tau_1 \leq \tau_1' \wedge \tau_2' \leq \tau_2 \Rightarrow \tau_1' \rightarrow \tau_2' \leq \tau_1 \rightarrow \tau_2$. By IH, $[\![\tau_1]\!]^3_{\mathcal{P}} \subseteq [\![\tau_1']\!]^3_{\mathcal{P}}$ and $[\![\tau_2']\!]^3_{\mathcal{P}} \subseteq [\![\tau_2]\!]^3_{\mathcal{P}}$. If $\tau_1 \rightarrow \tau_2 \in \mathsf{Type}^3$ then $\tau_2 \in \mathsf{Type}^3$ and by 3, $\tau_2' \in \mathsf{Type}^3$, so $\tau_1' \rightarrow \tau_2' \in \mathsf{Type}^3$ and $[\![\tau_1' \rightarrow \tau_2']\!]^3_{\mathcal{P}} = \{M \in \mathcal{P} \mid \forall N \in [\![\tau_1']\!]^3_{\mathcal{P}}.\ MN \in [\![\tau_2']\!]^3_{\mathcal{P}}\} \subseteq \{M \in \mathcal{P} \mid \forall N \in [\![\tau_1]\!]^3_{\mathcal{P}}.\ MN \in [\![\tau_2]\!]^3_{\mathcal{P}}\} = [\![\tau_1 \rightarrow \tau_2]\!]^3_{\mathcal{P}}$. Otherwise, $[\![\tau_1' \rightarrow \tau_2']\!]^3_{\mathcal{P}} \subseteq [\![\tau_1 \rightarrow \tau_2]\!]^3_{\mathcal{P}} = \Lambda$.

5. Assume $\mathrm{VAR}(\mathcal{P}, \mathcal{P})$. Let $n \geq 0$, $x \in \mathcal{V}$ and for all $i \in \{1, \ldots, n\}, M_i \in \mathcal{P}$. By the hypothesis, $xM_1 \cdots M_n \in \mathcal{P}$. We prove that $xM_1 \cdots M_n \in [\![\varphi]\!]^3_{\mathcal{P}}$ by induction on the structure of φ.
   - If $\varphi = \alpha$ then $xM_1 \cdots M_n \in \mathcal{P} = [\![\alpha]\!]^3_{\mathcal{P}}$.
   - If $\varphi = \Omega$ then $xM_1 \cdots M_n \in \Lambda = [\![\Omega]\!]^3_{\mathcal{P}}$.
   - If $\varphi = \tau \cap \varphi'$. By IH, $xM_1 \cdots M_n \in [\![\varphi']\!]^3_{\mathcal{P}}$, so by 1, $xM_1 \cdots M_n \in [\![\tau]\!]^3_{\mathcal{P}} \cap [\![\varphi']\!]^3_{\mathcal{P}} = [\![\tau \cap \varphi']\!]^3_{\mathcal{P}}$.
   - If $\varphi = \varphi' \cap \tau$. By IH, $xM_1 \cdots M_n \in [\![\varphi']\!]^3_{\mathcal{P}}$, so by 1, $xM_1 \cdots M_n \in [\![\varphi']\!]^3_{\mathcal{P}} \cap [\![\tau]\!]^3_{\mathcal{P}} = [\![\varphi' \cap \tau]\!]^3_{\mathcal{P}}$.
   - If $\varphi = \rho \to \varphi'$.
     - If $\varphi \in \mathsf{Type}^3$ then $\varphi' \in \mathsf{Type}^3$. Let $N \in [\![\rho]\!]^3_{\mathcal{P}}$, so by 2, $N \in \mathcal{P}$. By IH, $xM_1 \cdots M_n N \in [\![\varphi']\!]^3_{\mathcal{P}}$. So $xM_1 \cdots M_n \in [\![\rho \to \varphi']\!]^3_{\mathcal{P}}$.
     - If $\varphi \notin \mathsf{Type}^3$ then $xM_1 \cdots M_n \in [\![\rho \to \varphi']\!]^3_{\mathcal{P}} = \Lambda$.

6. Assume $\mathrm{SAT}(\mathcal{P}, \mathcal{P})$. Let $n \geq 0$, $x \in \mathcal{V}$, $M, N \in \Lambda$ and for all $i \in \{1, \ldots, n\}, N_i \in \Lambda$. We prove that if $M[x := N]N_1 \cdots N_n \in [\![\tau]\!]^3_{\mathcal{P}}$ then $(\lambda x.M)NN_1 \cdots N_n \in [\![\tau]\!]^3_{\mathcal{P}}$ by induction on the structure of τ.
   - If $\tau = \alpha$ then $[\![\alpha]\!]^3_{\mathcal{P}} = \mathcal{P}$ and we conclude using the hypothesis $\mathrm{SAT}(\mathcal{P}, \mathcal{P})$.
   - If $\tau = \Omega$ then $(\lambda x.M)NN_1 \cdots N_n \in \Lambda = [\![\Omega]\!]^3_{\mathcal{P}}$.
   - If $\tau = \tau_1 \cap \tau_2$. Assume $M[x := N]N_1 \cdots N_n \in [\![\tau]\!]^3_{\mathcal{P}} =^1 [\![\tau_1]\!]^3 \cap [\![\tau_2]\!]^3$, then by IH, $(\lambda x.M)NN_1 \cdots N_n \in [\![\tau_1]\!]^3 \cap [\![\tau_2]\!]^3 =^1 [\![\tau]\!]^3$.
   - If $\tau = \tau_1 \to \tau_2$.
     - If $\tau \in \mathsf{Type}^3$ then $\tau_2 \in \mathsf{Type}^3$. Let $P \in [\![\tau_1]\!]^3_{\mathcal{P}}$ and $M[x := N]N_1 \cdots N_n \in [\![\tau]\!]^3_{\mathcal{P}}$ then by 2, $M[x := N]N_1 \cdots N_n \in \mathcal{P}$. By hypothesis, $(\lambda x.M)NN_1 \cdots N_n \in \mathcal{P}$. Moreover, $M[x := N]N_1 \cdots N_n P \in [\![\tau_2]\!]^3_{\mathcal{P}}$. By IH, $(\lambda x.M)NN_1 \cdots N_n P \in [\![\tau_2]\!]^3_{\mathcal{P}}$, so $(\lambda x.M)NN_1 \cdots N_n \in [\![\tau]\!]^3_{\mathcal{P}}$.
     - Let $\tau \notin \mathsf{Type}^3$ then $(\lambda x.M)NN_1 \cdots N_n \in [\![\tau]\!]^3_{\mathcal{P}} = \Lambda$.

□

Lemma 4.5. *If* $\mathrm{VAR}(\mathcal{P}, \mathcal{P})$*,* $\mathrm{SAT}(\mathcal{P}, \mathcal{P})$*,* $\mathrm{CLO}(\mathcal{P}, \mathcal{P})$ *and* $\Gamma \vdash^3 M : \tau$ *then* $\Gamma \models^3_{\mathcal{P}} M : \tau$ □

Proof. We prove this lemma by induction on the size of the derivation of $\Gamma \vdash^3 M : \tau$ and then by case on the last rule used in the derivation. In each case, if $\tau \notin \mathsf{Type}^3$, it is trivial since $[\![\tau]\!]^3_{\mathcal{P}} = \Lambda$. So let us consider in each case that $\tau \in \mathsf{Type}^3$.

- (ax): Let $\nu \models^3_{\mathcal{P}} \Gamma, x : \varphi$ then $\nu(x) \in [\![\varphi]\!]^3_{\mathcal{P}}$.
- $(\to_E)$: By IH, $\Gamma \models^3 M : \tau_1 \to \tau_2$ and $\Gamma \models^2 N : \tau_1$, so by lemma 3.2.1, $\Gamma \models^3_{\mathcal{P}} MN : \tau_2$ (because if $\tau_2 \in \mathsf{Type}^3$ then $\tau_1 \to \tau_2 \in \mathsf{Type}^3$).

- $(\to_I)$: By IH, $\Gamma, x : \tau_1 \models^3_{\mathcal{P}} M : \tau_2$. Let $\nu \models^3_{\mathcal{P}} \Gamma$ and $N \in [\![\tau_1]\!]^3_{\mathcal{P}}$. Then $\nu(x := N) \models^3_{\mathcal{P}} \Gamma$ since $x \notin \mathrm{dom}(\Gamma)$ and $\nu(x := N) \models^3_{\mathcal{P}} x : \tau_1$ since $N \in [\![\tau_1]\!]^3_{\mathcal{P}}$. Therefore $\nu(x := N) \models^3_{\mathcal{P}} M : \tau_2$, i.e. $[\![M]\!]_{\nu(x:=N)} \in [\![\tau_2]\!]^3_{\mathcal{P}}$. Hence, by lemma 3.2.1, $[\![M]\!]_{\nu(x:=x)}[x := N] \in [\![\tau_2]\!]^3_{\mathcal{P}}$. Hence by applying $\mathrm{SAT}(\mathcal{P}, \mathcal{P})$ and 4.4.6, we get $(\lambda x.[\![M]\!]_{\nu(x:=x)})N \in [\![\tau_2]\!]^3_{\mathcal{P}}$. Again by lemma 3.2.1, $([\![\lambda x.M]\!]_\nu)N \in [\![\tau_2]\!]^3_{\mathcal{P}}$. Hence $[\![\lambda x.M]\!]_\nu \in \{M \mid \forall N \in [\![\tau_1]\!]^3_{\mathcal{P}}.\ MN \in [\![\tau_2]\!]^3_{\mathcal{P}}\}$.

  Since $\tau_1 \in \mathsf{Type}^4$, by $\mathrm{VAR}(\mathcal{P}, \mathcal{P})$ and 4.4.5, $x \in [\![\tau_1]\!]^3_{\mathcal{P}}$, hence by the same argument as above we obtain $[\![M]\!]_{\nu(x:=x)} \in [\![\tau_2]\!]^3_{\mathcal{P}}$. Since $\tau_1 \to \tau_2 \in \mathsf{Type}^3$ then $\tau_2 \in \mathsf{Type}^3$, so by $\mathrm{CLO}(\mathcal{P}, \mathcal{P})$ and 4.4.2, $\lambda x.[\![M]\!]_{\nu(x:=x)} \in \mathcal{P}$ and by lemma 3.2.1, $[\![\lambda x.M]\!]_\nu \in \mathcal{P}$. Hence, we conclude that $[\![\lambda x.M]\!]_\nu \in [\![\tau_1 \to \tau_2]\!]^3_{\mathcal{P}}$.

- $(\leq^3)$: We conclude by IH and 4.4.4

- (Ω): This case is trivial because $\Omega \notin \mathsf{Type}^3$. □

The next lemma states that the set of terms satisfying the Church-Rosser, the weak head normalisation or the standardisation properties satisfies the variable, saturation and closure predicates.

Lemma 4.6. *Let* $\mathcal{P} \in \{\mathsf{CR}, \mathsf{S}, \mathsf{W}\}$. *Then* $\mathrm{VAR}(\mathcal{P}, \mathcal{P})$, $\mathrm{SAT}(\mathcal{P}, \mathcal{P})$ *and* $\mathrm{CLO}(\mathcal{P}, \mathcal{P})$. □

We obtain the following proof method. However, we strongly believe that the set of terms typable in our type system with a type ρ is no more than the set of strongly normalisable terms.

Proposition 4.7. *If* $\Gamma \vdash^3 M : \rho$ *then* $M \in \mathsf{CR}$, $M \in \mathsf{S}$, *and* $M \in \mathsf{W}$. □

Proof. By lemma 4.6, lemma 4.4.2 and lemma 4.5 □

5 Adapting the CR proof of Koletsos and Stavrinos [KS08] to βI-reduction

[KS08] gave a proof of Church-Rosser for β-reduction for the intersection type system D of Definition 2.25 (studied in detail in [Kri90]) and showed that this can be used to establish confluence of β-developments without using strong normalisation. In this section, we adapt his proof to βI and at the same time, set the formal ground for generalising the method for $\beta\eta$ in the next section. First, we adapt and formalise a number of definitions and lemmas given in [Kri90] in order to make them applicable to βI-developments. Then, we define type interpretations for both βI and $\beta\eta$, establish the soundness and Church-Rosser of both systems D and D_I (for $\beta\eta$- resp. βI-reduction), and finally, adapt [KS08] to establish the confluence of βI-developments.

All proofs from this section are located in appendix B.

5.1 Formalising βI-developments

The next definition, taken from [Kri90] (and used in [KS08]) uses the variable c to destroy the βI-redexes of M which are not in the set $\mathcal{F}$ of βI-redex occurrences in M, and to neutralise applications so that they cannot be transformed into redexes after βI-reduction. For example, in $c(\lambda x.x)y$, c is used to destroy the βI-redex $(\lambda x.x)y$.

Definition 5.1 ($\Phi^c(-,-)$)**.** Let $M \in \Lambda\mathrm{I}$, such that $c \notin \mathrm{fv}(M)$ and $\mathcal{F} \subseteq \mathcal{R}^{\beta I}_M$.

1. If $M = x$ then $\mathcal{F} = \varnothing$ and $\Phi^c(x, \mathcal{F}) = x$
2. If $M = \lambda x.N$ such that $x \neq c$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_N$ then $\Phi^c(\lambda x.N, \mathcal{F}) = \lambda x.\Phi^c(N, \mathcal{F}')$
3. If $M = NP$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_N$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_P$ then
$$\Phi^c(NP, \mathcal{F}) = \begin{cases} c\Phi^c(N, \mathcal{F}_1)\Phi^c(P, \mathcal{F}_2) & \text{if } 0 \notin \mathcal{F} \\ \Phi^c(N, \mathcal{F}_1)\Phi^c(P, \mathcal{F}_2) & \text{otherwise} \end{cases}$$

□

The next lemma is an adapted version of a lemma which appears in [KS08] and which in turns adapts a lemma from [Kri90].

Lemma 5.2.

1. *If $M \in \Lambda I$, $c \notin \mathrm{fv}(M)$, and $\mathcal{F} \subseteq \mathcal{R}^{\beta I}_M$ then*
   (a) $\mathrm{fv}(M) = \mathrm{fv}(\Phi^c(M, \mathcal{F})) \setminus \{c\}$.
   (b) $\Phi^c(M, \mathcal{F}) \in \Lambda I_c$.
   (c) $|\Phi^c(M, \mathcal{F})|^c = M$.
   (d) $|\langle \Phi^c(M, \mathcal{F}), \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F})} \rangle|^c = \mathcal{F}$.
2. *Let $M \in \Lambda I_c$.*
   (a) $|\langle M, \mathcal{R}^{\beta I}_M \rangle|^c \subseteq \mathcal{R}^{\beta I}_{|M|^c}$ *and* $M = \Phi^c(|M|^c, |\langle M, \mathcal{R}^{\beta I}_M \rangle|^c)$.
   (b) $\langle |M|^c, |\langle M, \mathcal{R}^{\beta I}_M \rangle|^c \rangle$ *is the one and only pair* $\langle N, \mathcal{F} \rangle$ *such that* $N \in \Lambda I$, $c \notin \mathrm{fv}(N)$, $\mathcal{F} \subseteq \mathcal{R}^{\beta I}_N$ *and* $\Phi^c(N, \mathcal{F}) = M$.

□

The next lemma is needed to define βI-developments.

Lemma 5.3. *Let $M \in \Lambda I$, such that $c \notin \mathrm{fv}(M)$, $\mathcal{F} \subseteq \mathcal{R}^{\beta I}_M$, $p \in \mathcal{F}$ and $M \xrightarrow{p}_{\beta I} M'$. Then, there exists a unique set $\mathcal{F}' \subseteq \mathcal{R}^{\beta I}_{M'}$ such that $\Phi^c(M, \mathcal{F}) \xrightarrow{p'}_{\beta I} \Phi^c(M', \mathcal{F}')$ and $|\langle \Phi^c(M, \mathcal{F}), p' \rangle|^c = p$.* □

We follow [Kri90] and define the set of βI-residuals of a set of βI-redexes $\mathcal{F}$ relative to a sequence of βI-redexes. First, we give the definition relative to one redex.

Definition 5.4. Let $M \in \Lambda\mathrm{I}$, such that $c \notin \mathrm{fv}(M)$, $\mathcal{F} \subseteq \mathcal{R}^{\beta I}_{M}$, $p \in \mathcal{F}$ and $M \xrightarrow{p}_{\beta I} M'$. By lemma 5.3, there exists a unique $\mathcal{F}' \subseteq \mathcal{R}^{\beta I}_{M'}$ such that $\Phi^c(M,\mathcal{F}) \xrightarrow{p'}_{\beta I} \Phi^c(M',\mathcal{F}')$ and $|\langle \Phi^c(M,\mathcal{F}), p'\rangle|^c = p$. We call $\mathcal{F}'$ the set of **βI-residuals in M' of the set of βI-redexes $\mathcal{F}$ in M relative to p.** $\square$

Definition 5.5 (βI-development)**.** Let $M \in \Lambda\mathrm{I}$ where $c \notin \mathrm{fv}(M)$ and $\mathcal{F} \subseteq \mathcal{R}^{\beta I}_{M}$. A one-step βI-development of $\langle M,\mathcal{F}\rangle$, denoted $\langle M,\mathcal{F}\rangle \rightarrow_{\beta Id} \langle M',\mathcal{F}'\rangle$, is a βI-reduction $M \xrightarrow{p}_{\beta I} M'$ where $p \in \mathcal{F}$ and $\mathcal{F}'$ is the set of βI-residuals in M' of the set of βI-redexes $\mathcal{F}$ in M relative to p. A **βI-development** is the transitive closure of a one-step βI-development. We write also $M \xrightarrow{\mathcal{F}}_{\beta Id} M_n$ for the βI-development $\langle M,\mathcal{F}\rangle \rightarrow^*_{\beta Id} \langle M_n,\mathcal{F}_n\rangle$. $\square$

The next two lemmas are informative about developments.

Lemma 5.6. *Let $M \in \Lambda I$, such that $c \notin \mathrm{fv}(M)$ and $\mathcal{F} \subseteq \mathcal{R}^{\beta I}_{M}$. Then: $\langle M,\mathcal{F}\rangle \rightarrow^*_{\beta Id} \langle M',\mathcal{F}'\rangle \iff \Phi^c(M,\mathcal{F}) \rightarrow^*_{\beta I} \Phi^c(M',\mathcal{F}')$.* $\square$

Lemma 5.7. *Let $M \in \Lambda I$, such that $c \notin \mathrm{fv}(M)$ and $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{R}^{\beta I}_{M}$. If $\langle M,\mathcal{F}_1\rangle \rightarrow_{\beta Id} \langle M',\mathcal{F}_1'\rangle$ then there exists $\mathcal{F}_2' \subseteq \mathcal{R}^{\beta I}_{M'}$ such that $\mathcal{F}_1 \subseteq \mathcal{F}_2'$ and $\langle M,\mathcal{F}_2\rangle \rightarrow_{\beta Id} \langle M',\mathcal{F}_2'\rangle$.* $\square$

5.2 Confluence of βI-developments, hence of βI-reduction

Definition 5.8.

1. Let $r \in \{\beta I, \beta\eta\}$. We define the type interpretation $[\![-]\!]^r : \mathsf{Type}^1 \rightarrow 2^\Lambda$ by:
   - $[\![\alpha]\!]^r = \mathsf{CR}^r$, where $\alpha \in \mathcal{A}$.
   - $[\![\sigma \cap \tau]\!]^r = [\![\sigma]\!]^r \cap [\![\tau]\!]^r$.
   - $[\![\sigma \rightarrow \tau]\!]^r = \{M \in \mathsf{CR}^r \mid \forall N \in [\![\sigma]\!]^r.\ MN \in [\![\tau]\!]^r\}$.
2. A set $\mathcal{X} \subseteq \Lambda$ is saturated iff $\forall n \geq 0.\ \forall M, N, M_1, \ldots, M_n \in \Lambda.\ \forall x \in \mathcal{V}.$
$$M[x := N]M_1 \ldots M_n \in \mathcal{X} \Rightarrow (\lambda x.M)NM_1 \ldots M_n \in \mathcal{X}$$
3. A set $\mathcal{X} \subseteq \Lambda\mathrm{I}$ is I-saturated iff $\forall n \geq 0.\ \forall M, N, M_1, \ldots, M_n \in \Lambda.\ \forall x \in \mathcal{V}.$
$$x \in \mathrm{fv}(M) \Rightarrow M[x := N]M_1 \ldots M_n \in \mathcal{X} \Rightarrow (\lambda x.M)NM_1 \ldots M_n \in \mathcal{X}$$

$\square$

Here is a background lemma:

Lemma 5.9.

1. *If $\Gamma \vdash^{\beta I} M : \sigma$ then $M \in \Lambda I$ and $\mathrm{fv}(M) = \mathrm{dom}(\Gamma)$.*

2. *Let* $\Gamma \vdash^{\beta\eta} M : \sigma$*. Then* $\mathrm{fv}(M) \subseteq \mathrm{dom}(\Gamma)$ *and if* $\Gamma \subseteq \Gamma'$ *then* $\Gamma' \vdash^{\beta\eta} M : \sigma$*.*
3. *Let* $r \in \{\beta I, \beta\eta\}$*. If* $\Gamma \vdash^r M : \sigma$*,* $\sigma \sqsubseteq \sigma'$ *and* $\Gamma' \sqsubseteq \Gamma$ *then* $\Gamma' \vdash^r M : \sigma'$*.* □

The next lemma states that the interpretations of types are saturated and only contain terms that are Church-Rosser. Krivine [Kri90] proved a similar result for $r = \beta$ and where CR_0^r and CR^r were replaced by the corresponding sets of strongly normalising terms. Koletsos and Stavrinos [KS08] adapted Krivine's lemma for Church-Rosser w.r.t. β-reduction instead of strong normalisation. Here, we adapt the result to βI and $\beta\eta$.

Lemma 5.10. *Let* $r \in \{\beta I, \beta\eta\}$*.*

1. $\forall \sigma \in \mathsf{Type}^1.\ \mathsf{CR}_0^r \subseteq [\![\sigma]\!]^r \subseteq \mathsf{CR}^r$*.*
2. $\mathsf{CR}^{\beta I}$ *is I-saturated.*
3. $\mathsf{CR}^{\beta\eta}$ *is saturated.*
4. $\forall \sigma \in \mathsf{Type}^1.\ [\![\sigma]\!]^{\beta I}$ *is I-saturated.*
5. $\forall \sigma \in \mathsf{Type}^1.\ [\![\sigma]\!]^{\beta\eta}$ *is saturated.* □

Next we adapt the soundness lemma of [Kri90] to both $\vdash^{\beta I}$ and $\vdash^{\beta\eta}$.

Lemma 5.11. *Let* $r \in \{\beta I, \beta\eta\}$*. If* $x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash^r M : \sigma$ *and* $\forall i \in \{1, \ldots, n\}$*,* $N_i \in [\![\sigma_i]\!]^r$ *then* $M[(x_i := N_i)_1^n] \in [\![\sigma]\!]^r$*.* □

Finally, we adapt a corollary from [KS08] to show that every term of Λ typable in system D has the $\beta\eta$ Church-Rosser property and every term of Λ typable in system D_I has the βI Church-Rosser property.

Corollary 5.12. *Let* $r \in \{\beta I, \beta\eta\}$*. If* $\Gamma \vdash^r M : \sigma$ *then* $M \in \mathsf{CR}^r$*.* □

Proof. Let $\Gamma = (x_i : \sigma_i)_n$. By lemma 5.10, $\forall i \in \{1, \ldots, n\}, x_i \in [\![\sigma_i]\!]^r$, so by lemma 5.11 and again by lemma 5.10, $M \in [\![\sigma]\!]^r \subseteq \mathsf{CR}^r$. □

In order to accommodate βI- and $\beta\eta$-reduction, the next lemma generalises a lemma given in [Kri90] (and used in [KS08]). Basically this lemma states that every term of $\Lambda\mathrm{I}_c$ is typable in system D and every term of $\Lambda\eta_c$ is typable in D_I.

Lemma 5.13. *Let* $\mathrm{fv}(M) \setminus \{c\} = \{x_1, \ldots, x_n\} \subseteq \mathrm{dom}(\Gamma)$ *where* $c \notin \mathrm{dom}(\Gamma)$*.*

1. *If* $M \in \Lambda I_c$ *then for* $\Gamma' = \Gamma \upharpoonright \mathrm{fv}(M)$*,* $\exists \sigma, \tau \in \mathsf{Type}^1$ *such that if* $c \in \mathrm{fv}(M)$ *then* $\Gamma', c : \sigma \vdash^{\beta I} M : \tau$*, and if* $c \notin \mathrm{fv}(M)$ *then* $\Gamma' \vdash^{\beta I} M : \tau$*.*
2. *If* $M \in \Lambda\eta_c$ *then* $\exists \sigma, \tau \in \mathsf{Type}^1$ *such that* $\Gamma, c : \sigma \vdash^{\beta\eta} M : \tau$*.* □

The next lemma is an adaptation of the main theorem in [KS08] where as far as we know appears for the first time.

Lemma 5.14 (confluence of the βI-developments)**.** *Let $M \in \Lambda I$, such that $c \notin \mathrm{fv}(M)$. If $M \xrightarrow{\mathcal{F}_1}_{\beta Id} M_1$ and $M \xrightarrow{\mathcal{F}_2}_{\beta Id} M_2$, then there exist $\mathcal{F}'_1 \subseteq \mathcal{R}^{\beta I}_{M_1}$, $\mathcal{F}'_2 \subseteq \mathcal{R}^{\beta I}_{M_2}$ and $M_3 \in \Lambda I$ such that $M_1 \xrightarrow{\mathcal{F}'_1}_{\beta Id} M_3$ and $M_2 \xrightarrow{\mathcal{F}'_2}_{\beta Id} M_3$.* □

We follow [Bar84] and [KS08] and define one reduction as follows:

Notation 5.15. Let $M, M' \in \Lambda$I, such that $c \notin \mathrm{fv}(M)$. We define one reduction by: $M \to_{1I} M' \iff \exists \mathcal{F}, \mathcal{F}', (M, \mathcal{F}) \to^*_{\beta Id} (M', \mathcal{F}')$. □

Lemma 5.16. *Let $c \notin \mathrm{fv}(M)$. Then, $\mathcal{R}^{\beta I}_{\Phi^c(M,\varnothing)} = \varnothing$.* □

Lemma 5.17. *Let $c \notin \mathrm{fv}(MN)$ and $x \neq c$. Then, $\mathcal{R}^{\beta I}_{\Phi^c(M,\varnothing)[x:=\Phi^c(N,\varnothing)]} = \varnothing$.* □

Lemma 5.18. *Let $c \notin \mathrm{fv}(M)$. If $p \in \mathcal{R}^{\beta I}_M$ and $\Phi^c(M, \{p\}) \to_{\beta I} M'$ then $\mathcal{R}^{\beta I}_{M'} = \varnothing$.* □

Lemma 5.19. *Let $M \in \Lambda I$ such that $c \notin \mathrm{fv}(M)$. If $M \xrightarrow{p}_{\beta I} M'$ then $\langle M, \{p\}\rangle \to_{\beta Id} \langle M', \varnothing\rangle$.* □

Lemma 5.20. $\to^*_{\beta I} = \to^*_{1I}$. □

Finally, we achieve what we started to do: the confluence of βI-reduction on ΛI.

Lemma 5.21. $\Lambda I \subseteq \mathsf{CR}^{\beta I}$. □

6 Generalisation of the method to $\beta\eta$-reduction

In this section, we generalise the method of [KS08] to handle $\beta\eta$-reduction. This generalisation is not trivial since we needed to develop developments involving η-reduction and to establish the important result of the closure under η-reduction of a defined set of frozen terms. It is for reasons like this that we extended the various definitions related to developments. For example, clause (R4) of the definition of $\Lambda\eta_c$ in Definition 2.3 aims to ensure closure under η-reduction. The definition of Λ_c in [Kri90] exluded such a rule and hence we lose closure under η-reduction as can be seen in the following example: Let $M = \lambda x.cNx \in \Lambda_c$ where $x \notin \mathrm{fv}(N)$ and $N \in \Lambda_c$, then $M \to_\eta cN \notin \Lambda_c$.

Again here, the proofs are moved to appendix C.

A full common definition of a $\beta\eta$-residual is given by Curry and Feys [CF58] (p. 117, 118). Another definition of $\beta\eta$-residual (called λ-residual) is presented by Klop [Klo80] (definition 2.4, p. 254). Klop [Klo80] shows that both definitions enable to prove different properties of developments. Following the definition of a $\beta\eta$-residual given by Curry and Feys [CF58] (and as pointed out in [CF58, Klo80, BBKV76]), if the η-redex $\lambda x.(\lambda y.M)x$, where $x \notin \mathrm{fv}(\lambda y.M)$, is reduced in the term $P = (\lambda x.(\lambda y.M)x)N$ to give the term $Q = (\lambda y.M)N$, then Q is not a $\beta\eta$-residual of P in P (note that following the definition of a

λ-residual given by Klop [Klo80], Q is a λ-residual of the redex $(\lambda y.M)x$ in P since the λ of the redex Q is the same than the λ of the redex $(\lambda y.M)x$ in P). Moreover, if the β-redex $(\lambda y.My)x$, where $y \notin \mathrm{fv}(M)$, is reduced in the term $P = \lambda x.(\lambda y.My)x$ to give the term $Q = \lambda x.Mx$, then Q is not a $\beta\eta$-residual of P in P (note that following the definition of a λ-residual given by Klop [Klo80], Q is a λ-residual of the redex P in P since the λ of the redex Q is the same than the λ of the redex P in P). Our definition 6.5 differs from the common one stated by Curry and Feys [CF58] by these cases as we illustrate in the following example: $\Psi^c((\lambda x.(\lambda y.M)x)N, \{1, 1.0, 1.1.0\}) = \{c^n((\lambda x.(\lambda y.P[y := c(cy)])x)Q) \mid n \geq 0 \wedge P \in \Psi^c(M, \varnothing) \wedge Q \in \Psi^c(N, \varnothing)\}$, where $x \notin \mathrm{fv}(\lambda y.M)$. Let $p = 1.0$ then $(\lambda x.(\lambda y.M)x)N \xrightarrow{p}_{\beta\eta} (\lambda y.M)N$. Moreover, $P_0 = c^n((\lambda x.(\lambda y.P[y := c(cy)])x)Q) \xrightarrow{p'}_{\beta\eta} c^n((\lambda y.P[y := c(cy)])Q)$ such that $n \geq 0$, $P \in \Psi^c(M, \varnothing)$, $Q \in \Psi^c(N, \varnothing)$, $|\langle P_0, p'\rangle|^c = |\langle P_0, 2^n.1.0\rangle|^c = p$ (using a lemma stated and proved in the long version of this article) and $c^n((\lambda y.P[y := c(cy)])Q) \in \Psi^c((\lambda y.M)N, \{0\})$.

The next two definitions adapt definition 5.1 to deal with $\beta\eta$-reduction. The variable c enables to destroy the $\beta\eta$-redexes of M which are not in the set $\mathcal{F}$ of $\beta\eta$-redex occurrences in M; to neutralise applications so that they cannot be transformed into redexes after $\beta\eta$-reduction; and to neutralise bound variables so λ-abstraction cannot be transformed into redexes after $\beta\eta$-reduction. For example, in $\lambda x.y(c(cx))$ $(x \neq x)$, c is used to destroy the η-redex $\lambda x.yx$.

Definition 6.1 ($\Psi^c(-,-), \Psi^c_0(-,-)$)**.** Let $c \notin \mathrm{fv}(M)$ and $\mathcal{F} \subseteq \mathcal{R}^{\beta\eta}_M$.

(P1) If $M \in \mathcal{V} \setminus \{c\}$ then $\mathcal{F} =^{2.6} \varnothing$ and

$$\Psi^c(M, \mathcal{F}) = \{c^n(M) \mid n > 0\}$$

$$\Psi^c_0(M, \mathcal{F}) = \{M\}$$

(P2) If $M = \lambda x.N$ and $x \neq c$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq^{2.6} \mathcal{R}^{\beta\eta}_N$:

$$\Psi^c(M, \mathcal{F}) = \begin{cases} \{c^n(\lambda x.P[x := c(cx)]) \mid n \geq 0 \wedge P \in \Psi^c(N, \mathcal{F}')\} & \text{if } 0 \notin \mathcal{F} \\ \{c^n(\lambda x.N') \mid n \geq 0 \wedge N' \in \Psi^c_0(N, \mathcal{F}')\} & \text{otherwise} \end{cases}$$

$$\Psi^c_0(M, \mathcal{F}) = \begin{cases} \{\lambda x.N'[x := c(cx)] \mid N' \in \Psi^c(N, \mathcal{F}')\} & \text{if } 0 \notin \mathcal{F} \\ \{\lambda x.N' \mid N' \in \Psi^c_0(N, \mathcal{F}')\} & \text{otherwise} \end{cases}$$

(P3) If $M = NP$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq^{2.6} \mathcal{R}^{\beta\eta}_N$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq^{2.6} \mathcal{R}^{\beta\eta}_P$ then:
$\Psi^c(M, \mathcal{F}) =$
$$\begin{cases} \{c^n(cN'P') \mid n \geq 0 \wedge N' \in \Psi^c(N, \mathcal{F}_1) \wedge P' \in \Psi^c(P, \mathcal{F}_2)\} & \text{if } 0 \notin \mathcal{F} \\ \{c^n(N'P') \mid n \geq 0 \wedge N' \in \Psi^c_0(N, \mathcal{F}_1) \wedge P' \in \Psi^c(P, \mathcal{F}_2)\} & \text{otherwise} \end{cases}$$

$$\Psi^c_0(M, \mathcal{F}) = \begin{cases} \{cN'P' \mid N' \in \Psi^c(N, \mathcal{F}_1) \wedge P' \in \Psi^c_0(P, \mathcal{F}_2)\} & \text{if } 0 \notin \mathcal{F} \\ \{N'P' \mid N' \in \Psi^c_0(N, \mathcal{F}_1) \wedge P' \in \Psi^c_0(P, \mathcal{F}_2) & \text{otherwise} \end{cases}$$

$\square$

Lemma 6.2. *If $M \in \Lambda\eta_c$ and $n \geq 0$ then $c^n(M) \in \Lambda\eta_c$.* □

Proof. By induction on $n \geq 0$ using $(R4)$. □

Lemma 6.3.

1. *Let $c \notin \mathrm{fv}(M)$ and $\mathcal{F} \subseteq \mathcal{R}^{\beta\eta}_M$. We have:*
   (a) $\Psi^c_0(M, \mathcal{F}) \subseteq \Psi^c(M, \mathcal{F})$.
   (b) $\forall N \in \Psi^c(M, \mathcal{F}).\ \mathrm{fv}(M) = \mathrm{fv}(N) \setminus \{c\}$.
   (c) $\Psi^c(M, \mathcal{F}) \subseteq \Lambda\eta_c$.
   (d) *Let $M = Nx$ such that $x \notin \mathrm{fv}(N) \cup \{c\}$ and $P \in \Psi^c_0(M, \mathcal{F})$. Then, $\mathcal{R}^{\beta\eta}_{\lambda x.P} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_P\}$.*
   (e) *Let $M = Nx$. If $Px \in \Psi^c(Nx, \mathcal{F})$ then $Px \in \Psi^c_0(Nx, \mathcal{F})$.*
   (f) $\forall N \in \Psi^c(M, \mathcal{F}).\ \forall n \geq 0.\ c^n(N) \in \Psi^c(M, \mathcal{F})$.
   (g) $\forall N \in \Psi^c(M, \mathcal{F}).\ |N|^c = M$.
   (h) $\forall N \in \Psi^c(M, \mathcal{F}).\ \mathcal{F} = |\langle N, \mathcal{R}^{\beta\eta}_N\rangle|^c$.

2. *Let $M \in \Lambda\eta_c$. We have:*
   (a) $|\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c \subseteq \mathcal{R}^{\beta\eta}_{|M|^c}$ *and* $M \in \Psi^c(|M|^c, |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c)$.
   (b) $\langle |M|^c, |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c\rangle$ *is the one and only pair $\langle N, \mathcal{F}\rangle$ such that $c \notin \mathrm{fv}(N)$, $\mathcal{F} \subseteq \mathcal{R}^{\beta\eta}_N$ and $M \in \Psi^c(N, \mathcal{F})$.*

□

Lemma 6.4. *Let $M \in \Lambda$, such that $c \notin \mathrm{fv}(M)$, $\mathcal{F} \subseteq \mathcal{R}^{\beta\eta}_M$, $p \in \mathcal{F}$ and $M \xrightarrow{p}_{\beta\eta} M'$. Then, there exists a unique set $\mathcal{F}' \subseteq \mathcal{R}^{\beta\eta}_{M'}$, such that for all $N \in \Psi^c(M, \mathcal{F})$ there exists $N' \in \Psi^c(M', \mathcal{F}')$ and $p' \in \mathcal{R}^{\beta\eta}_N$ such that $N \xrightarrow{p'}_{\beta\eta} N'$ and $|\langle N, p'\rangle|^c = p$.* □

Definition 6.5. Let $M \in \Lambda$, $\mathcal{F} \subseteq \mathcal{R}^{\beta\eta}_M$, $p \in \mathcal{F}$ and $M \xrightarrow{p}_{\beta\eta} M'$. By lemma 6.4, there exists a unique $\mathcal{F}' \subseteq \mathcal{R}^{\beta\eta}_{M'}$, such that for all $N \in \Psi^c(M, \mathcal{F})$, there exist $N' \in \Psi^c(M', \mathcal{F}')$ and $p' \in \mathcal{R}^{\beta\eta}_N$ such that $N \xrightarrow{p'}_{\beta\eta} N'$ and $|\langle N, p'\rangle|^c = p$. We call $\mathcal{F}'$ the set of **$\beta\eta$-residuals in M' of the set of $\beta\eta$-redexes $\mathcal{F}$ in M relative to** p. □

Definition 6.6 ($\beta\eta$-development)**.** Let $M \in \Lambda$, where $c \notin \mathrm{fv}(M)$, and $\mathcal{F} \subseteq \mathcal{R}^{\beta\eta}_M$. A one-step $\beta\eta$-development of $\langle M, \mathcal{F}\rangle$, denoted $\langle M, \mathcal{F}\rangle \rightarrow_{\beta\eta d} \langle M', \mathcal{F}'\rangle$, is a $\beta\eta$-reduction $M \xrightarrow{p}_{\beta\eta} M'$ where $p \in \mathcal{F}$ and $\mathcal{F}'$ is the set of $\beta\eta$-residuals in M' of the set of $\beta\eta$-redexes $\mathcal{F}$ in M relative to p. A **$\beta\eta$-development** is the transitive closure of a one-step $\beta\eta$-development. We write also $M \xrightarrow{\mathcal{F}}_{\beta\eta d} M'$ for the $\beta\eta$-development $\langle M, \mathcal{F}\rangle \rightarrow^*_{\beta\eta d} \langle M', \mathcal{F}'\rangle$. □

REMARK 6.7. Let us now compare our definition of $\beta\eta$-residuals to the one given by Klop [Klo80] (λ-residuals). We believe that we accept more redexes as residuals of a set of redexes than Curry and Feys [CF58] (as our examples given at the beginning of section 6, tend to prove) and less than Klop.

In order to do so, let us introduce the two calculus $\bar{\Lambda}$ and $\bar{\Lambda\eta_c}$ which are labelled versions of the two calculus Λ and $\Lambda\eta_c$:

$$\begin{array}{llll} t & \in & \bar{\Lambda} & ::= \quad x \mid \lambda_n x.t \mid t_1 t_2 \\ v & \in & \mathsf{ABS}_c & ::= \quad \lambda_n \bar{x}.w\bar{x} \mid \lambda_n \bar{x}.u[\bar{x} := c(c\bar{x})], \text{ where } \bar{x} \notin \mathrm{fv}(w) \\ w & \in & \mathsf{APP}_c & ::= \quad v \mid cu \\ u & \in & \bar{\Lambda\eta_c} & ::= \quad \bar{x} \mid v \mid wu \mid cu \end{array}$$

where $\bar{x}, \bar{y} \in \mathcal{V} \setminus \{c\}$. Note that $\mathsf{ABS}_c \subseteq \mathsf{APP}_c \subseteq \bar{\Lambda\eta_c} \subseteq \bar{\Lambda}$.

The labels enable to distinguish two different occurrences of a λ.

Since these two calculus are only labelled versions of Λ and $\Lambda\eta_c$, let us assume in this remark that the work done so far is true when Λ ans $\Lambda\eta_c$ are replaced by $\bar{\Lambda}$ and $\bar{\Lambda\eta_c}$.

Klop [Klo80] defines his λ-residuals as follows:

> "Let $\mathcal{R} = M_0 \to M_1 \to \ldots \to M_k \to \ldots$ be a $\beta\eta$-reduction, R_0 a redex in M_0 and R_k a redex in M_k such that the head-λ of R_k descends from that of R_0.
>
> Then, regardless whether R_0, R_k are β- or η-redexes, R_k is called a λ-residual of R_0 via $\mathcal{R}$."

We are now going to our own definition of the head-λ of a $\beta\eta$-redex, slightly different from Klop's ones, as we intend to prove below.

Let us define the head-λ of a $\beta\eta$-redex as follows: $\mathrm{headlam}((\lambda_n x.t_1)t_2) = \langle 1, n\rangle$ and $\mathrm{headlam}(\lambda_n x.t_0 x) = \langle 2, n\rangle$, if $x \notin \mathrm{fv}(t_0)$. If $\mathcal{F} \subseteq \mathcal{R}_t^{\beta\eta}$ we define $\mathrm{headlamred}(t, \mathcal{F})$ to be $\{\langle i, n\rangle \mid \exists p \in \mathcal{F}.\ \mathrm{headlam}(t|_p) = \langle i, n\rangle\}$. We define $\mathrm{hlr}(t)$ to be $\mathrm{headlamred}(t, \mathcal{R}_t^{\beta\eta})$.

Let $c \notin \mathrm{fv}(t)$, $\mathcal{F} \subseteq \mathcal{R}_t^{\beta\eta}$ and $t \xrightarrow{p}_{\beta\eta} t'$ then by definition 6.5, there exists a unique $\mathcal{F}' \subseteq \mathcal{R}_{t'}^{\beta\eta}$, such that for all $u \in \Psi^c(t, \mathcal{F})$ (by lemma 6.3.1c, $u \in \bar{\Lambda\eta_c}$), there exist $u' \in \Psi^c(t', \mathcal{F}')$ and $p' \in \mathcal{R}_u^{\beta\eta}$ such that $u \xrightarrow{p'}_{\beta\eta} u'$ and $|\langle u, p'\rangle|^c = p$. The set $\mathcal{F}'$ is the set of $\beta\eta$-residuals in t' of the set of redexes $\mathcal{F}$ in t relative to p. By lemma 2.2.3, $c \notin \mathrm{fv}(t')$. By definition $\Psi^c(t, \mathcal{F})$ is not empty. Let $u \in \Psi^c(t, \mathcal{F})$ then there exist $u' \in \Psi^c(t', \mathcal{F}')$ and $p' \in \mathcal{R}_u^{\beta\eta}$ such that $u \xrightarrow{p'}_{\beta\eta} u'$ and $|\langle u, p'\rangle|^c = p$. By lemma 6.10, $\mathrm{hlr}(u') \subseteq \mathrm{hlr}(u)$. So, by lemma 6.8, $\mathrm{headlamred}(t', \mathcal{F}') \subseteq \mathrm{headlamred}(t, \mathcal{F})$.

However we can find t and $\mathcal{F}$ such that, following Klop's definition [Klo80], $p_0 \in \mathcal{R}_{t'}^{\beta\eta}$ and p_0 is a λ-residual of $\mathcal{F}$ via p but $p_0 \notin \mathcal{F}'$.

For example: Let $t = (\lambda_0 x.xy)(\lambda_1 z.yz) \xrightarrow{0}_{\beta\eta} (\lambda_1 z.yz)y = t'$. Let $\mathcal{F} = \{0, 2.0\}$. Then $\Psi^c(t, \mathcal{F}) = \{c^{n_1}((\lambda_0 x.c^{n_2}(c^3(x)y))(c^{n_3}(\lambda_1 z.c^{n_4+1}(y)z))) \mid n_1, n_2, n_3, n_4 \geq 0\}$. Let $u \in \Psi^c(t, \mathcal{F})$, then $u = c^{n_1}((\lambda_0 x.c^{n_2}(c^3(x)y))(c^{n_3}(\lambda_1 z.c^{n_4+1}(y)z)))$ such that $n_1, n_2, n_3, n_4 \geq 0$. We obtain $u = c^{n_1}((\lambda_0 x.c^{n_2}(c^3(x)y))(c^{n_3}(\lambda_1 z.c^{n_4+1}(y)z))) \xrightarrow{p_0}_{\beta\eta} c^{n_1+n_2}(c^{n_3+3}(\lambda_1 z.c^{n_4+1}(y)z)y) = u'$ such that $p_0 = 2^{n_1}.0$. Then $\mathcal{F}' = \{1.0\}$ is the set of $\beta\eta$-residuals in t' of the set of redexes $\mathcal{F}$ in t relative to p. But 0 is a λ-residual of $\mathcal{F}$ via 0 and $0 \notin \mathcal{F}'$.

We could think that our definition of $\beta\eta$-redexes capture the Klop's definition without the condition: "regardless whether R_0, R_k are β- or η-redexes", But it is not the case . For example: $t = \lambda_n\bar{x}.(\lambda_m\bar{y}.z\bar{y})\bar{x} \xrightarrow{1.0}_\beta \lambda_n\bar{x}.z\bar{x} = t'$ and $0 \in \mathcal{R}^{\beta\eta}_{t'}$, but $u = \lambda_n\bar{x}.(\lambda_m\bar{y}.cz(c(c\bar{y})))\bar{x} \in \Psi^c(t, \{0, 1.0\})$ and $u = \lambda_n\bar{x}.(\lambda_m\bar{y}.cz(c(c\bar{y})))\bar{x} \xrightarrow{1.0}_{\beta\eta} \lambda_n\bar{x}.cz(c(c\bar{x})) = u'$ and $0 \notin \mathcal{R}^{\beta\eta}_{u'}$.

So, we believe that our $\beta\eta$-residuals are only a subset of Klop's λ-residuals (in its definition and without "regardless whether R_0, R_k are β- or η-redexes"). $\square$

Let now show that our definition of $\beta\eta$-residuals corresponds to a restriction (in the sense that we believe we accept more redexes as residuals of a set of redexes than Curry and Feys [CF58] and less than Klop [Klo80]) of the definition of λ-residuals given by Klop [Klo80].

In order to do so, let us introduce the two calculus $\bar{\Lambda}$ and $\bar{\Lambda\eta_c}$ which are labelled versions of the two calculus Λ and $\Lambda\eta_c$:

$$
\begin{array}{lcll}
t & \in & \bar{\Lambda} & ::= \quad x \mid \lambda_n x.t \mid t_1 t_2 \\
v & \in & \mathsf{ABS}_c & ::= \quad \lambda_n\bar{x}.w\bar{x} \mid \lambda_n\bar{x}.u[\bar{x} := c(c\bar{x})], \text{ where } \bar{x} \notin \mathrm{fv}(w) \\
w & \in & \mathsf{APP}_c & ::= \quad v \mid cu \\
u & \in & \bar{\Lambda\eta_c} & ::= \quad \bar{x} \mid v \mid wu \mid cu
\end{array}
$$

where $\bar{x}, \bar{y} \in \mathcal{V} \setminus \{c\}$. Note that $\mathsf{ABS}_c \subseteq \mathsf{APP}_c \subseteq \bar{\Lambda\eta_c} \subseteq \bar{\Lambda}$.

The labels enable to distinguish two different occurrences of a λ.

Since these two calculus are only labelled versions of Λ and $\Lambda\eta_c$, let us assume here that the work done so far is true when Λ ans $\Lambda\eta_c$ are replaced by $\bar{\Lambda}$ and $\bar{\Lambda\eta_c}$.

Klop [Klo80] defines his λ-residuals as follows:

> "Let $\mathcal{R} = M_0 \to M_1 \to \ldots \to M_k \to \ldots$ be a $\beta\eta$-reduction, R_0 a redex in M_0 and R_k a redex in M_k such that the head-λ of R_k descends from that of R_0.
>
> Then, regardless whether R_0, R_k are β- or η-redexes, R_k is called a λ-residual of R_0 via $\mathcal{R}$."

Let us define the head-λ of a $\beta\eta$-redex as follows: $\mathrm{headlam}((\lambda_n x.t_1)t_2) = \langle 1, n\rangle$ and $\mathrm{headlam}(\lambda_n x.t_0 x) = \langle 2, n\rangle$, if $x \notin \mathrm{fv}(t_0)$. If $\mathcal{F} \subseteq \mathcal{R}^{\beta\eta}_t$ we define $\mathrm{headlamred}(t, \mathcal{F})$ to be $\{\langle i, n\rangle \mid \exists p \in \mathcal{F}.\ \mathrm{headlam}(t|_p) = \langle i, n\rangle\}$. We define $\mathrm{hlr}(t)$ to be $\mathrm{headlamred}(t, \mathcal{R}^{\beta\eta}_t)$.

Lemma 6.8. *Let $c \notin \mathrm{fv}(t)$ and $\mathcal{F} \subseteq \mathcal{R}^{\beta\eta}_t$. If $u \in \Psi^c(t, \mathcal{F})$ then* $\mathrm{hlr}(u) = \mathrm{headlamred}(t, \mathcal{F})$. $\square$

Proof. We prove this lemma by induction on the structure of t.

- Let $t = x \neq c$ then by lemma 2.5, $\mathcal{F} = \varnothing$ and $u = c^n(x)$ such that $n \geq 0$. Then, $\mathrm{hlr}(u) =^{6.13} \varnothing = \mathrm{headlamred}(t, \mathcal{F})$.

- Let $t = \lambda_n x.t_1$ such that $x \neq c$ and $\mathcal{F}_1 = p \mid 1.p \in \mathcal{F}$.

– If $0 \in \mathcal{F}$ then $t_1 = t'_1 x$ such that $x \notin \mathrm{fv}(t'_1)$, and $u = c^n(\lambda_n x.u_1)$ such that $n \geq 0$ and $u_1 \in \Psi^c_0(t_1, \mathcal{F}_1)$. By IH and lemma 6.3.1a, $\mathrm{hlr}(u_1) = \mathrm{headlamred}(t_1, \mathcal{F}_1)$. Then, $\mathrm{hlr}(u) =^{6.3.1d,6.13} \mathrm{hlr}(u_1) \cup \{\langle 2, n\rangle\} = \mathrm{headlamred}(t_1, \mathcal{F}_1) \cup \{\langle 2, n\rangle\} =^{6.13} \mathrm{headlamred}(t, \mathcal{F})$.

– Else, $u = c^n(\lambda_n x.u_1[x := c(cx)])$ such that $n \geq 0$ and $u_1 \in \Psi^c(t_1, \mathcal{F}_1)$. By IH, $\mathrm{hlr}(u_1) = \mathrm{headlamred}(t_1, \mathcal{F}_1)$. Then, $\mathrm{hlr}(u) =^{6.13} \mathrm{hlr}(u_1) = \mathrm{headlamred}(t_1, \mathcal{F}_1) =^{6.13} \mathrm{headlamred}(t, \mathcal{F})$.

- Let $t = t_1 t_2$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\}$.

  – If $0 \in \mathcal{F}$ then $t_1 = \lambda_n y.t'_1$, and $u = c^n(u_1 u_2)$ such that $n \geq 0$, $u_1 \in \Psi^c_0(t_1, \mathcal{F}_1)$ and $u_2 \in \Psi^c(t_2, \mathcal{F}_2)$. By definition, $u_1 = \lambda_n y.u'_1$. By IH and lemma 6.3.1a, $\mathrm{hlr}(u_1) = \mathrm{headlamred}(t_1, \mathcal{F}_1)$ and $\mathrm{hlr}(u_2) = \mathrm{headlamred}(t_2, \mathcal{F}_2)$. Then, $\mathrm{hlr}(u) =^{6.13} \mathrm{hlr}(u_1) \cup \mathrm{hlr}(u_2) \cup \{\langle 1, n\rangle\} = \mathrm{headlamred}(t_1, \mathcal{F}_1) \cup \mathrm{headlamred}(t_2, \mathcal{F}_2) \cup \{\langle 1, n\rangle\} =^{6.13} \mathrm{headlamred}(t, \mathcal{F})$.

  – Else, $u = c^n(cu_1 u_2)$ such that $n \geq 0$, $u_1 \in \Psi^c(t_1, \mathcal{F}_1)$ and $u_2 \in \Psi^c(t_2, \mathcal{F}_2)$. By IH, $\mathrm{hlr}(u_1) = \mathrm{headlamred}(t_1, \mathcal{F}_1)$ and $\mathrm{hlr}(u_2) = \mathrm{headlamred}(t_2, \mathcal{F}_2)$. Then, $\mathrm{hlr}(u) =^{6.13} \mathrm{hlr}(u_1) \cup \mathrm{hlr}(u_2) = \mathrm{headlamred}(t_1, \mathcal{F}_1) \cup \mathrm{headlamred}(t_2, \mathcal{F}_2) =^{6.13} \mathrm{headlamred}(t, \mathcal{F})$.

$\square$

Lemma 6.9. *If* $|u|^c = t$, $p \in \mathcal{R}^{\beta\eta}_u$, $p' \in \mathcal{R}^{\beta\eta}_t$ *and* $|\langle u, p\rangle|^c = p'$ *then* $\mathrm{headlam}(t|_{p'}) = \mathrm{headlam}(u|_p)$. $\square$

Lemma 6.10. *If* $u \in \bar{\Lambda\eta}_c$ *and* $u \xrightarrow{p}_{\beta\eta} u'$ *then* $\mathrm{hlr}(u') \subseteq \mathrm{hlr}(u)$. $\square$

Proof. We prove this lemma by induction on the size of u and then by case on the structure of u.

- Let $u = \bar{x}$ then it is done because $\bar{x}$ does not reduce by $\to_{\beta\eta}$.

- Let $u = \lambda_n \bar{x}.u_1[\bar{x} := c(c\bar{x})]$. Because $u \xrightarrow{p}_{\beta\eta} u'$, then by lemma 2.2.8, lemma 2.7.3 and lemma 2.4.12a, $p = 1.p'$, $u' = \lambda_n \bar{x}.u'_1[\bar{x} := c(c\bar{x})]$ and $u_1 \xrightarrow{p'}_{\beta\eta} u'_1$. By IH, $\mathrm{hlr}(u'_1) \subseteq \mathrm{hlr}(u_1)$. So, by lemma 6.13, $\mathrm{hlr}(u') = \mathrm{hlr}(u'_1) \subseteq \mathrm{hlr}(u_1) = \mathrm{hlr}(u)$.

- Let $u = \lambda_n \bar{x}.w\bar{x}$ and $\bar{x} \notin \mathrm{fv}(w)$. Because $u \xrightarrow{p}_{\beta\eta} u'$, by lemma 2.2.8 and lemma 2.5:

  – Either $p = 0$ and $u' = w$. So $\mathrm{hlr}(u') \subseteq^{6.14} \mathrm{hlr}(u)$.

  – Or $p = 1.p'$, $w\bar{x} \xrightarrow{p'}_{\beta\eta} u'_1$ and $u' = \lambda_n \bar{x}.u'_1$. By IH, $\mathrm{hlr}(u'_1) \subseteq \mathrm{hlr}(w\bar{x})$. So, $\mathrm{hlr}(u') \subseteq^{6.13} \mathrm{hlr}(u'_1) \cup \{\langle 2, n\rangle\} \subseteq \mathrm{hlr}(w\bar{x}) \cup \{\langle 2, n\rangle\} =^{6.13} \mathrm{hlr}(t)$.

- Let $u = (\lambda_n \bar{x}.w\bar{x})u_1$ such that $\bar{x} \notin \mathrm{fv}(w)$. Because $u \xrightarrow{p}_{\beta\eta} u'$, by lemma 2.2.8 and lemma 2.5:

  – Either $p = 0$. So $u' = wu_1$. By case on w:

        * Either w is a v and so $u' \in \mathcal{R}^{\beta\eta}$. Let $\langle 1, m\rangle = \mathrm{headlam}(u')$ then $\mathrm{hlr}(u') =^{6.13} \mathrm{hlr}(w) \cup \mathrm{hlr}(u_1) \cup \{\langle 1, m\rangle\} \subseteq^{6.13} \mathrm{hlr}(u)$.
        * Or $w = cu_2$ and so $u' \notin \mathcal{R}^{\beta\eta}$. Then $\mathrm{hlr}(u') =^{6.13} \mathrm{hlr}(w) \cup \mathrm{hlr}(u_1) \subseteq^{6.13} \mathrm{hlr}(u)$.
    - Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{\lambda_n\bar{x}.w\bar{x}}$. So $u' = u_1' u_1$ such that $\lambda_n\bar{x}.w\bar{x} \xrightarrow{p'}_{\beta\eta} u_1'$. By IH, $\mathrm{hlr}(u_1') \subseteq \mathrm{hlr}(\lambda_n\bar{x}.w\bar{x})$. By lemma 2.5:
        * Either $p' = 0$ and $u_1' = w$, so $u' = wu_1$. By case on w:
            · Either w is a v and so $u' \in \mathcal{R}^{\beta\eta}$. Let $\langle 1, m\rangle = \mathrm{headlam}(u')$ then $\mathrm{hlr}(u') =^{6.13} \mathrm{hlr}(w) \cup \mathrm{hlr}(u_1) \cup \{\langle 1, m\rangle\} \subseteq^{6.13} \mathrm{hlr}(u)$.
            · Or $w = cu_2$ and so $u' \notin \mathcal{R}^{\beta\eta}$. Then $\mathrm{hlr}(u') =^{6.13} \mathrm{hlr}(w) \cup \mathrm{hlr}(u_1) \subseteq^{6.13} \mathrm{hlr}(u)$.
        * Or $p' = 1.p''$, $u_1' = \lambda_n\bar{x}.u_2$ and $w\bar{x} \xrightarrow{p''}_{\beta\eta} u_2$. Then, $\mathrm{hlr}(u') =^{6.13} \mathrm{hlr}(u_1') \cup \mathrm{hlr}(u_1) \cup \{\langle 1, n\rangle\} \subseteq \mathrm{hlr}(\lambda_n\bar{x}.w\bar{x}) \cup \mathrm{hlr}(u_1) \cup \{\langle 1, n\rangle\} =^{6.13} \mathrm{hlr}(t)$.
    - Or $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{u_1}$. So $u' = (\lambda_n\bar{x}.wx)u_1'$ such that $u_1 \xrightarrow{p'}_{\beta\eta} u_1'$. By IH, $\mathrm{hlr}(u_1') \subseteq \mathrm{hlr}(u_1)$. So, $\mathrm{hlr}(u') =^{6.13} \mathrm{hlr}(\lambda_n\bar{x}.w\bar{x}) \cup \mathrm{hlr}(u_1') \cup \{\langle 1, n\rangle\} \subseteq \mathrm{hlr}(\lambda_n\bar{x}.w\bar{x}) \cup \mathrm{hlr}(u_1) \cup \{\langle 1, n\rangle\} =^{6.13} \mathrm{hlr}(u)$.

- Let $u = (\lambda_n\bar{x}.u_1[\bar{x} := c(c\bar{x})])u_2$. Because $u \xrightarrow{p}_{\beta\eta} u'$, by lemma 2.2.8 and lemma 2.5:
    - Either $p = 0$. So $u' = u_1[\bar{x} := c(cu_2)]$. By lemma 6.11, $\mathrm{hlr}(u') \subseteq \mathrm{hlr}(u)$.
    - Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{\lambda_n\bar{x}.u_1[\bar{x}:=c(c\bar{x})]}$. So $u' = u_1' u_2$ such that $\lambda_n\bar{x}.u_1[\bar{x} := c(c\bar{x})] \xrightarrow{p'}_{\beta\eta} u_1'$. By IH, $\mathrm{hlr}(u_1') \subseteq \mathrm{hlr}(\lambda_n\bar{x}.u_1[\bar{x} := c(c\bar{x})])$. By lemma 2.2.8, lemma 2.7.3, lemma 2.7.4 and lemma 2.4.12a, $p' = 1.p''$, $u_1' = \lambda_n\bar{x}.u_1''[\bar{x} := c(c\bar{x})]$ and $u_1 \xrightarrow{p''}_{\beta\eta} u_1''$. Then, $\mathrm{hlr}(u') =^{6.13} \mathrm{hlr}(u_1') \cup \mathrm{hlr}(u_2) \cup \{\langle 1, n\rangle\} \subseteq \mathrm{hlr}(\lambda_n\bar{x}.u_1[\bar{x} := c(c\bar{x})]) \cup \mathrm{hlr}(u_2) \cup \{\langle 1, n\rangle\} =^{6.13} \mathrm{hlr}(u)$.
    - Or $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{u_2}$. So $u' = (\lambda_n\bar{x}.u_1[\bar{x} := c(c\bar{x})])u_2'$ such that $u_2 \xrightarrow{p'}_{\beta\eta} u_2'$. By IH, $\mathrm{hlr}(u_2') \subseteq \mathrm{hlr}(u_2)$. So, $\mathrm{hlr}(u') =^{6.13} \mathrm{hlr}(\lambda_n\bar{x}.u_1[\bar{x} := c(c\bar{x})]) \cup \mathrm{hlr}(u_2') \cup \{\langle 1, n\rangle\} \subseteq \mathrm{hlr}(\lambda_n\bar{x}.u_1[\bar{x} := c(c\bar{x})]) \cup \mathrm{hlr}(u_2) \cup \{\langle 1, n\rangle\} =^{6.13} \mathrm{hlr}(u)$.

- Let $u = cu_1u_2$. Because $u \xrightarrow{p}_{\beta\eta} u'$, by lemma 2.2.8 and lemma 2.5:
    - Either $p = 1.2.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{u_1}$. So $u' = cu_1'u_2$ such that $u_1 \xrightarrow{p'}_{\beta\eta} u_1'$. By IH, $\mathrm{hlr}(u_1') \subseteq \mathrm{hlr}(u_1)$. So, $\mathrm{hlr}(u') =^{6.13} \mathrm{hlr}(u_1') \cup \mathrm{hlr}(u_2) \subseteq \mathrm{hlr}(u_1) \cup \mathrm{hlr}(u_2) =^{6.13} \mathrm{hlr}(u)$.
    - Or $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{u_2}$. So $u' = cu_1u_2'$ such that $u_2 \xrightarrow{p'}_{\beta\eta} u_2'$. By IH, $\mathrm{hlr}(u_2') \subseteq \mathrm{hlr}(u_2)$. So, $\mathrm{hlr}(u') =^{6.13} \mathrm{hlr}(u_1) \cup \mathrm{hlr}(u_2') \subseteq \mathrm{hlr}(u_1) \cup \mathrm{hlr}(u_2) =^{6.13} \mathrm{hlr}(u)$.

- Let $u = cu_1$. Because $u \xrightarrow{p}_{\beta\eta} u'$, by lemma 2.2.8 and lemma 2.5 $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{u_1}$. So $u' = cu'_1$ such that $u_1 \xrightarrow{p'}_{\beta\eta} u'_1$. By IH, $\mathrm{hlr}(u'_1) \subseteq \mathrm{hlr}(u_1)$. So, $\mathrm{hlr}(u') =^{6.13} \mathrm{hlr}(u'_1) \subseteq \mathrm{hlr}(u_1) =^{6.13} \mathrm{hlr}(u)$. $\square$

Lemma 6.11. $\mathrm{hlr}(u_1[\bar{x} := c(cu_2)]) \subseteq \mathrm{hlr}((\lambda_n \bar{x}.u_1[\bar{x} := c(c\bar{x})])u_2)$. $\square$

Proof. We prove the lemma by induction on the size of u_1 and then by case on the structure of u_1.

- Let $u_1 \in \mathcal{V}$. Either $u_1 = \bar{x}$ then $\mathrm{hlr}(u_1[\bar{x} := c(cu_2)]) = \mathrm{hlr}(c(cu_2)) =^{6.13} \mathrm{hlr}(u_2) \subseteq^{6.14} \mathrm{hlr}((\lambda_n \bar{x}.u_1[\bar{x} := c(c\bar{x})])u_2)$. Or $u_1 = y \neq \bar{x}$ then $\mathrm{hlr}(u_1[\bar{x} := c(cu_2)]) = \mathrm{hlr}(u_1) \subseteq^{6.14,6.13} \mathrm{hlr}((\lambda_n \bar{x}.u_1[\bar{x} := c(c\bar{x})])u_2)$.

- Let $u_1 = \lambda_m \bar{y}.u'_1[\bar{y} := c(c\bar{y})]$. Then $\mathrm{hlr}(u_1[\bar{x} := c(cu_2)]) = \mathrm{hlr}((\lambda_m \bar{y}.u'_1[\bar{y} := c(c\bar{y})])[\bar{x} := c(cu_2)]) = \mathrm{hlr}(\lambda_m \bar{y}.u'_1[\bar{x} := c(cu_2)][\bar{y} := c(c\bar{y})]) =^{6.13} \mathrm{hlr}(u'_1[\bar{x} := c(cu_2)]) \subseteq^{IH} \mathrm{hlr}((\lambda_n \bar{x}.u'_1[\bar{x} := c(c\bar{x})])u_2) =^{6.13} \mathrm{hlr}(u'_1) \cup \mathrm{hlr}(u_2) \cup \{\langle 1, n\rangle\} =^{6.13} \mathrm{hlr}(\lambda_m \bar{y}.u'_1[\bar{y} := c(c\bar{y})]) \cup \mathrm{hlr}(u_2) \cup \{\langle 1, n\rangle\} =^{6.13} \mathrm{hlr}((\lambda_n \bar{x}.u_1[\bar{x} := c(c\bar{x})])u_2)$ such that $\bar{y} \notin \mathrm{fv}(u_2) \cup \{\bar{x}\}$.

- Let $u_1 = \lambda_m \bar{y}.w\bar{y}$ such that $\bar{y} \notin \mathrm{fv}(w)$. Then, $\mathrm{hlr}(u_1[\bar{x} := c(cu_2)]) = \mathrm{hlr}(\lambda_m \bar{y}.(w\bar{y})[\bar{x} := c(cu_2)]) =^{6.13} \mathrm{hlr}((w\bar{y})[\bar{x} := c(cu_2)]) \cup \{\langle 2, m\rangle\} \subseteq^{IH} \mathrm{hlr}((\lambda_n \bar{x}.(w\bar{y})[\bar{x} := c(c\bar{x})])u_2) \cup \{\langle 2, m\rangle\} =^{6.13} \mathrm{hlr}(w\bar{y}) \cup \mathrm{hlr}(u_2) \cup \{\langle 1, n\rangle, \langle 2, m\rangle\} =^{6.13} \mathrm{hlr}((\lambda_n \bar{x}.(\lambda_m \bar{y}.w\bar{y})[\bar{x} := c(c\bar{x})])u_2)$ such that $\bar{y} \notin \mathrm{fv}(u_2) \cup \{\bar{x}\}$.

- Let $u_1 = cu'_1u''_1$. Then, $\mathrm{hlr}(u_1[\bar{x} := c(cu_2)]) = \mathrm{hlr}(cu'_1[\bar{x} := c(cu_2)]u''_1[\bar{x} := c(cu_2)]) =^{6.13} \mathrm{hlr}(u'_1[\bar{x} := c(cu_2)]) \cup \mathrm{hlr}(u''_1[\bar{x} := c(cu_2)]) \subseteq^{IH} \mathrm{hlr}((\lambda_n \bar{x}.u'_1[\bar{x} := c(c\bar{x})])u_2) \cup \mathrm{hlr}((\lambda_n \bar{x}.u''_1[\bar{x} := c(c\bar{x})])u_2) =^{6.13} \mathrm{hlr}(u'_1) \cup \mathrm{hlr}(u''_1) \cup \mathrm{hlr}(u_2) \cup \{\langle 1, n\rangle\} =^{6.13} \mathrm{hlr}((\lambda_n \bar{x}.(cu'_1u''_1)[\bar{x} := c(c\bar{x})])u_2)$.

- Let $u_1 = vu''_1$ (such that $v = \lambda_m \bar{y}.w\bar{y}$ and $\bar{y} \notin \mathrm{fv}(w)$ or $v = \lambda_m \bar{y}.u'_1[\bar{y} := c(c\bar{y})]$). Then, $\mathrm{hlr}(u_1[\bar{x} := c(cu_2)]) = \mathrm{hlr}(v[\bar{x} := c(cu_2)]u''_1[\bar{x} := c(cu_2)]) =^{6.13} \mathrm{hlr}(v[\bar{x} := c(cu_2)]) \cup \mathrm{hlr}(u''_1[\bar{x} := c(cu_2)]) \cup \{\langle 1, m\rangle\} \subseteq^{IH} \mathrm{hlr}((\lambda_n \bar{x}.v[\bar{x} := c(c\bar{x})])u_2) \cup \mathrm{hlr}((\lambda_n \bar{x}.u''_1[\bar{x} := c(c\bar{x})])u_2) \cup \{\langle 1, m\rangle\} =^{6.13} \mathrm{hlr}(v) \cup \mathrm{hlr}(u''_1) \cup \mathrm{hlr}(u_2) \cup \{\langle 1, n\rangle, \langle 1, m\rangle\} =^{6.13} \mathrm{hlr}((\lambda_n \bar{x}.(vu''_1)[\bar{x} := c(c\bar{x})])u_2)$.

- Let $u_1 = cu'_1$. Then, $\mathrm{hlr}(u_1[\bar{x} := u_2]) = \mathrm{hlr}(cu'_1[\bar{x} := c(cu_2)]) =^{6.13} \mathrm{hlr}(u'_1[\bar{x} := c(cu_2)]) \subseteq^{IH} \mathrm{hlr}((\lambda_n \bar{x}.u'_1[\bar{x} := c(c\bar{x})])u_2) =^{6.13} \mathrm{hlr}(u'_1) \cup \mathrm{hlr}(u_2) \cup \{\langle 1, n\rangle\} =^{6.13} \mathrm{hlr}((\lambda_n \bar{x}.(cu'_1)[\bar{x} := c(c\bar{x})])u_2)$. $\square$

Lemma 6.12. *If* $p \in \mathcal{R}^{\beta\eta}_t$ *then* $\mathrm{headlam}(t|_p[\bar{x} := c(c\bar{x})]) = \mathrm{headlam}(t|_p)$. $\square$

Proof. We prove this lemma by induction on the structure of t.

- Let $t \in \mathcal{V}$ then by lemma 2.5, $\mathcal{R}^{\beta\eta}_t = \varnothing$.
- Let $t = \lambda_n y.t'$ then by lemma 2.5:
  - Either $p = 0$ if $t' = t''y$ and $y \notin \mathrm{fv}(t'')$. Then $\mathrm{headlam}(t|_p[\bar{x} := c(c\bar{x})]) = \mathrm{headlam}(t[\bar{x} := c(c\bar{x})]) = \mathrm{headlam}(\lambda_n y.t''[\bar{x} := c(c\bar{x})]y) = \langle 2, n\rangle = \mathrm{headlam}(t)$ such that $y \notin \{c, \bar{x}\}$.

- Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{t'}$. Then $\text{headlam}(t|_p[\bar{x} := c(c\bar{x})]) = \text{headlam}(t'|_{p'}[\bar{x} := c(c\bar{x})]) =^{IH} \text{headlam}(t'|_{p'}) = \text{headlam}(t|_p)$.

- Let $t = t_1 t_2$ then by lemma 2.5:
  - Either $p = 0$ if $t_1 = \lambda_n y.t_0$. Then $\text{headlam}(t|_p[\bar{x} := c(c\bar{x})]) = \text{headlam}(t[\bar{x} := c(c\bar{x})]) = \text{headlam}((\lambda_n y.t_0[\bar{x} := c(c\bar{x})])t_2[\bar{x} := c(c\bar{x})]) = \langle 1, n\rangle = \text{headlam}(t)$ such that $y \notin \{c, \bar{x}\}$.
  - Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{t_1}$. Then $\text{headlam}(t|_p[\bar{x} := c(c\bar{x})]) = \text{headlam}(t_1|_{p'}[\bar{x} := c(c\bar{x})]) =^{IH} \text{headlam}(t_1|_{p'}) = \text{headlam}(t|_p)$.
  - Or $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{t_2}$. Then $\text{headlam}(t|_p[\bar{x} := c(c\bar{x})]) = \text{headlam}(t_2|_{p'}[\bar{x} := c(c\bar{x})]) =^{IH} \text{headlam}(t_2|_{p'}) = \text{headlam}(t|_p)$.

$\square$

Lemma 6.13. *Let $t \in \bar{\Lambda}$ and $\mathcal{F} \subseteq \mathcal{R}^{\beta\eta}_t$.*

- *If $t = x$ then* $\text{headlamred}(t, \mathcal{F}) = \text{hlr}(t) = \varnothing$.
- *If $t = \lambda_n x.t_1$ then if $t \in \mathcal{R}^{\beta\eta}$ then* $\text{hlr}(t) = \text{hlr}(t_1) \cup \{\langle 2, n\rangle\}$ *else* $\text{hlr}(t) = \text{hlr}(t_1)$.
- *If $t = \lambda_n x.t_1$ and $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\}$ then if $0 \in \mathcal{F}$ then* $\text{headlamred}(t, \mathcal{F}) = \text{headlamred}(t_1, \mathcal{F}_1) \cup \{\langle 2, n\rangle\}$ *else* $\text{headlamred}(t, \mathcal{F}) = \text{headlamred}(t_1, \mathcal{F}_1)$.
- *If $t = t_1 t_2$ then if $t \in \mathcal{R}^{\beta\eta}$ then* $\text{hlr}(t) = \text{hlr}(t_1) \cup \text{hlr}(t_2) \cup \{\text{headlam}(t)\}$ *else* $\text{hlr}(t) = \text{hlr}(t_1) \cup \text{hlr}(t_2)$.
- *If $t = t_1 t_2$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\}$ then if $0 \in \mathcal{F}$ then* $\text{headlamred}(t, \mathcal{F}) = \text{headlamred}(t_1, \mathcal{F}_1) \cup \text{headlamred}(t_2, \mathcal{F}_2) \cup \{\text{headlam}(t)\}$ *else* $\text{headlamred}(t, \mathcal{F}) = \text{headlamred}(t_1, \mathcal{F}_1) \cup \text{headlamred}(t_2, \mathcal{F}_2)$.
- *If $t = \lambda_n \bar{x}.t_1[\bar{x} := c(c\bar{x})]$ then* $\text{hlr}(t) = \text{hlr}(t_1)$.
- *If $t = c^n(t_1)$, then* $\text{hlr}(t) = \text{hlr}(t_1)$. $\square$

Proof. By definition $\text{hlr}(t) = \{\langle i, n\rangle \mid \exists p \in \mathcal{R}^{\beta\eta}_t.\ \text{headlam}(t|_p) = \langle i, n\rangle\}$ and $\text{headlamred}(t, \mathcal{F}) = \{\langle i, n\rangle \mid \exists p \in \mathcal{F}.\ \text{headlam}(t|_p) = \langle i, n\rangle\}$. We prove the frist three items of this lemma by induction on the size of t and then by case on the structure of t.

- Let $t = x$. By lemma 2.5, $\mathcal{F} = \mathcal{R}^{\beta\eta}_x = \varnothing$, then $\text{headlamred}(x, \mathcal{F}) = \text{hlr}(x) = \varnothing$.
- Let $t = \lambda_n x.t_1$.
  - Let $t \in \mathcal{R}^{\beta\eta}$ then $t_1 = t_0 x$ such that $x \notin \text{fv}(t_0)$.
    * Let $\langle j, m\rangle \in \text{hlr}(t)$ then there exists $p \in \mathcal{R}^{\beta\eta}_t$ such that $\text{headlam}(t|_p) = \langle j, m\rangle$. By lemma 2.5:

- $\cdot$ Either $p = 0$, so $\langle j, m\rangle = \text{headlam}(t|_0) = \text{headlam}(t) = \langle 2, n\rangle$.
- $\cdot$ Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{t_1}$. Then, $\langle j, m\rangle = \text{headlam}(t|_p) = \text{headlam}(t_1|_{p'})$. So $\langle j, m\rangle \in \text{hlr}(t_1)$.

$*$ Let $\langle j, m\rangle \in \text{hlr}(t_1) \cup \{\langle 2, n\rangle\}$.

- $\cdot$ Either $\langle j, m\rangle \in \text{hlr}(t_1)$. Then there exists $p \in \mathcal{R}^{\beta\eta}_{t_1}$ such that $\text{headlam}(t_1|_p) = \langle j, m\rangle$. By lemma 2.5, $1.p \in \mathcal{R}^{\beta\eta}_t$ and $\langle j, m\rangle = \text{headlam}(t_1|_p) = \text{headlam}(t|_{1.p})$. So $\langle j, m\rangle \in \text{hlr}(t)$.
- $\cdot$ Or $\langle j, m\rangle = \langle 2, n\rangle$. By lemma 2.5, $0 \in \mathcal{R}^{\beta\eta}_t$ and $\text{headlam}(t|_0) = \text{headlam}(t) = \langle 2, n\rangle$. So $\langle j, m\rangle \in \text{hlr}(t)$.

– Let $t \notin \mathcal{R}^{\beta\eta}$.

$*$ Let $\langle j, m\rangle \in \text{hlr}(t)$ then there exists $p \in \mathcal{R}^{\beta\eta}_t$ such that $\text{headlam}(t|_p) = \langle j, m\rangle$. By lemma 2.5, $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{t_1}$. Then, $\langle j, m\rangle = \text{headlam}(t|_p) = \text{headlam}(t_1|_{p'})$. So $\langle j, m\rangle \in \text{hlr}(t_1)$.

$*$ Let $\langle j, m\rangle \in \text{hlr}(t_1)$ then there exists $p \in \mathcal{R}^{\beta\eta}_{t_1}$ such that $\text{headlam}(t_1|_p) = \langle j, m\rangle$. By lemma 2.5, $1.p \in \mathcal{R}^{\beta\eta}_t$ and $\langle j, m\rangle = \text{headlam}(t_1|_p) = \text{headlam}(t|_{1.p})$. So $\langle j, m\rangle \in \text{hlr}(t)$.

- Let $t = \lambda_n x.t_1$ and $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\}$.

– Let $0 \in \mathcal{F}$ then $t \in \mathcal{R}^{\beta\eta}$.

$*$ Let $\langle j, m\rangle \in \text{headlamred}(t, \mathcal{F})$ then there exists $p \in \mathcal{F}$ such that $\text{headlam}(t|_p) = \langle j, m\rangle$. By lemma 2.6:

- $\cdot$ Either $p = 0$, so $\langle j, m\rangle = \text{headlam}(t|_0) = \text{headlam}(t) = \langle 2, n\rangle$.
- $\cdot$ Or $p = 1.p'$ such that $p' \in \mathcal{F}_1$. Then, $\langle j, m\rangle = \text{headlam}(t|_p) = \text{headlam}(t_1|_{p'})$. So $\langle j, m\rangle \in \text{headlamred}(t_1, \mathcal{F}_1)$.

$*$ Let $\langle j, m\rangle \in \text{headlamred}(t_1, \mathcal{F}_1) \cup \{\langle 2, n\rangle\}$.

- $\cdot$ Either $\langle j, m\rangle \in \text{headlamred}(t_1, \mathcal{F}_1)$. Then there exists $p \in \mathcal{F}_1$ such that $\text{headlam}(t_1|_p) = \langle j, m\rangle$. So, $1.p \in \mathcal{F}$ and $\langle j, m\rangle = \text{headlam}(t_1|_p) = \text{headlam}(t|_{1.p})$. Hence, $\langle j, m\rangle \in \text{headlamred}(t, \mathcal{F})$.
- $\cdot$ Or $\langle j, m\rangle = \langle 2, n\rangle$. Because $0 \in \mathcal{F}$ and $\text{headlam}(t|_0) = \text{headlam}(t) = \langle 2, n\rangle$ then $\langle j, m\rangle \in \text{headlamred}(t, \mathcal{F})$.

– Let $0 \notin \mathcal{F}$.

$*$ Let $\langle j, m\rangle \in \text{headlamred}(t, \mathcal{F})$ then there exists $p \in \mathcal{F}$ such that $\text{headlam}(t|_p) = \langle j, m\rangle$. By lemma 2.6, $p = 1.p'$ such that $p' \in \mathcal{F}_1$. Then, $\langle j, m\rangle = \text{headlam}(t|_p) = \text{headlam}(t_1|_{p'})$. So $\langle j, m\rangle \in \text{headlamred}(t_1, \mathcal{F}_1)$.

$*$ Let $\langle j, m\rangle \in \text{headlamred}(t_1, \mathcal{F}_1)$ then there exists $p \in \mathcal{F}_1$ such that $\text{headlam}(t_1|_p) = \langle j, m\rangle$. By lemma 2.6, $1.p \in \mathcal{F}$ and $\langle j, m\rangle = \text{headlam}(t_1|_p) = \text{headlam}(t|_{1.p})$. So $\langle j, m\rangle \in \text{headlamred}(t, \mathcal{F})$.

- Let $t = t_1 t_2$.
  - Let $t \in \mathcal{R}^{\beta\eta}$ then $t_1 = \lambda_n x.t_0$. So $\langle 1, n\rangle = \text{headlam}(t)$.
    * Let $\langle j, m\rangle \in \text{hlr}(t)$ then there exists $p \in \mathcal{R}_t^{\beta\eta}$ such that $\text{headlam}(t|_p) = m$. By lemma 2.5:
      · Either $p = 0$, so $\langle j, m\rangle = \text{headlam}(t|_0) = \text{headlam}(t) = \langle 1, n\rangle$.
      · Or $p = 1.p'$ such that $p' \in \mathcal{R}_{t_1}^{\beta\eta}$. Then, $\langle j, m\rangle = \text{headlam}(t|_p) = \text{headlam}(t_1|_{p'})$. So $\langle j, m\rangle \in \text{hlr}(t_1)$.
      · Or $p = 2.p'$ such that $p' \in \mathcal{R}_{t_2}^{\beta\eta}$. Moreover, $\langle j, m\rangle = \text{headlam}(t|_p) = \text{headlam}(t_2|_{p'})$. So $\langle j, m\rangle \in \text{hlr}(t_2)$.
    * Let $\langle j, m\rangle \in \text{hlr}(t_1) \cup \text{hlr}(t_2) \cup \{\langle 1, n\rangle\}$.
      · Either $\langle j, m\rangle \in \text{hlr}(t_1)$. Then there exists $p \in \mathcal{R}_{t_1}^{\beta\eta}$ such that $\text{headlam}(t_1|_p) = \langle j, m\rangle$. By lemma 2.5, $1.p \in \mathcal{R}_t^{\beta\eta}$ and $\langle j, m\rangle = \text{headlam}(t_1|_p) = \text{headlam}(t|_{1.p})$. So $\langle j, m\rangle \in \text{hlr}(t)$.
      · Or $\langle j, m\rangle \in \text{hlr}(t_2)$. Then there exists $p \in \mathcal{R}_{t_2}^{\beta\eta}$ such that $\text{headlam}(t_2|_p) = \langle j, m\rangle$. By lemma 2.5, $2.p \in \mathcal{R}_t^{\beta\eta}$ and $\langle j, m\rangle = \text{headlam}(t_2|_p) = \text{headlam}(t|_{2.p})$. So $\langle j, m\rangle \in \text{hlr}(t)$.
      · Or $\langle j, m\rangle = \langle 1, n\rangle$. By lemma 2.5, $0 \in \mathcal{R}_t^{\beta\eta}$ and $\text{headlam}(t|_0) = \text{headlam}(t) = \langle 1, n\rangle$. So $\langle j, m\rangle \in \text{hlr}(t)$.
  - Let $t \notin \mathcal{R}^{\beta\eta}$.
    * Let $\langle j, m\rangle \in \text{hlr}(t)$ then there exists $p \in \mathcal{R}_t^{\beta\eta}$ such that $\text{headlam}(t|_p) = \langle j, m\rangle$. By lemma 2.5:
      · Either $p = 1.p'$ such that $p' \in \mathcal{R}_{t_1}^{\beta\eta}$. Moreover, $\langle j, m\rangle = \text{headlam}(t|_p) = \text{headlam}(t_1|_{p'})$. So $\langle j, m\rangle \in \text{hlr}(t_1)$.
      · Or $p = 2.p'$ such that $p' \in \mathcal{R}_{t_2}^{\beta\eta}$. Moreover, $\langle j, m\rangle = \text{headlam}(t|_p) = \text{headlam}(t_2|_{p'})$. So $\langle j, m\rangle \in \text{hlr}(t_2)$.
    * Let $\langle j, m\rangle \in \text{hlr}(t_1) \cup \text{hlr}(t_2)$.
      · Either $\langle j, m\rangle \in \text{hlr}(t_1)$. Then there exists $p \in \mathcal{R}_{t_1}^{\beta\eta}$ such that $\text{headlam}(t_1|_p) = \langle j, m\rangle$. By lemma 2.5, $1.p \in \mathcal{R}_t^{\beta\eta}$ and $\langle j, m\rangle = \text{headlam}(t_1|_p) = \text{headlam}(t|_{1.p})$. So $\langle j, m\rangle \in \text{hlr}(t)$.
      · Or $\langle j, m\rangle \in \text{hlr}(t_2)$. Then there exists $p \in \mathcal{R}_{t_2}^{\beta\eta}$ such that $\text{headlam}(t_2|_p) = \langle j, m\rangle$. By lemma 2.5, $2.p \in \mathcal{R}_t^{\beta\eta}$ and $\langle j, m\rangle = \text{headlam}(t_2|_p) = \text{headlam}(t|_{2.p})$. So $\langle j, m\rangle \in \text{hlr}(t)$.
- Let $t = t_1 t_2$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\}$.
  - Let $0 \in \mathcal{F}$ then $t \in \mathcal{R}^{\beta\eta}$.
    * Let $\langle j, m\rangle \in \text{headlamred}(t, \mathcal{F})$ then there exists $p \in \mathcal{F}$ such that $\text{headlam}(t|_p) = m$. By lemma 2.6:
      · Either $p = 0$, so $\langle j, m\rangle = \text{headlam}(t|_0) = \text{headlam}(t)$.

· Or $p = 1.p'$ such that $p' \in \mathcal{F}_1$. Then, $\langle j, m\rangle = \text{headlam}(t|_p) = \text{headlam}(t_1|_{p'})$. So $\langle j, m\rangle \in \text{headlamred}(t_1, \mathcal{F}_1)$.

· Or $p = 2.p'$ such that $p' \in \mathcal{F}_2$. Then, $\langle j, m\rangle = \text{headlam}(t|_p) = \text{headlam}(t_2|_{p'})$. So $\langle j, m\rangle \in \text{headlamred}(t_2, \mathcal{F}_2)$.

∗ Let $\langle j, m\rangle \in \text{headlamred}(t_1, \mathcal{F}_1) \cup \text{headlamred}(t_2, \mathcal{F}_2) \cup \{\text{headlam}(t)\}$.

· Either $\langle j, m\rangle \in \text{headlamred}(t_1, \mathcal{F}_1)$. Then there exists $p \in \mathcal{F}_1$ such that $\text{headlam}(t_1|_p) = \langle j, m\rangle$. So, $1.p \in \mathcal{F}$ and $\langle j, m\rangle = \text{headlam}(t_1|_p) = \text{headlam}(t|_{1.p})$. Hence, $\langle j, m\rangle \in \text{headlamred}(t, \mathcal{F})$.

· Or $\langle j, m\rangle \in \text{headlamred}(t_2, \mathcal{F}_2)$. Then there exists $p \in \mathcal{F}_2$ such that $\text{headlam}(t_2|_p) = \langle j, m\rangle$. So, $2.p \in \mathcal{F}$ and $\langle j, m\rangle = \text{headlam}(t_2|_p) = \text{headlam}(t|_{2.p})$. Hence, $\langle j, m\rangle \in \text{headlamred}(t, \mathcal{F})$.

· Or $\langle j, m\rangle = \text{headlam}(t)$. Because $0 \in \mathcal{F}$ and $\text{headlam}(t|_0) = \text{headlam}(t)$, then $\langle j, m\rangle \in \text{headlamred}(t, \mathcal{F})$.

– Let $0 \notin \mathcal{F}$.

∗ Let $\langle j, m\rangle \in \text{headlamred}(t, \mathcal{F})$ then there exists $p \in \mathcal{F}$ such that $\text{headlam}(t|_p) = \langle j, m\rangle$. By lemma 2.6:

· Either $p = 1.p'$ such that $p' \in \mathcal{F}_1$. Moreover, $\langle j, m\rangle = \text{headlam}(t|_p) = \text{headlam}(t_1|_{p'})$. So $\langle j, m\rangle \in \text{headlamred}(t_1, \mathcal{F}_1)$.

· Or $p = 2.p'$ such that $p' \in \mathcal{F}_2$. Moreover, $\langle j, m\rangle = \text{headlam}(t|_p) = \text{headlam}(t_2|_{p'})$. So $\langle j, m\rangle \in \text{headlamred}(t_2, \mathcal{F}_2)$.

∗ Let $\langle j, m\rangle \in \text{headlamred}(t_1, \mathcal{F}_1) \cup \text{headlamred}(t_2, \mathcal{F}_2)$.

· Either $\langle j, m\rangle \in \text{headlamred}(t_1, \mathcal{F}_1)$. Then there exists $p \in \mathcal{F}_1$ such that $\text{headlam}(t_1|_p) = \langle j, m\rangle$. So, $1.p \in \mathcal{F}$ and $\langle j, m\rangle = \text{headlam}(t_1|_p) = \text{headlam}(t|_{1.p})$. Hence, $\langle j, m\rangle \in \text{headlamred}(t, \mathcal{F})$.

· Or $\langle j, m\rangle \in \text{headlamred}(t_2, \mathcal{F}_2)$. Then there exists $p \in \mathcal{F}_2$ such that $\text{headlam}(t_2|_p) = \langle j, m\rangle$. So, $2.p \in \mathcal{F}$ and $\langle j, m\rangle = \text{headlam}(t_2|_p) = \text{headlam}(t|_{2.p})$. Hence, $\langle j, m\rangle \in \text{headlamred}(t, \mathcal{F})$.

Let $t = \lambda_n \bar{x}.t_1[\bar{x} := c(c\bar{x})]$.

- Let $\langle j, m\rangle \in \text{hlr}(t)$ then there exists $p \in \mathcal{R}^{\beta\eta}_t$ such that $\text{headlam}(t|_p) = \langle j, m\rangle$. By lemma 2.7.3 and lemma 2.7.4, $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{t_1}$. Moreover, $\langle j, m\rangle = \text{headlam}(t|_p) = \text{headlam}(t_1[\bar{x} := c(c\bar{x})]|_{p'}) =^{2.7.2} \text{headlam}(t_1|_{p'}[\bar{x} := c(c\bar{x})]) =^{6.12} \text{headlam}(t_1|_{p'})$. So $\langle j, m\rangle \in \text{hlr}(t_1)$.

- Let $\langle j, m\rangle \in \text{hlr}(t_1)$ then there exists $p \in \mathcal{R}^{\beta\eta}_{t_1}$ such that $\text{headlam}(t_1|_p) = \langle j, m\rangle$. By lemma 2.7.3 and lemma 2.7.4, $1.p \in \mathcal{R}^{\beta\eta}_t$. Moreover, $\langle j, m\rangle = \text{headlam}(t_1|_p) =^{6.12} \text{headlam}(t_1|_p[\bar{x} := c(c\bar{x})]) =^{2.7.2} \text{headlam}(t_1[\bar{x} := c(c\bar{x})]|_p) = \text{headlam}(t|_{1.p})$. So $\langle j, m\rangle \in \text{hlr}(t)$.

Let $t = c^n(t_1)$. We prove that $\text{hlr}(t) = \text{hlr}(t_1)$ by induction on n.

- Let $n = 0$ then it is done.

- Let $n = m + 1$ such that $m \geq 0$ then $\mathrm{hlr}(t) =^{6.13} \mathrm{hlr}(c^m(t_1)) =^{IH} \mathrm{hlr}(t_1)$.

$\square$

Lemma 6.14. *If* $t_1 \subseteq t_2$ *then* $\mathrm{hlr}(t_1) \subseteq \mathrm{hlr}(t_2)$. $\square$

Proof. We prove the lemma by induction on the structure of t_2.

- Let $t_2 = x$, then it is done because by definition $t_1 = x$.
- Let $t_2 = \lambda_n x.t_0$ then by definition:
  - Either $t_1 = t_2$ so it is done.
  - Or $t_1 \subseteq t_0$. Then $\mathrm{hlr}(t_1) \subseteq^{IH} \mathrm{hlr}(t_0) \subseteq^{6.13} \mathrm{hlr}(t_2)$.
- Let $t_2 = t_3 t_4$ then by definition:
  - Either $t_1 = t_2$ so it is done.
  - Or $t_1 \subseteq t_3$. Then $\mathrm{hlr}(t_1) \subseteq^{IH} \mathrm{hlr}(t_3) \subseteq^{6.13} \mathrm{hlr}(t_2)$.
  - Or $t_1 \subseteq t_4$. Then $\mathrm{hlr}(t_1) \subseteq^{IH} \mathrm{hlr}(t_4) \subseteq^{6.13} \mathrm{hlr}(t_2)$.

$\square$

Lemma 6.15. *Let* $M \in \Lambda$, *where* $c \notin \mathrm{fv}(M)$, *and* $\mathcal{F} \subseteq \mathcal{R}^{\beta\eta}_M$. *Then:*

$$\langle M, \mathcal{F}\rangle \to^*_{\beta\eta d} \langle M', \mathcal{F}'\rangle \iff \exists N \in \Psi^c(M, \mathcal{F}).\ \exists N' \in \Psi^c(M', \mathcal{F}').\ N \to^*_{\beta\eta} N'$$

and

$$\langle M, \mathcal{F}\rangle \to^*_{\beta\eta d} \langle M', \mathcal{F}'\rangle \iff \forall N \in \Psi^c(M, \mathcal{F}).\ \exists N' \in \Psi^c(M', \mathcal{F}').\ N \to^*_{\beta\eta} N'$$

$\square$

Lemma 6.16. *Let* $M \in \Lambda$, *such that* $c \notin \mathrm{fv}(M)$ *and* $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{R}^{\beta\eta}_M$. *If* $\langle M, \mathcal{F}_1\rangle \to_{\beta\eta d} \langle M', \mathcal{F}'_1\rangle$ *then there exists* $\mathcal{F}'_2 \subseteq \mathcal{R}^{\beta\eta}_{M'}$ *such that* $\mathcal{F}_1 \subseteq \mathcal{F}'_2$ *and* $\langle M, \mathcal{F}_2\rangle \to_{\beta\eta d} \langle M', \mathcal{F}'_2\rangle$. $\square$

Lemma 6.17 (confluence of the $\beta\eta$-developments)**.** *Let* $M \in \Lambda$ *such that* $c \notin \mathrm{fv}(M)$. *If* $M \xrightarrow{\mathcal{F}_1}_{\beta\eta d} M_1$ *and* $M \xrightarrow{\mathcal{F}_2}_{\beta\eta d} M_2$, *then there exist* $\mathcal{F}'_1 \subseteq \mathcal{R}^{\beta\eta}_{M_1}$, $\mathcal{F}'_2 \subseteq \mathcal{R}^{\beta\eta}_{M_2}$ *and* $M_3 \in \Lambda$ *such that* $M_1 \xrightarrow{\mathcal{F}'_1}_{\beta\eta d} M_3$ *and* $M_2 \xrightarrow{\mathcal{F}'_2}_{\beta\eta d} M_3$. $\square$

Notation 6.18. Let $c \notin \mathrm{fv}(M)$. $M \to_1 M' \iff \exists \mathcal{F}, \mathcal{F}', \langle M, \mathcal{F}\rangle \to^*_{\beta\eta d} \langle M', \mathcal{F}'\rangle$. $\square$

Lemma 6.19. *Let* $c \notin \mathrm{fv}(M)$. $\forall P \in \Psi^c(M, \varnothing)$. $\mathcal{R}^{\beta\eta}_P = \varnothing$. $\square$

Lemma 6.20. *Let* $c \notin \mathrm{fv}(M) \cup \mathrm{fv}(N)$ *and* $x \neq c$. $\forall P \in \Psi^c(M, \varnothing)$. $\forall Q \in \Psi^c(N, \varnothing)$. $\mathcal{R}^{\beta\eta}_{P[x:=Q]} = \varnothing$. $\square$

Lemma 6.21. *Let* $c \notin \mathrm{fv}(M)$. *If* $p \in \mathcal{R}^{\beta\eta}_M$, $P \in \Psi^c(M, \{p\})$ *and* $P \to_{\beta\eta} Q$ *then* $\mathcal{R}^{\beta\eta}_Q = \varnothing$. $\square$

Lemma 6.22. *Let* $c \notin \mathrm{fv}(M)$. *If* $M \xrightarrow{p}_{\beta\eta} M'$ *then* $\langle M, \{p\}\rangle \to_{\beta\eta d} \langle M', \varnothing\rangle$. $\square$

Lemma 6.23. $\to^*_{\beta\eta} = \to^*_1$. $\square$

Lemma 6.24. $\Lambda \subseteq \mathsf{CR}^{\beta\eta}$. $\square$

7 Conclusion

Reducibility is a powerful method and has been applied to prove using a single method, a number of properties of the λ-calculus (Church-Rosser, strong normalisation, etc.). This paper studied two reducibility methods which exploit the passage from typed (in an intersection type system) to untyped terms. We showed that a first method given by Ghilezan and Likavec [GL02] fails in its aim and we have only been able to provide a partial solution. We adapted a second method given by Koletsos and Stavrinos [KS08] from β to βI-reduction and we generalised it to $\beta\eta$-reduction. There are differences in the type systems chosen and the methods of reducibility used by Ghilezan and Likavec on one side and by Koletsos and Stavrinos on the other. Koletsos and Stavrinos use system D [Kri90], which has elimination rules for intersection types whereas Ghilezan and Likavec use $\lambda\cap$ and $\lambda\cap^{\Omega}$ with subtyping. Moreover, Koletsos and Stavrinos's method depends on the inclusion of typable λ-terms in the set of λ-terms possessing the Church-Rosser property, whereas Ghilezan and Likavec's method (the working part of their method) is to prove the inclusion of typable terms in an arbitrary subset of the untyped λ-calculus closed by some properties. Moreover, Ghilezan and Likavec consider the $\mathrm{VAR}(\mathcal{P})$, $\mathrm{SAT}(\mathcal{P})$ and $\mathrm{CLO}(\mathcal{P})$ predicates whereas Koletsos and Stavrinos use standard reducibility methods through saturated sets. Koletsos and Stavrinos prove the confluence of developments using the confluence of typable λ-terms in system D (the authors prove that even a simple type system is sufficient). The advantage of Koletsos and Stavrinos's proof of confluence of developments is that strong normalisation is not needed.

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Contents

A Proofs of section 2

Lemma 2.2 . 1 We prove the lemma by induction on p.

- Let $p = 0$.

  Let $M \xrightarrow{0}_{\beta\eta} M'$ then either $M = (\lambda x.P)Q$ and $M' = P[x := Q]$ and so $M \xrightarrow{0}_{\beta} M'$. Or $M = \lambda x.M'x$ such that $x \notin \mathrm{fv}(M')$ and so $M \xrightarrow{0}_{\eta} M'$.

  Let $M \to_{\eta} 0M'$ then $M = \lambda x.M'x$ such that $x \notin \mathrm{fv}(M')$ and so $M \xrightarrow{0}_{\beta\eta} M'$.

  Let $M \to_{\beta} 0M'$ then $M = (\lambda x.P)Q$ and $M' = P[x := Q]$ and so $M \xrightarrow{0}_{\beta\eta} M'$.

- Let $p = 1.p'$.

  Let $M \xrightarrow{p}_{\beta\eta} M'$ then either $M = \lambda x.N$, $M' = \lambda x.N'$ and $N \xrightarrow{p'}_{\beta\eta} N'$. By IH, $N \xrightarrow{p}_{\beta} N'$ or $N \xrightarrow{p'}_{\eta} N'$. So $M \xrightarrow{p}_{\beta} M'$ or $M \xrightarrow{p}_{\eta} M'$. Or $M = PQ$, $M' = P'Q$ and $P \xrightarrow{p'}_{\beta\eta} P'$. By IH, $P \xrightarrow{p}_{\beta} P'$ or $P \xrightarrow{p'}_{\eta} P'$. So $M \xrightarrow{p}_{\beta} M'$ or $M \xrightarrow{p}_{\eta} M'$.

  Let $M \xrightarrow{p}_{\eta} M'$ then either $M = \lambda x.N$, $M' = \lambda x.N'$ and $N \xrightarrow{p'}_{\eta} N'$. By IH, $N \xrightarrow{p}_{\beta\eta} N'$, so $M \xrightarrow{p}_{\beta\eta} M'$. Or $M = PQ$, $M' = P'Q$ and $P \xrightarrow{p'}_{\eta} P'$. By IH, $P \xrightarrow{p}_{\beta\eta} P'$, so $M \xrightarrow{p}_{\beta\eta} M'$.

  Let $M \xrightarrow{p}_{\beta} M'$ then either $M = \lambda x.N$, $M' = \lambda x.N'$ and $N \xrightarrow{p'}_{\beta} N'$. By IH, $N \xrightarrow{p}_{\beta\eta} N'$, so $M \xrightarrow{p}_{\beta\eta} M'$. Or $M = PQ$, $M' = P'Q$ and $P \xrightarrow{p'}_{\beta} P'$. By IH, $P \xrightarrow{p}_{\beta\eta} P'$, so $M \xrightarrow{p}_{\beta\eta} M'$.

- Let $p = 2.p'$.

  Let $M \xrightarrow{p}_{\beta\eta} M'$ then $M = PQ$, $M' = PQ'$ and $Q \xrightarrow{p'}_{\beta\eta} Q'$. By IH, $Q \xrightarrow{p}_{\beta} Q'$ or $Q \xrightarrow{p'}_{\eta} Q'$. So $M \xrightarrow{p}_{\beta} M'$ or $M \xrightarrow{p}_{\eta} M'$.

  Let $M \xrightarrow{p}_{\eta} M'$ then $M = PQ$, $M' = PQ'$ and $Q \xrightarrow{p'}_{\eta} Q'$. By IH, $Q \xrightarrow{p}_{\beta\eta} Q'$, so $M \xrightarrow{p}_{\beta\eta} M'$.

  Let $M \xrightarrow{p}_{\beta} M'$ then $M = PQ$, $M' = PQ'$ and $Q \xrightarrow{p'}_{\beta} Q'$. By IH, $Q \xrightarrow{p}_{\beta\eta} Q'$, so $M \xrightarrow{p}_{\beta\eta} M'$.

2 We prove this lemma by induction on the structure of M_1.

- Either $M_1 = x$, then $\mathrm{fv}((\lambda x.M_1)M_2) = \mathrm{fv}(M_2) = \mathrm{fv}(M_1[x := M_2])$. If $(\lambda x.M_1)M_2 \in \Lambda\mathrm{I}$ then $M_2 = M_1[x := M_2] \in \Lambda\mathrm{I}$.
- Or $M_1 = \lambda y.M_0$ then $\mathrm{fv}((\lambda x.\lambda y.M_0)M_2) = \mathrm{fv}((\lambda x.M_0)M_2)\setminus\{y\} =^{IH} \mathrm{fv}(M_0[x := M_2])\setminus\{y\} = \mathrm{fv}(M_1[x := M_2])$ such that $y \notin \mathrm{fv}(M_2)\cup\{x\}$. If $(\lambda x.\lambda y.M_0)M_2 \in \Lambda\mathrm{I}$ then $M_0, M_2 \in \Lambda\mathrm{I}$ and $x, y \in \mathrm{fv}(M_0)$. So

$(\lambda x.M_0)M_2 \in \Lambda\mathrm{I}$. By IH, $M_0[x := M_2] \in \Lambda\mathrm{I}$. Hence, $M_1[x := M_2] \in \Lambda\mathrm{I}$ such that $y \notin \mathrm{fv}(M_2) \cup \{x\}$.

- Or $M_1 = PQ$ then $\mathrm{fv}((\lambda x.PQ)M_2) = \mathrm{fv}(\lambda x.P)M_2 \cup \mathrm{fv}((\lambda x.Q)M_2) =^{IH} \mathrm{fv}(P[x := M_2]) \cup \mathrm{fv}(Q[x := M_2]) = \mathrm{fv}((PQ)[x := M_2])$.

3. We prove the lemma by induction on the length of the reduction $M \to^*_{\beta\eta} M'$.

   - If $M = M'$ then $\mathrm{fv}(M) = \mathrm{fv}(M')$
   - Let $M \to^*_{\beta\eta} M'' \to_{\beta\eta} M'$. By IH, $\mathrm{fv}(M) \subseteq \mathrm{fv}(M'')$. By definition there exists p such that $M'' \xrightarrow{p}_{\beta\eta} M'$. We prove that $\mathrm{fv}(M'') \subseteq \mathrm{fv}(M')$ by induction on p.
     * Let $p = 0$.
       · either $M'' = (\lambda x.M_1)M_2$ and $M' = M_1[x := M_2]$. We prove that $\mathrm{fv}(M') \subseteq (\mathrm{fv}(M_1)\setminus\{x\}) \cup \mathrm{fv}(M_2) = \mathrm{fv}(M'')$ by induction on the structure of M_1.
         1. Let $M_1 = y$. If $y = x$ then $M' = M_2$ and $\mathrm{fv}(M') = \mathrm{fv}(M'')$. If $y \neq x$ then $M' = y$ and $\mathrm{fv}(M') = \{y\} \subseteq \{y\} \cup \mathrm{fv}(M_2) = \mathrm{fv}(M'')$.
         2. Let $M_1 = \lambda y.M'_1$ then $M' = \lambda y.M'_1[x := M_2]$ such that $y \notin \mathrm{fv}(M_2) \cup \{x\}$. By IH, $\mathrm{fv}(M'_1[x := M_2]) \subseteq \mathrm{fv}((\lambda x.M'_1)M_2)$. Hence, $\mathrm{fv}(M') = \mathrm{fv}(M'_1[x := M_2])\setminus\{y\} \subseteq \mathrm{fv}((\lambda x.M'_1)M_2)\setminus \{y\} = (\mathrm{fv}(M'_1) \setminus \{x, y\}) \cup (\mathrm{fv}(M_2) \setminus \{y\}) = \mathrm{fv}(M'')$.
         3. Let $M_1 = M'_1M''_1$ then $M' = M'_1[x := M_2]M''_1[x := M_2]$. By IH, $\mathrm{fv}(M'_1[x := M_2]) \subseteq \mathrm{fv}((\lambda x.M'_1)M_2)$ and $\mathrm{fv}(M''_1[x := M_2]) \subseteq \mathrm{fv}((\lambda x.M''_1)M_2)$. Hence, $\mathrm{fv}(M') = \mathrm{fv}(M'_1[x := M_2]) \cup \mathrm{fv}(M''_1[x := M_2]) \subseteq \mathrm{fv}((\lambda x.M'_1)M_2) \cup \mathrm{fv}((\lambda x.M''_1)M_2) = ((\mathrm{fv}(M'_1) \cup \mathrm{fv}(M''_1)) \setminus \{x\}) \cup \mathrm{fv}(M_2) = \mathrm{fv}(M'')$.

       · Or $M'' = \lambda x.M'x$ such that $x \notin \mathrm{fv}(M')$, so $\mathrm{fv}(M'') = \mathrm{fv}(M')$.
     * Let $p = 1.p'$ then either $M'' = \lambda x.M_1$, $M' = \lambda x.M_2$ and $M_1 \xrightarrow{p'}_{\beta\eta} M_2$. By IH, $\mathrm{fv}(M_1) \subseteq \mathrm{fv}(M_2)$, so $\mathrm{fv}(M'') = \mathrm{fv}(M_1) \setminus \{x\} \subseteq \mathrm{fv}(M_2) \setminus \{x\} = \mathrm{fv}(M')$. Or $M'' = M_1M_2$, $M' = M'_1M_2$ and $M_1 \xrightarrow{p'}_{\beta\eta} M'_1$. By IH, $\mathrm{fv}(M_1) \subseteq \mathrm{fv}(M'_1)$, so $\mathrm{fv}(M'') = \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2) \subseteq \mathrm{fv}(M'_1) \cup \mathrm{fv}(M_2) = \mathrm{fv}(M')$.
     * Let $p = 2.p'$ then $M'' = M_1M_2$, $M' = M_1M'_2$ and $M_2 \xrightarrow{p'}_{\beta\eta} M'_2$. By IH, $\mathrm{fv}(M_2) \subseteq \mathrm{fv}(M'_2)$, so $\mathrm{fv}(M'') = \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2) \subseteq \mathrm{fv}(M_1) \cup \mathrm{fv}(M'_2) = \mathrm{fv}(M')$.

4. We prove the lemma by induction on the length of the reduction $M \to^*_{\beta I} M'$.

   - If $M = M'$ then $\mathrm{fv}(M) = \mathrm{fv}(M')$

- Let $M \to^*_{\beta I} M'' \to_{\beta I} M'$. By IH, $\mathrm{fv}(M) = \mathrm{fv}(M'')$ and if $M \in \Lambda\mathrm{I}$ then $M'' \in \Lambda\mathrm{I}$. By definition there exists p such that $M'' \xrightarrow{p}_{\beta I} M'$. We prove that $\mathrm{fv}(M'') = \mathrm{fv}(M')$ and that if $M'' \in \Lambda\mathrm{I}$ then $M' \in \Lambda\mathrm{I}$ by induction on p.
  * Let $p = 0$ then $M'' = (\lambda x.M_1)M_2$ and $M' = M_1[x := M_2]$ such that $x \in \mathrm{fv}(M_1)$. So, by lemmma 2.2.2, $\mathrm{fv}(M') = \mathrm{fv}(M'')$ and if $M'' \in \Lambda\mathrm{I}$ then $M' \in \Lambda\mathrm{I}$.
  * Let $p = 1.p'$ then either $M'' = \lambda x.M_1$, $M' = \lambda x.M_2$ and $M_1 \xrightarrow{p'}_{\beta I} M_2$. By IH, $\mathrm{fv}(M_1) = \mathrm{fv}(M_2)$ and if $M_1 \in \Lambda\mathrm{I}$ then $M_2 \in \Lambda\mathrm{I}$, so $\mathrm{fv}(M'') = \mathrm{fv}(M_1) \setminus \{x\} = \mathrm{fv}(M_2) \setminus \{x\} = \mathrm{fv}(M')$ and if $M'' \in \Lambda\mathrm{I}$ then $x \in \mathrm{fv}(M_1) = \mathrm{fv}(M_2)$ and so $M' \in \Lambda\mathrm{I}$. Or $M'' = M_1M_2$, $M' = M'_1M_2$ and $M_1 \xrightarrow{p'}_{\beta\eta} M'_1$. By IH, $\mathrm{fv}(M_1) = \mathrm{fv}(M'_1)$ and if $M_1 \in \Lambda\mathrm{I}$ then $M'_1 \in \Lambda\mathrm{I}$, so $\mathrm{fv}(M'') = \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2) = \mathrm{fv}(M'_1) \cup \mathrm{fv}(M_2) = \mathrm{fv}(M')$ and if $M'' \in \Lambda\mathrm{I}$ then $M' \in \Lambda\mathrm{I}$.
  * Let $p = 2.p'$ then $M'' = M_1M_2$, $M' = M_1M'_2$ and $M_2 \xrightarrow{p'}_{\beta\eta} M'_2$. By IH, $\mathrm{fv}(M_2) = \mathrm{fv}(M'_2)$ and if $M_2 \in \Lambda\mathrm{I}$ then $M'_2 \in \Lambda\mathrm{I}$, so $\mathrm{fv}(M'') = \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2) = \mathrm{fv}(M_1) \cup \mathrm{fv}(M'_2) = \mathrm{fv}(M')$ and if $M'' \in \Lambda\mathrm{I}$ then $M' \in \Lambda\mathrm{I}$.

5. $\Rightarrow$) Let $\lambda x.M \xrightarrow{p}_{\beta\eta} P$. We prove the result by case on p. Either $p = 0$ and $M = Px$ such that $x \notin \mathrm{fv}(P)$. Or $p = 1.p'$, $P = \lambda x.M'$ and $M \xrightarrow{p'}_{\beta\eta} M'$.

   $\Leftarrow$) If $P = \lambda x.M'$ and $M \to_{\beta\eta} pM'$. So, $\lambda x.M \xrightarrow{1.p}_{\beta\eta} P$ and $\lambda x.M \to_{\beta\eta} P$. If $M = Px$ and $x \notin fvP$ then $\lambda x.M = \lambda x.Px \xrightarrow{0}_{\beta\eta} P$, so $\lambda x.M \to_{\beta\eta} P$.

6a. If $k = 0$ then $P = (\lambda x.M)N_1N_1 \ldots N_n$ is a direct r-reduct of $(\lambda x.M)N_0N_1 \ldots N_n$, absurd. So $k \geq 1$. Assume $k = 1$, we prove $P = M[x := N_0]N_1 \ldots N_n$ by induction on $n \geq 0$.

- Let $n = 0$ and $r = \beta I$. By definition there exists p such that $(\lambda x.M)N_0 \xrightarrow{p}_{\beta I} P$. We prove the result by case on p.
  * Let $p = 0$ then $P = M[x := N_0]$ and $x \in \mathrm{fv}(M)$.
  * Let $p = 1.p'$ then $\lambda x.M \xrightarrow{p'}_{\beta I} \lambda x.M'$ and $P = (\lambda x.M')N_0$ is a direct βI-reduct of $(\lambda x.M)N_0$, absurd.
  * Let $p = 2.p'$ then $N_0 \xrightarrow{p'}_{\beta I} N'$ and $P = (\lambda x.M)N'$ is a direct βI-reduct of $(\lambda x.M)N_0$, absurd.
- Let $n = 0$ and $r = \beta\eta$. By definition there exists p such that $(\lambda x.M)N_0 \xrightarrow{p}_{\beta I} P$. We prove the result by case on p.
  * Let $p = 0$ then $P = M[x := N_0]$.
  * Let $p = 1.p'$ then $\lambda x.M \xrightarrow{p'}_{\beta\eta} Q$ and $P = QN_0$. By lemma 2.2.5:

        · Either $p' = 1.p''$, $Q = \lambda x.M'$ and $M \xrightarrow{p''}_{\beta\eta} M'$. Hence $P = (\lambda x.M')N_0$ is a direct $\beta\eta$-reduct of $(\lambda x.M)N_0$, absurd.
        · Or $p = 0$, $M = Qx$ and $x \notin \text{fv}(Q)$. Hence, $P = QN_0 = M[x := N_0]$.
    * Let $p = 2.p'$ then $N_0 \xrightarrow{p'}_{\beta\eta} N'$ and $P = (\lambda x.M)N'$ is a direct $\beta\eta$-reduct of $(\lambda x.M)N_0$, absurd.
  – Let $n = m + 1$ where $m \geq 0$. By definition there exists p such that $(\lambda x.M)N_0 \ldots N_{m+1} \xrightarrow{p}_r P$. We prove the result by case on p.
    * Either $p = 1.p'$ then $(\lambda x.M)N_0 \ldots N_m \xrightarrow{p'}_r Q$ and $P = QN_{m+1}$.
        · If Q is a direct r-reduct of $(\lambda x.M)N_0 \ldots N_m$ then P is a direct r-reduct of $(\lambda x.M)N_0 \ldots N_{m+1}$, absurd.
        · If Q is not a direct r-reduct of $(\lambda x.M)N_0 \ldots N_m$ then it is done by IH.
    * Or $p = 2.p'$ then $N_{m+1} \xrightarrow{p'}_r N'_{m+1}$ and $P = (\lambda x.M)N_0 \ldots N_m N'_{m+1}$ which is a direct r-reduct of $(\lambda x.M)N_0 \ldots N_{m+1}$, absurd.

6b. By 6a, $k \geq 1$. We prove the statement by induction on $k \geq 1$.
  – If $k = 1$ then we conclude by 6a.
  – Let $(\lambda x.M)N_0 \ldots N_n \rightarrow^*_r Q \rightarrow_r P$.
    * If Q is a direct r-reduct of $(\lambda x.M)N_0 \ldots N_n$, then $Q = (\lambda x.M')N'_0 \ldots N'_n$, such that $M \rightarrow^*_r M'$ and $\forall i \in \{0, \ldots, n\}$, $N_i \rightarrow^*_r N'_i$. Since P is not a direct r-reduct of $(\lambda x.M)N_0 \ldots N_n$, P is not a direct r-reduct of Q. Hence by 6a, $P = M'[x := N'_0]N'_1 \ldots N'_n$.
    * If Q is not a direct r-reduct of $(\lambda x.M)N_0 \ldots N_n$, then by IH, there exists a direct r-reduct $(\lambda x.M')N'_0 \ldots N'_n$ of $(\lambda x.M)N_0 \ldots N_n$ such that $M'[x := N'_0]N'_1 \ldots N'_n \rightarrow^*_r Q \rightarrow_r P$.

7. If P is a direct r-reduct of $(\lambda x.M)N_0 \ldots N_n$ then $P = (\lambda x.M')N'_0 \ldots N'_n$ such that $M \rightarrow^*_r M'$ and $\forall i \in \{0, \ldots, n\}, N_i \rightarrow^*_r N'_i$. So $P \rightarrow_r M'[x := N'_0]N'_1 \ldots N'_n$ (if $r = \beta I$, note that $x \in \text{fv}(M')$ by lemma 2.2.4) and $M[x := N_0]N_1 \ldots N_n \rightarrow^*_r M'[x := N'_0]N'_1 \ldots N'_n$. If P is not a direct r-reduct of $(\lambda x.M)N_0 \ldots N_n$ then by lemma 6.6b, there exists a direct r-reduct, $(\lambda x.M')N'_0 \ldots N'_n$, such that $M \rightarrow^*_r M'$ and $\forall i \in \{0, \ldots, n\}, N_i \rightarrow^*_r N'_i$, of $(\lambda x.M)N_0 \ldots N_n$. We have $M[x := N_0]N_1 \ldots N_n \rightarrow^*_r M'[x := N'_0]N'_1 \ldots N'_n \rightarrow^*_r P$.

8 We prove this lemma by induction on ths structure of p.
  – Let $p = 0$ it is done by definition.
  – Let $p = 1.p'$. Then:
    * Either $M = \lambda x.M_1 \xrightarrow{1.p'}_r \lambda x.M'_1 = M'$ such that $M_1 \xrightarrow{p'}_r M'_1$. By IH, $p' \in \mathcal{R}^r_{M_1}$. So $p \in \mathcal{R}^r_M$. If $p \in \mathcal{R}^r_M$ then $M|_p = M_1|_{p'} \in \mathcal{R}^r$. By IH, there exists M'_1 such that $M_1 \xrightarrow{p'}_r M'_1$, so $M \xrightarrow{p}_r \lambda x.M'_1$.

* Or $M = M_1M_2 \xrightarrow{1.p}_r M_1'M_2 = M'$ such that $M_1 \xrightarrow{p'}_r M_1'$. By IH, $p' \in \mathcal{R}^r_{M_1}$. So $p \in \mathcal{R}^r_M$. If $p \in \mathcal{R}^r_M$ then $M|_p = M_1|_{p'} \in \mathcal{R}^r$. By IH, there exists M_1' such that $M_1 \xrightarrow{p'}_r M_1'$, so $M \xrightarrow{p}_r M_1'M_2$.

– Let $p = 2.p'$. Then, $M = M_1M_2 \xrightarrow{1.p}_r M_1M_2' = M'$ such that $M_2 \xrightarrow{p'}_r M_2'$. By IH, $p' \in \mathcal{R}^r_{M_2}$. So $p \in \mathcal{R}^r_M$. If $p \in \mathcal{R}^r_M$ then $M|_p = M_2|_{p'} \in \mathcal{R}^r$. By IH, there exists M_2' such that $M_2 \xrightarrow{p'}_r M_2'$, so $M \xrightarrow{p}_r M_1M_2'$.

9 We prove this lemma by induction on ths structure of p.

– Let $p = 0$ it is done by definition.

– Let $p = 1.p'$. Then either $M = \lambda x.M' \xrightarrow{1.p'}_r \lambda x.M_1' = M_1$ such that $M' \xrightarrow{p'}_r M_1'$. By definition, $M_2 = \lambda x.M_2'$ and $M' \xrightarrow{p'}_r M_2'$. By IH, $M_1' = M_2'$, so $M_1 = M_2$. Or $M = M'N \xrightarrow{1.p}_r M_1'N = M_1$ such that $M' \xrightarrow{p'}_r M_1'$. By definition, $M_2 = M_2'N$ and $M' \xrightarrow{p'}_r M_2'$. By IH, $M_1' = M_2'$, so $M_1 = M_2$.

– Let $p = 2.p'$. Then $M = NM' \xrightarrow{1.p}_r NM_1' = M_1$ such that $M' \xrightarrow{p'}_r M_1'$. By definition, $M_2 = NM_2'$ and $M' \xrightarrow{p'}_r M_2'$. By IH, $M_1' = M_2'$, so $M_1 = M_2$. □

Lemma 2.4 .

1. We prove the lemma by induction on the structure of M.
   - Let $M = y$.
     – Either $y = x$ then $M[x := c(cx)] = c(cx) \neq x$ and for any N, $M[x := c(cx)] = c(cx) \neq Nx$ because $cx \neq x$.
     – Or $y \neq x$ then $M[x := c(cx)] = y \neq x$ and for any N, $M[x := c(cx)] = y \neq Nx$.
   - Let $M = \lambda y.P$. Then, $M[x := c(cx)] = \lambda y.P[x := c(cx)] \neq x$ (such that $y \notin \{c, x\}$) and for any N, $M[x := c(cx)] \neq Nx$.
   - Let $M = PQ$. Then, $M[x := c(cx)] = P[x := c(cx)]Q[x := c(cx)] \neq x$. Assume $M[x := c(cx)] = Nx$, so $Q[x := c(cx)] = x$ and by IH, absurd.
2. We prove this lemma by induction on the structure of M.
   - Let $M = z$.
     – Either $z = y$ then $M[y := c(cx)] = c(cx) \neq x$ and for any N, $M[y := c(cx)] = c(cx) \neq Nx$ because $cx \neq x$.
     – Or $z \neq y$ then $M[y := c(cx)] = z \neq x$ by hypothesis and for any N, $M[y := c(cx)] = z \neq Nx$.

- Let $M = \lambda z.P$. Then, $M[y := c(cx)] = \lambda z.P[y := c(cx)] \neq x$ (such that $y \notin \{c, x, y\}$) and for any N, $M[y := c(cx)] \neq Nx$.
- Let $M = PQ$. Then, $M[y := c(cx)] = P[x := c(cx)]Q[x := c(cx)] \neq x$. Assume $M[y := c(cx)] = Nx$, so $Q[y := c(cx)] = x$ and by IH, absurd.

3. By cases on the derivation of $M \in \mathcal{M}_c$.

4. By cases on the structure of M using 3.

5. By cases on the derivation of $MN \in \mathcal{M}_c$.

6. We prove this result by induction on n.

   - If $n = 0$ then it is done.
   - Let $n = m + 1$ such that $m \geq 0$. By lemma 2.4.5, $c^m(M) \in \mathcal{M}_c$ then by IH, $M \in \mathcal{M}_c$.

7. By cases on the derivation of $\lambda x.P \in \Lambda\eta_c$.

8. By cases on the derivation of $\lambda x.P \in \Lambda\mathrm{I}_c$.

9. We prove the lemma by induction on the structure of $M \in \mathcal{M}_c$.

   - Case (R1)1. Either $M = x$ then $M[x := N] = N \in \mathcal{M}_c$. Or $M = y \neq x$ then $M[x := N] = M \in \mathcal{M}_c$.
   - Case (R1)2. Let $M = \lambda y.P \in \Lambda\mathrm{I}_c$ such that $y \neq c$, $P \in \Lambda\mathrm{I}_c$ and $y \in \mathrm{fv}(P)$. We have $M[x := N] = \lambda y.M[x := N]$ such that $y \notin \mathrm{fv}(N) \cup \{x\}$. By IH, $P[x := N] \in \Lambda\mathrm{I}_c$, so $M[x := N] \in \Lambda\mathrm{I}_c$.
   - Case (R1)3. Let $M = \lambda y.P[y := c(cy)] \in \Lambda\eta_c$ such that $y \neq c$ and $P \in \Lambda\eta_c$. By IH, $P[x := N] \in \Lambda\eta_c$. So by (R1).3 $M[x := N] = \lambda y.P[y := c(cy)][x := N] = \lambda y.P[x := N][y := c(cy)] \in \Lambda\eta_c$ such that $y \notin \mathrm{fv}(N) \cup \{x\}$.
   - Case (R1)4. Let $M = \lambda y.Py$ such that $Py \in \Lambda\eta_c$, $y \notin \mathrm{fv}(P) \cup \{c\}$ and $P \neq c$. We have $M[x := N] = \lambda y.(Py)[x := N] = \lambda y.P[x := N]y$, such that $y \notin \mathrm{fv}(N) \cup \{x\}$. By IH, $P[x := N]y \in \Lambda\eta_c$. By lemma 2.4.4, $P[x := N] \neq c$. Hence, because $y \notin \mathrm{fv}(P[x := N])$, $M[x := N] \in \Lambda\eta_c$.
   - Case (R2) Let $M = cM_1M_2$ such that $M_1, M_2 \in \mathcal{M}_c$. Then by IH, $M_1[x := N], M_2[x := N] \in \mathcal{M}_c$. Hence, $cM_1[x := N]M_2[x := N] \in \mathcal{M}_c$.
   - Case (R3) Let $M = M_1M_2$ such that $M_1, M_2 \in \mathcal{M}_c$ and M_1 is a λ-abstraction. Then by IH, $M_1[x := N], M_2[x := N] \in \mathcal{M}_c$. Hence, $M_1[x := N]M_2[x := N] \in \mathcal{M}_c$, since $M_1[x := N]$ is a λ-abstraction.
   - Case (R4) Let $M = cP$ such that $P \in \Lambda\eta_c$. Then by IH, $P[x := N] \in \Lambda\eta_c$ and by (R4), $M[x := N] \in \Lambda\eta_c$.

10. By case on the structure of M.

   - let $M \in \mathcal{V}$.
     - Either $M = x$ then, $M[x := c(cx)] = c(cx)$. Hence, $c(cx) \neq y$, $c(cx) \neq Py$ since $cx \neq y$, $c(cx) \neq \lambda y.P$ and $c(cx) \neq (\lambda y.P)Q$. If $M[x := c(cx)] = PQ$ then $P = c$ and $Q = cx$.
     - Or $M = z \neq x$ then $M[x := c(cx)] = z$. Hence, if $z = y$ then $M = y$, $z \neq Py$, $z \neq \lambda y.P$, $z \neq PQ$ and $z \neq (\lambda y.P)Q$.
   - Let $M = \lambda z.M'$ then $M[x := c(cx)] = \lambda z.M'[x := c(cx)]$, where $z \notin \{x, c\}$. Hence, $\lambda z.M'[x := c(cx)] \neq y$, $\lambda z.M'[x := c(cx)] \neq Py$, $\lambda z.M'[x := c(cx)] \neq PQ$ and $\lambda z.M'[x := c(cx)] \neq (\lambda y.P)Q$. Let $\lambda z.M'[x := c(cx)] = \lambda y.P$. By α-converions, assume $y = z$. So $M'[x := c(cx)] = P$.
   - Let $M = M_1M_2$ then $M[x := c(cx)] = M_1[x := c(cx)]M_2[x := c(cx)]$. Hence, $M_1[x := c(cx)]M_2[x := c(cx)] \neq y$ and $M_1[x := c(cx)]M_2[x := c(cx)] \neq \lambda y.P$. If $M_1[x := c(cx)]M_2[x := c(cx)] = Py$ then $P = M_1[x := c(cx)]$ and $M_2[x := c(cx)] = y$. So $M_2 = y$. If $M_1[x := c(cx)]M_2[x := c(cx)] = PQ$ then $P = M_1[x := c(cx)]$ and $Q = M_2[x := c(cx)]$. If $M_1[x := c(cx)]M_2[x := c(cx)] = (\lambda y.P)Q$ then $\lambda y.P = M_1[x := c(cx)]$ and $Q = M_2[x := c(cx)]$. So $M_1 = \lambda y.M_0$ and $P = M_0[x := c(cx)]$

11. (a) By definition, $x \neq c$. By lemma 2.4.7, either $P = Nx$ where $Nx \in \Lambda\eta_c$ or $P = N[x := c(cx))]$ where $N \in \Lambda\eta_c$. In the second case since by (R4) $c(cx) \in \Lambda\eta_c$, we get by lemma 2.4.9 that $N[x := c(cx))] \in \Lambda\eta_c$.

    (b) By lemma 2.4.1 and lemma 2.4.7.

12. (a) $\Rightarrow$) We prove the lemma by induction on the structure of p.
    - Let $p = 0$ then:
      - either $M[x := c(cx)] = (\lambda y.P)Q$ and $M' = P[y := Q]$. By lemma 2.4.10, $M = (\lambda y.P')Q'$, $P = P'[x := c(cx)]$ and $Q = Q'[x := c(cx)]$ such that $y \notin \{c, x\}$. So $M' = P'[y := Q'][x := c(cx)]$ and $M \xrightarrow{0}_{\beta\eta} P'[y := Q']$.
      - Or $M[x := c(cx)] = \lambda y.M'y$ such that $y \notin \mathrm{fv}(M')$. By lemma 2.4.10, $M = \lambda y.N$ and $M'y = N[x := c(cx)]$ such that $y \notin \{x, c\}$. Again by lemma 2.4.10, $N = N'y$ and $M' = N'[x := c(cx)]$. Because $y \notin \mathrm{fv}(M')$, we obatin $y \notin \mathrm{fv}(N')$ and so $M = \lambda y.N'y \xrightarrow{0}_{\beta\eta} N'$.
    - Let $p = 1.p'$. Then:
      - Either $M[x := c(cx)] = \lambda y.P \xrightarrow{1.p'}_{\beta\eta} \lambda y.P' = M'$ such that $P \xrightarrow{p'}_{\beta\eta} P'$. By lemma 2.4.10, $M = \lambda y.N$ and $P = N[x := c(cx)]$ such that $y \notin \{c, x\}$. By IH, $P' = N'[x := c(cx)]$

and $N \xrightarrow{p'}_{\beta\eta} N'$. So $M' = (\lambda y.N')[x := c(cx)]$ and $M \xrightarrow{1.p}_{\beta\eta} \lambda y.N'$.

  - Or $M[x := c(cx)] = PQ \xrightarrow{1.p'}_{\beta\eta} P'Q = M'$ such that $P \xrightarrow{p'}_{\beta\eta} P'$. Then by lemma 2.4.10, either $M = x$ and $P = c$ and $Q = cx$ but then $P \xrightarrow{p'}_{\beta\eta} P'$ is wrong. Or $M = P_0Q_0$, $P = P_0[x := c(cx)]$ and $Q = Q_0[x := c(cx)]$. By IH, $P' = P'_0[x := c(cx)]$ and $P_0 \xrightarrow{p'}_{\beta\eta} P'_0$. So $M' = (P'_0Q_0)[x := c(cx)]$ and $P_0Q_0 \xrightarrow{1.p'}_{\beta\eta} P'_0Q_0$.

- Let $p = 2.p'$ then $M[x := c(cx)] = PQ \xrightarrow{2.p'}_{\beta\eta} PQ' = M'$ such that $Q \xrightarrow{p'}_{\beta\eta} Q'$. Then by lemma 2.4.10, either $M = x$ and $P = c$ and $Q = cx$ but then $Q \xrightarrow{p'}_{\beta\eta} Q'$ is wrong. Or $M = P_0Q_0$, $P = P_0[x := c(cx)]$ and $Q = Q_0[x := c(cx)]$. By IH, $Q' = Q'_0[x := c(cx)]$ and $Q_0 \xrightarrow{p'}_{\beta\eta} Q'_0$. So $M' = (P_0Q'_0)[x := c(cx)]$ and $P_0Q_0 \xrightarrow{2.p'}_{\beta\eta} P_0Q'_0$.

$\Leftarrow$) We prove the lemma by induction on the structure of p.

- Let $p = 0$ then:
  - Either $M = \lambda y.Ny$ such that $y \notin \mathrm{fv}(N)$. Then $M[x := c(cx)] = \lambda y.N[x := c(cx)]y \xrightarrow{0}_{\beta\eta} N[x := c(cx)]$ such that $y \notin \{c, x\}$.
  - Or $M = (\lambda y.P)Q$ and $M' = P[y := Q]$. Then $M[x := c(cx)] = (\lambda y.P[x := c(cx)])Q[x := c(cx)] \xrightarrow{0}_{\beta\eta} P[x := c(cx)][y := Q[x := c(cx)]] = P[y := Q][x := c(cx)]$ such that $y \notin \{c, x\}$.
- Let $p = 1.p'$.
  - Either $M = \lambda y.N \xrightarrow{p}_{\beta\eta} \lambda y.N' = M'$ such that $N \xrightarrow{p'}_{\beta\eta} N'$. By IH, $N[x := c(cx)] \xrightarrow{p'}_{\beta\eta} N'[x := c(cx)]$. So, $M[x := c(cx)] \xrightarrow{p}_{\beta\eta} M'[x := c(cx)]$ such that $y \notin \{c, x\}$.
  - Or $M = PQ \xrightarrow{p}_{\beta\eta} P'Q = M'$ such that $P \xrightarrow{p'}_{\beta\eta} P'$. By IH, $P[x := c(cx)] \xrightarrow{p'}_{\beta\eta} P'[x := c(cx)]$. So, $M[x := c(cx)] \xrightarrow{p}_{\beta\eta} M'[x := c(cx)]$.
- Let $p = 2.p'$ then $M = PQ \xrightarrow{p}_{\beta\eta} PQ' = M'$ such that $Q \xrightarrow{p'}_{\beta\eta} Q'$. By IH, $Q[x := c(cx)] \xrightarrow{p'}_{\beta\eta} Q'[x := c(cx)]$. So, $M[x := c(cx)] \xrightarrow{p}_{\beta\eta} M'[x := c(cx)]$.

(b) We prove this lemma by induction on n.

- Let $n = 0$ then it is done.
- Let $n = m + 1$ such that $m \geq 0$. Then $c^n(M) = c(c^m(M)) \xrightarrow{p}_{\beta\eta} M'$. By case on p we obtain that $p = 2.p'$ and $M' = c(N')$ and

$c^m(M) \xrightarrow{p'}_{\beta\eta} N'$. By IH, $p' = 2^m.p''$ and there exists $N'' \in \Lambda\eta_c$ such that $N' = c^m(N'')$ and $M \xrightarrow{p''}_{\beta\eta} N''$. So $p = 2^n.p''$ and $M' = c^n(N'')$.

$\square$

Lemma 2.5. We prove this lemma by case on the structure of M.

- Let $M \in \mathcal{V}$ and $p \in \mathcal{R}^r_M$. So $M|_p \in \mathcal{R}^r$. We prove by case on the structure of p that there is no such p.
  - Let $p = 0$ then $M|_p = M \notin \mathcal{R}^r$.
  - Let $p = 1.p'$ then $M|_p$ is undefined.
  - Let $p = 2.p'$ then $M|_p$ is undefined.
- Let $M = \lambda x.N$.
  - Let $M \in \mathcal{R}^r$. We prove by case on the structure of p that if $p \in \mathcal{R}^r_M$ then $p \in \{0\} \cup \{1.p' \mid p' \in \mathcal{R}^r_N\}$.
    - Let $p = 0$ then $M|_p = M \in \mathcal{R}^r$.
    - Let $p = 1.p'$ then $M|_p = N|_{p'} \in \mathcal{R}^r$, so $p' \in \mathcal{R}^r_N$.
    - Let $p = 2.p'$ then $M|_p$ is undefined.

    Let $p \in \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_N\}$, we prove that $p \in \mathcal{R}^r_M$.
    - Let $p = 0$. Since $M = M|_p \in \mathcal{R}^r$, by definition, $p \in \mathcal{R}^r_M$.
    - Let $p = 1.p'$ such that $p' \in \mathcal{R}^r_N$. By definition $M|_p = N|_{p'} \in \mathcal{R}^r$.
  - Let $M \notin \mathcal{R}^r$. We prove by case on the structure of p that if $p \in \mathcal{R}^r_M$ then $p \in \{1.p' \mid p' \in \mathcal{R}^r_N\}$.
    - Let $p = 0$ then $M|_p = M \notin \mathcal{R}^r$.
    - Let $p = 1.p'$ then $M|_p = N|_{p'} \in \mathcal{R}^r$, so $p' \in \mathcal{R}^r_N$.
    - Let $p = 2.p'$ then $M|_p$ is undefined.

    Let $p \in \{1.p' \mid p' \in \mathcal{R}^r_N\}$, we prove that $p \in \mathcal{R}^r_M$. Then, $p = 1.p'$ such that $p' \in \mathcal{R}^r_N$. By definition $M|_p = N|_{p'} \in \mathcal{R}^r$.
- Let $M = PQ$.
  - Let $M \in \mathcal{R}^r$. We prove by case on the structure of p that if $p \in \mathcal{R}^r_M$ then $p \in \{0\} \cup \{1.p' \mid p' \in \mathcal{R}^r_P\} \cup \{2.p' \mid p' \in \mathcal{R}^r_Q\}$.
    - Let $p = 0$ then $M|_p = M \in \mathcal{R}^r$.
    - Let $p = 1.p'$ then $M|_p = P|_{p'} \in \mathcal{R}^r$, so $p' \in \mathcal{R}^r_P$.
    - Let $p = 2.p'$ then $M|_p = Q|_{p'} \in \mathcal{R}^r$, so $p' \in \mathcal{R}^r_Q$.

    Let $p \in \{0\} \cup \{1.p' \mid p' \in \mathcal{R}^r_P\} \cup \{2.p' \mid p' \in \mathcal{R}^r_Q\}$, we prove that $p \in \mathcal{R}^r_M$.
    - Let $p = 0$. Since $M|_p = M \in \mathcal{R}^r$, so $p \in \mathcal{R}^r_M$.

    - Let $p = 1.p'$ such that $p' \in \mathcal{R}^r_P$. Since $M|_p = P|_{p'} \in \mathcal{R}^r$, $p \in \mathcal{R}^r_M$
    - Let $p = 2.p'$ such that $p' \in \mathcal{R}^r_Q$. Since $M|_p = Q|_{p'} \in \mathcal{R}^r$, $p \in \mathcal{R}^r_M$
  - Let $M \notin \mathcal{R}^r$. We prove by induction on the structure of p that if $p \in \mathcal{R}^r_M$ then $p \in \{1.p' \mid p' \in \mathcal{R}^r_P\} \cup \{1.p' \mid p' \in \mathcal{R}^r_Q\}$.
    - Let $p = 0$ then $M|_p = M \notin \mathcal{R}^r$.
    - Let $p = 1.p'$ then $M|_p = P|_{p'} \in \mathcal{R}^r$, so $p' \in \mathcal{R}^r_P$.
    - Let $p = 2.p'$ then $M|_p = Q|_{p'} \in \mathcal{R}^r$, so $p' \in \mathcal{R}^r_Q$.

    Let $p \in \{1.p' \mid p' \in \mathcal{R}^r_P\} \cup \{2.p' \mid p' \in \mathcal{R}^r_Q\}$, we prove that $p \in \mathcal{R}^r_M$.
    - Let $p = 1.p'$ such that $p' \in \mathcal{R}^r_P$. Since $M|_p = P|_{p'} \in \mathcal{R}^r$, $p \in \mathcal{R}^r_M$
    - Let $p = 2.p'$ such that $p' \in \mathcal{R}^r_Q$. Since $M|_p = Q|_{p'} \in \mathcal{R}^r$, $p \in \mathcal{R}^r_M$

□

Lemma 2.6. We prove the statement by case on the structure of M.

- Let $M \in \mathcal{V}$, by lemma 2.5, $\mathcal{R}^r_M = \varnothing$, so $\mathcal{F} = \varnothing$.
- Let $M = \lambda y.N$ then by lemma 2.5:
  - If $M \in \mathcal{R}^r$ then $\mathcal{R}^r_M = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_N\}$. Let $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\}$. Let $p \in \mathcal{F}'$ then $1.p \in \mathcal{F}$, so $p \in \mathcal{R}^r_N$.
    - Let $p \in \mathcal{F} \setminus \{0\}$ then $p = 1.p'$ such that $p' \in \mathcal{R}^r_N$. So $p' \in \mathcal{F}'$ and it is done.
    - Let $p \in \{1.p' \mid p' \in \mathcal{F}'\}$ then $p = 1.p'$ such that $p' \in \mathcal{F}'$. So $1.p' = p \in \mathcal{F} \setminus \{0\}$.
  - If $M \notin \mathcal{R}^r$ then $\mathcal{R}^r_M = \{1.p \mid p \in \mathcal{R}^r_N\}$. Let $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\}$. Let $p \in \mathcal{F}'$ then $1.p \in \mathcal{F}$, so $p \in \mathcal{R}^r_N$.
    - Let $p \in \mathcal{F}$ then $p = 1.p'$ such that $p' \in \mathcal{R}^r_N$. So $p' \in \mathcal{F}'$ and it is done.
    - Let $p \in \{1.p' \mid p' \in \mathcal{F}'\}$ then $p = 1.p'$ such that $p' \in \mathcal{F}'$. So $1.p' = p \in \mathcal{F}$.
- Let $M = PQ$ then by lemma 2.5:
  - If $M \in \mathcal{R}^r$ then $\mathcal{R}^r_M = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_P\} \cup \{2.p \mid p \in \mathcal{R}^r_Q\}$. Let $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\}$ and $\mathcal{F}_2 = \{2.p \mid p \in \mathcal{F}\}$. Let $p \in \mathcal{F}_1$ then $1.p \in \mathcal{F}$, so $p \in \mathcal{R}^r_P$. Let $p \in \mathcal{F}_2$ then $2.p \in \mathcal{F}$, so $p \in \mathcal{R}^r_Q$.
    - Let $p \in \mathcal{F} \setminus \{0\}$. Either $p = 1.p'$ such that $p' \in \mathcal{R}^r_P$, so $p' \in \mathcal{F}_1$ and it is done. Or $p = 2.p'$ such that $p' \in \mathcal{R}^r_Q$, so $p' \in \mathcal{F}_2$ and it is done.

* Let $p \in \{1.p' \mid p' \in \mathcal{F}_1\} \cup \{2.p' \mid p' \in \mathcal{F}_2\}$. Either $p = 1.p'$ such that $p' \in \mathcal{F}_1$, so $1.p' \in \mathcal{F} \setminus \{0\}$. Or $p = 2.p'$ such that $p' \in \mathcal{F}_2$, so $2.p' \in \mathcal{F} \setminus \{0\}$.

- If $M \notin \mathcal{R}^r$ then $\mathcal{R}^r_M = \{1.p \mid p \in \mathcal{R}^r_P\} \cup \{2.p \mid p \mid p \in \mathcal{R}^r_Q\}$. Let $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\}$. Let $p \in \mathcal{F}_1$ then $1.p \in \mathcal{F}$, so $p \in \mathcal{R}^r_P$. Let $p \in \mathcal{F}_2$ then $2.p \in \mathcal{F}$, so $p \in \mathcal{R}^r_Q$.
  * Let $p \in \mathcal{F}$. Either $p = 1.p'$ such that $p' \in \mathcal{R}^r_P$, so $p' \in \mathcal{F}_1$ and it is done. Or $p = 2.p'$ such that $p' \in \mathcal{R}^r_Q$, so $p' \in \mathcal{F}_2$ and it is done.
  * Let $p \in \{1.p' \mid p' \in \mathcal{F}_1\} \cup \{2.p' \mid p' \in \mathcal{F}_2\}$. Either $p = 1.p'$ such that $p' \in \mathcal{F}_1$, so $1.p' \in \mathcal{F}$. Or $p = 2.p'$ such that $p' \in \mathcal{F}_2$, so $2.p' \in \mathcal{F}$.

□

Lemma 2.7.

1. By case on the structure of M.
   - Let $M \in \mathcal{V}$ then $M, M[x := c(cx)] \notin \mathcal{R}^{\beta\eta}$.
   - Let $M = \lambda y.N$ then $M[x := c(cx)] = \lambda y.N[x := c(cx)]$, where $y \notin \{x, c\}$.
     - If $M \in \mathcal{R}^{\beta\eta}$ then $N = Py$ such that $y \notin \mathrm{fv}(P)$. $N[x := c(cx)] = P[x := c(cx)]y$ and $y \notin \mathrm{fv}(P[x := c(cx)])$, so $M[x := c(cx)] \in \mathcal{R}^{\beta\eta}$.
     - If $M[x := c(cx)] \in \mathcal{R}^{\beta\eta}$ then $N[x := c(cx)] = Py$ such that $y \notin \mathrm{fv}(P)$. By 2.4.10, $N = Qy$ and $P = Q[x := c(cx)]$. So $M = \lambda y.Qy$. Because $y \notin \mathrm{fv}(P)$, we obtain $y \notin \mathrm{fv}(Q)$ and so $M \in \mathcal{R}^{\beta\eta}$.
   - Let $M = M_1M_2$ then $M[x := c(cx)] = M_1[x := c(cx)]M_2[x := c(cx)]$.
     - If $M \in \mathcal{R}^{\beta\eta}$ then $M_1 = \lambda y.M_0$. So $M[x := c(cx)] = (\lambda y.M_0[x := c(cx)])M_2[x := c(cx)] \in \mathcal{R}^{\beta\eta}$, where $y \notin \{x, c\}$.
     - If $M[x := c(cx)] \in \mathcal{R}^{\beta\eta}$ then $M_1[x := c(cx)] = \lambda y.P$. By 2.4.10, $M_1 = \lambda y.M_0$ and $P = M_0[x := c(cx)]$ such that $y \notin \{c, x\}$. So, $M \in \mathcal{R}^{\beta\eta}$
2. We prove this result by inducion on the structure of M.
   - If $M \in \mathcal{V}$ then by lemma 2.5, $\mathcal{R}^{\beta\eta}_M = \varnothing$.
   - Let $M = \lambda y.M'$. Then $M[x := c(cx)] = \lambda y.M'[x := c(cx)]$ where $y \notin \{x, c\}$. By lemma 2.5:
     - If $M \in \mathcal{R}^{\beta\eta}$ then let $p = 0$. Then, $M[x := c(cx)]|_p = M[x := c(cx)] = M|_p[x := c(cx)]$
     - Let $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{M'}$. Then, $M[x := c(cx)]|_p = M'[x := c(cx)]|_{p'} =^{IH} M'|_{p'}[x := c(cx)] = M|_p[x := c(cx)]$.

- Let $M = M_1M_2$. Then $M[x := c(cx)] = M_1[x := c(cx)]M_2[x := c(cx)]$. By lemma 2.5:
  - If $M \in \mathcal{R}^{\beta\eta}$ then let $p = 0$. Then, $M[x := c(cx)]|_p = M[x := c(cx)] = M|_p[x := c(cx)]$
  - Let $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{M_1}$. Then, $M[x := c(cx)]|_p = M_1[x := c(cx)]|_{p'} =^{IH} M_1|_{p'}[x := c(cx)] = M|_p[x := c(cx)]$.
  - Let $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{M_2}$. Then, $M[x := c(cx)]|_p = M_2[x := c(cx)]|_{p'} =^{IH} M_2|_{p'}[x := c(cx)] = M|_p[x := c(cx)]$.

3. $\Rightarrow$) Let $p \in \mathcal{R}^{\beta\eta}_{\lambda x.M[x:=c(cx)]}$. By lemma 2.4.1, $\lambda x.M[x := c(cx)] \notin \mathcal{R}^{\beta\eta}$ so by lemma 2.5, $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{M[x:=c(cx)]}$.

   $\Leftarrow$) Let $p \in \mathcal{R}^{\beta\eta}_{M[x:=c(cx)]}$. By lemma 2.5, $1.p \in \mathcal{R}^{\beta\eta}_{\lambda x.M[x:=c(cx)]}$.

4. $\Rightarrow$) Let $p \in \mathcal{R}^{\beta\eta}_{M[x:=c(cx)]}$. We prove the statement by induction on the structure of M
   - $M \notin \mathcal{V}$ since by lemma 2.5, $\mathcal{R}^{\beta\eta}_{M[x:=c(cx)]} = \varnothing$.
   - Let $M = \lambda y.N$ so $M[x := c(cx)] = \lambda y.N[x := c(cx)]$, where $y \notin \{x, c\}$. By lemma 2.5:
     * Either if $M[x := c(cx)] \in \mathcal{R}^{\beta\eta}$, $p = 0$. By 1, $M \in \mathcal{R}^{\beta\eta}$, so $p \in \mathcal{R}^{\beta\eta}_M$.
     * Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{N[x:=c(cx)]}$. By IH, $p' \in \mathcal{R}^{\beta\eta}_N$. Hence by lemma 2.5, $p = 1.p' \in \mathcal{R}^{\beta\eta}_M$.
   - Let $M = M_1M_2$ so $M[x := c(cx)] = M_1[x := c(cx)]M_2[x := c(cx)]$. By lemma 2.5:
     * Either if $M[x := c(cx)] \in \mathcal{R}^{\beta\eta}$, $p = 0$. By 1, $M \in \mathcal{R}^{\beta\eta}$, so $0 \in \mathcal{R}^{\beta\eta}_M$.
     * Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{M_1[x:=c(cx)]}$. By IH, $p' \in \mathcal{R}^{\beta\eta}_{M_1}$. Hence by lemma 2.5, $p = 1.p' \in \mathcal{R}^{\beta\eta}_M$.
     * Or $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{M_2[x:=c(cx)]}$. By IH, $p' \in \mathcal{R}^{\beta\eta}_{M_2}$. Hence by lemma 2.5, $p = 2.p' \in \mathcal{R}^{\beta\eta}_M$.

   $\Leftarrow$) Let $p \in \mathcal{R}^r_M$. Then by definition $M|_p \in \mathcal{R}^{\beta\eta}$. By 1, $M|_p[x := c(cx)] \in \mathcal{R}^{\beta\eta}$. By 2, $M[x := c(cx)]|_p \in \mathcal{R}^{\beta\eta}$. So $p \in \mathcal{R}^{\beta\eta}_{M[x:=c(cx)]}$.

5. We prove this statement by induction on $n \geq 0$.
   - Let $n = 0$ then trivial.
   - Let $n = m+1$ such that $m \geq 0$. By lemma 2.5, $\mathcal{R}^{\beta\eta}_{c^m(M)} = \{1.p \mid p \in \mathcal{R}^{\beta\eta}_c\} \cup \{2.p \mid p \in \mathcal{R}^{\beta\eta}_{c^m(M)}\} =^{IH} \{2^n.p \mid p \in \mathcal{R}^{\beta\eta}_M\}$.

$\square$

Lemma 2.8. We prove the statement by case on r.

- Either $r = \beta I$. Since $M \in \Lambda\mathrm{I}_c$, $M \in \Lambda\mathrm{I}$, so $\lambda x.P, Q \in \Lambda\mathrm{I}$. Hence, $x \in \mathrm{fv}(P)$ and $M \in \mathcal{R}^{\beta I}$.
- Or $r = \beta\eta$. Trivial. □

Lemma 2.9. We prove the statement by induction on the structure of M.

- Let $M \in \mathcal{V} \setminus \{c\}$. By lemma 2.5, $\mathcal{R}^r_M = \varnothing$.
- Let $M = \lambda x.N \in \Lambda\mathrm{I}_c$ such that $N \in \Lambda\mathrm{I}_c$ and let $p \in \mathcal{R}^{\beta I}_M$. Since $M \notin \mathcal{R}^{\beta I}$, by lemma 2.5, $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta I}_N$. So by IH, $M|_p = N|_{p'} \in \Lambda\mathrm{I}_c$.
- Let $M = \lambda x.N[x := c(cx)] \in \Lambda\eta_c$ such that $N \in \Lambda\eta_c$ and let $p \in \mathcal{R}^{\beta\eta}_M$. By lemma 2.7.3, $p = 1.p'$ and $p' \in \mathcal{R}^{\beta\eta}_{N[x:=c(cx)]}$. By lemma 2.7.4, $p' \in \mathcal{R}^{\beta\eta}_N$. By IH, $N|_{p'} \in \Lambda\eta_c$. So, $M|_p = N[x := c(cx)]|_{p'} =^{2.7.2} N|_{p'}[x := c(cx)]$. By lemma 2.4.9, $N|_{p'}[x := c(cx)] \in \Lambda\eta_c$.
- Let $M = \lambda x.Nx \in \Lambda\eta_c$ such that $Nx \in \Lambda\eta_c$, $x \notin \mathrm{fv}(N)$ and $c \neq N$. Let $p \in \mathcal{R}^{\beta\eta}_M$. Since $M \in \mathcal{R}^{\beta\eta}$, by lemma 2.5:
  - Either $p = 0$ so $M|_p = M \in \Lambda\eta_c$.
  - Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{Nx}$. By IH, $M|_p = (Nx)|_{p'} \in \Lambda\eta_c$.
- Let $M = cNP \in \mathcal{M}_c$ such that $N, P \in \mathcal{M}_c$. Let $p \in \mathcal{R}^r_M$. Since $M, cN \notin \mathcal{R}^r$, by lemma 2.5:
  - Either $p = 1.2.p'$ such that $p' \in \mathcal{R}^r_N$. By IH, $M|_p = N|_{p'} \in \mathcal{M}_c$.
  - Or $p = 2.p'$ such that $p' \in \mathcal{R}^P_r$. By IH, $M|_p = P|_{p'} \in \mathcal{M}_c$.
- Let $M = (\lambda x.N)P \in \mathcal{M}_c$ such that $\lambda x.N, P \in \mathcal{M}_c$. Let $p \in \mathcal{R}^r_M$. Since by lemma 2.8, $M \in \mathcal{R}^r$, by lemma 2.5:
  - Either $p = 0$ so $M|_p = M \in \mathcal{M}_c$.
  - Or $p = 1.p'$ such that $p' \in \mathcal{R}^r_{\lambda x.N}$. By IH, $M|_p = (\lambda x.N)|_{p'} \in \mathcal{M}_c$.
  - Or $p = 2.p'$ such that $p' \in \mathcal{R}^r_P$. By IH, $M|_p = P|_{p'} \in \mathcal{M}_c$.
- Let $M = cN \in \Lambda\eta_c$ such that $N \in \Lambda\eta_c$. Let $p \in \mathcal{R}^{\beta\eta}_M$. Since $M \notin \mathcal{R}^{\beta\eta}$, by lemma 2.5, $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_N$. By IH, $M|_p = N|_{p'} \in \Lambda\eta_c$. □

Lemma 2.10.

1. Let $M \in \Lambda\eta_c$ and $M \to_{\beta\eta} M'$. Then there exists p such that $M \xrightarrow{p}_{\beta\eta} M'$. We prove that $M' \in \Lambda\eta_c$ by induction on the structure of p.
   - Let $p = 0$. Then:
     - either $M = \lambda x.M'x$ such that $x \notin \mathrm{fv}(M')$. Because $M \in \Lambda\eta_c$, then $M'x \in \Lambda\eta_c$ and $x \neq c$. By lemma 2.4.7, $M' \in \Lambda\eta_c$.

- or $M = (\lambda x.N)P$ and $M' = N[x := P]$. Since $M \in \Lambda\eta_c$ then $\lambda x.N, P \in \Lambda\eta_c$. By definition and lemmas 2.4.9, $N \in \Lambda\eta_c$ and $x \neq c$. By lemma 2.4.9, $M' \in \Lambda\eta_c$.

- Let $p = 1.p'$. Then:
  - either $M = \lambda x.N \xrightarrow{p}_{\beta\eta} \lambda x.N' = M'$ such that $N \xrightarrow{p'}_{\beta\eta} N'$. Since $M \in \Lambda\eta_c$:
    - Either $N = P[x := c(cx)]$ where $P \in \Lambda\eta_c$ and $x \neq c$. So by lemma 2.4.12a, $N' = N''[x := c(cx)]$ and $P \rightarrow_{\beta\eta} N''$. By IH, $N'' \in \Lambda\eta_c$ so by (R1).3, $M' = \lambda x.N''[x := c(cx)] \in \Lambda\eta_c$.
    - Or $N = Px$ where $Px \in \Lambda\eta_c$, $x \notin \text{fv}(P) \cup \{c\}$, $P \neq c$. By IH, $N' \in \Lambda\eta_c$. By lemma 2.4.7, $P \in \Lambda\eta_c$. By case on p':
      - Either $p' = 0$, $P = (\lambda y.Q)$ and $N' = Q[y := x]$. Hence $M' = \lambda x.Q[y := x] = P \in \Lambda\eta_c$.
      - Or $p' = 1.p''$, $N' = P'x$ and $P \xrightarrow{p''}_{\beta\eta} P'$. By lemma 2.2.3, $x \notin \text{fv}(P')$. By IH, $P' \in \Lambda\eta_c$, so by lemma 2.4.3, $P' \neq c$. Hence, $M' = \lambda x.P'x \in \Lambda\eta_c$.
  - or $M = M_1M_2 \xrightarrow{p}_{\beta\eta} M_1'M_2 = M'$ such that $M_1 \xrightarrow{p'}_{\beta\eta} M_1'$. By lemma 2.4.5, $M_2 \in \Lambda\eta_c$ and because $M_1 \neq c$ we obtain:
    - Either $M_1 = cM_0$ and $M_0 \in \Lambda\eta_c$. By case on p' we obtain $p' = 2.p''$, $M_1' = cM_0'$ and $M_0 \xrightarrow{p''}_{\beta\eta} M_0'$. By IH, $M_0' \in \Lambda\eta_c$, so by (R2), $M' = cM_0'M_2 \in \Lambda\eta_c$.
    - Or $M_1 = \lambda x.M_0$ and $M_1 \in \Lambda\eta_c$. By IH, $M_1' \in \Lambda\eta_c$. By lemma 2.4.11a, $M_0 \in \Lambda\eta_c$. lemma 2.4.7, $x \neq c$. By case on p':
      - Either $p' = 0$ and $M_0 = M_1'x$ such that $x \notin \text{fv}(M_1')$. Because $M_0 = M_1'x \in \Lambda\eta_c$, by definition and lemma 2.4.5 we obtain $M' = M_1'M_2 \in \Lambda\eta_c$.
      - Or $p' = 1.p''$ and $M_1' = \lambda x.M_0'$ such that $M_0 \xrightarrow{p''}_{\beta\eta} M_0'$. So $M' = (\lambda x.M_0')M_2 \in \Lambda\eta_c$.
- Let $p = 2.p'$. Then $M = M_1M_2 \xrightarrow{p}_{\beta\eta} M_1M_2' = M'$ such that $M_2 \xrightarrow{p'}_{\beta\eta} M_2'$. By lemma 2.4.5, $M_2 \in \Lambda\eta_c$ so by IH, $M_2' \in \Lambda\eta_c$. Because $M = M_1M_2 \in \Lambda\eta_c$, again by lemma 2.4.5 $M' = M_1M_2' \in \Lambda\eta_c$.

2. By induction on $M \rightarrow_{\beta I} M'$ in a similar fashion to the above. □

Lemma 2.12. We prove the statement by induction on $n \geq 0$.

- Let $n = 0$ then by definition $|c^n(M)|^c = |M|^c$.
- Let $n = m+1$ such that $m \geq 0$ then $|c^n(M)|^c = |c(c^m(M))|^c = |c^m(M)|^c =^{IH} |M|^c$. □

Lemma 2.13. We prove the lemma by induction on n.

- If $n = 0$ then it is done.
- Let $n = m+1$ such that $m \geq 0$. Then, $|\langle c^n(M), \mathcal{R}^{\beta\eta}_{c^n(M)}\rangle|^c = \{|\langle c^n(M), p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{c^n(M)}\} =^{2.5} \{|\langle c^n(M), 2.p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{c^m(M)}\} = \{|\langle c^m(M), p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{c^m(M)}\} =^{IH} |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c$. $\square$

Lemma 2.14. We prove the lemma by induction on n.

- If $n = 0$ then it is done.
- Let $n = m+1$ such that $m \geq 0$. Then, $|\langle c^n(M), 2^n.p\rangle|^c = |\langle c^m(M), 2^m.p\rangle|^c =^{IH} |\langle M, p\rangle|^c$ $\square$

Lemma 2.15.

- let $P \in \mathcal{V}$. We prove the statement by induction on the structure of M.
  - Let $M \in \mathcal{V}$ then $|M|^c = M = P$.
  - Let $M = \lambda x.N$ then $|M|^c = \lambda x.|N|^c \neq P$.
  - Let $M = M_1M_2$. If $M_1 = c$ then $|M|^c = |M_2|^c$. By IH, $\exists n \geq 0$ such that $M_2 = c^n(P)$. If $M_1 \neq c$ then $|M|^c = |M_1|^c|M_2|^c \neq P$.
- Let $P = \lambda x.Q$. We prove the statement by induction on the structure of M.
  - Let $M \in \mathcal{V}$ then $|M|^c = M \neq \lambda x.Q$.
  - Let $M = \lambda x.N$ then $|M|^c = \lambda x.|N|^c$ so $|N|^c = Q$.
  - Let $M = M_1M_2$. If $M_1 = c$ then $|M|^c = |M_2|^c$. By IH, $\exists n \geq 0$ such that $M_2 = c^n(\lambda x.N)$ and $|N|^c = Q$. If $M_1 \neq c$ then $|M|^c = |M_1|^c|M_2|^c \neq \lambda x.Q$.
- Let $P = P_1P_2$. We prove the statement by induction on the structure of M.
  - Let $M \in \mathcal{V}$ then $|M|^c = M \neq P_1P_2$.
  - Let $M = \lambda x.N$ then $|M|^c = \lambda x.|N|^c \neq P_1P_2$.
  - Let $M = M_1M_2$. If $M_1 = c$ then $|M|^c = |M_2|^c$. By IH, $\exists n \geq 0$ such that $M_2 = c^n(M_2'M_2'')$, $M_2' \neq c$, $|M_2'|^c = P_1$ and $|M_2''|^c = P_2$. If $M_1 \neq c$ then $|M|^c = |M_1|^c|M_2|^c = P_1P_2$ so $|M_1|^c = P_1$ and $|M_2|^c = P_2$.

$\square$

Lemma 2.16. We prove the statement by induction on M.

- Let $M \in \mathcal{V}$ then by lemma 2.5, $\mathcal{R}^r_M = \varnothing$.

- Let $M = \lambda x.N$ then by lemma 2.5:
  - Either $M \in \mathcal{R}^r$ then:
    - Either $p = p' = 0$ so it is done.
    - Or $p = 0$ and $p' = 1.p'_1$ such that $p'_1 \in \mathcal{R}^r_N$. Then, $|\langle M, 0\rangle|^c = 0 \neq |\langle M, p'\rangle|^c = 1.|\langle N, p'_1\rangle|^c$.
    - Or $p = 1.p_1$ and $p' = 1.p'_1$ such that $p_1, p_1 \in \mathcal{R}^r_N$. By hypothesis, $|\langle M, p\rangle|^c = 1.|\langle N, p_1\rangle|^c = 1.|\langle N, p'_1\rangle|^c = |\langle M, p'\rangle|^c$. So $|\langle N, p_1\rangle|^c = |\langle N, p'_1\rangle|^c$ and by IH, $p_1 = p'_1$ so $p = p'$.
  - Or $M \notin \mathcal{R}^r$ then $p = 1.p_1$ and $p' = 1.p'_1$ such that $p_1, p_1 \in \mathcal{R}^r_N$. By hypothesis, $|\langle M, p\rangle|^c = 1.|\langle N, p_1\rangle|^c = 1.|\langle N, p'_1\rangle|^c = |\langle M, p'\rangle|^c$. So $|\langle N, p_1\rangle|^c = |\langle N, p'_1\rangle|^c$ and by IH, $p_1 = p'_1$ so $p = p'$.
- Let $M = PQ$ then by lemma 2.5:
  - Either $M \in \mathcal{R}^r$, so P is a λ-abstraction and:
    - Either $p = p' = 0$ so it is done.
    - Or $p = 0$ and $p' = 1.p'_1$ such that $p'_1 \in \mathcal{R}^r_P$. Then $|\langle M, 0\rangle|^c = 0 \neq |\langle M, p'\rangle|^c = 1.|\langle P, p'_1\rangle|^c$.
    - Or $p = 0$ and $p' = 2.p'_1$ such that $p'_1 \in \mathcal{R}^r_Q$. Since P is a λ-abstraction, $|\langle M, 0\rangle|^c = 0 \neq |\langle M, p'\rangle|^c = 2.|\langle Q, p'_1\rangle|^c$.
    - Or $p = 1.p_1$ and $p' = 1.p'_1$ such that $p_1, p'_1 \in \mathcal{R}^r_P$. Since by hypothesis, $|\langle M, p\rangle|^c = 1.|\langle P, p_1\rangle|^c = 1.|\langle P, p'_1\rangle|^c = |\langle M, p'\rangle|^c$, then $|\langle P, p_1\rangle|^c = |\langle P, p'_1\rangle|^c$. By IH, $p_1 = p'_1$ so $p = p'$.
    - Or $p = 1.p_1$ and $p' = 2.p'_1$ such that $p_1 \in \mathcal{R}^r_P$ and $p'_1 \in \mathcal{R}^r_Q$. Since P is a λ-abstraction, $|\langle M, p\rangle|^c = 1.|\langle P, p_1\rangle|^c \neq 2.|\langle Q, p'_1\rangle|^c = |\langle M, p'\rangle|^c$.
    - Or $p = 2.p_1$ and $p' = 2.p'_1$ such that $p_1, p'_1 \in \mathcal{R}^r_Q$. Since P is a λ-abstraction, by hypothesis, $|\langle M, p\rangle|^c = 2.|\langle Q, p_1\rangle|^c = 2.|\langle Q, p'_1\rangle|^c = |\langle M, p'\rangle|^c$ so $|\langle Q, p_1\rangle|^c = |\langle Q, p'_1\rangle|^c$. By IH, $p_1 = p'_1$ so $p = p'$.
  - Or $M \notin \mathcal{R}^r$, then:
    - Or $p = 1.p_1$ and $p' = 1.p'_1$ such that $p_1, p'_1 \in \mathcal{R}^r_P$. Since by hypothesis, $|\langle M, p\rangle|^c = 1.|\langle P, p_1\rangle|^c = 1.|\langle P, p'_1\rangle|^c = |\langle M, p'\rangle|^c$, then $|\langle P, p_1\rangle|^c = |\langle P, p'_1\rangle|^c$. By IH, $p_1 = p'_1$ so $p = p'$.
    - Or $p = 1.p_1$ and $p' = 2.p'_1$ such that $p_1 \in \mathcal{R}^r_P$ and $p'_1 \in \mathcal{R}^r_Q$. $P =\neq c$, otherwise, by lemma 2.5, $\mathcal{R}^r_P = \varnothing$. Moreover, $|\langle M, p\rangle|^c = 1.|\langle P, p_1\rangle|^c \neq 2.|\langle Q, p'_1\rangle|^c = |\langle M, p'\rangle|^c$.
    - Or $p = 2.p_1$ and $p' = 2.p'_1$ such that $p_1, p'_1 \in \mathcal{R}^r_Q$. If $P \neq c$ then, by hypothesis, $|\langle M, p\rangle|^c = 2.|\langle Q, p_1\rangle|^c = 2.|\langle Q, p'_1\rangle|^c = |\langle M, p'\rangle|^c$ so $|\langle Q, p_1\rangle|^c = |\langle Q, p'_1\rangle|^c$. By IH, $p_1 = p'_1$ so $p = p'$. If $P = c$ then, by hypothesis, $|\langle M, p\rangle|^c = |\langle Q, p_1\rangle|^c = |\langle Q, p'_1\rangle|^c = |\langle M, p'\rangle|^c$ so $|\langle Q, p_1\rangle|^c = |\langle Q, p'_1\rangle|^c$. By IH, $p_1 = p'_1$ so $p = p'$.

$\square$

Lemma 2.17. We prove the statement by induction on the structure of M.

- Let $M \in \mathcal{V}$
  - Let $M = x$ then $|M[x := c(cx)]|^c = |c(cx)|^c = |x|^c$.
  - Let $M = y \neq x$ then $|M[x := c(cx)]|^c = |M|^c$.
- Let $M = \lambda y.N$ then $|M[x := c(cx)]|^c = \lambda y.|N[x := c(cx)]|^c =^{IH} \lambda y.|N|^c = |M|^c$, where $y \notin \{x, c\}$.
- Let $M = NP$.
  - Either $N = c$, so $N[x := c(cx)] = c$. Then, $|M[x := c(cx)]|^c = |P[x := c(cx)]|^c =^{IH} |P|^c = |M|^c$.
  - Or $N \neq c$, so $N[x := c(cx)] \neq c$. Then, $|M[x := c(cx)]|^c = |N[x := c(cx)]|^c|P[x := c(cx)]|^c =^{IH} |N|^c|P|^c = |M|^c$.

$\square$

Lemma 2.18. We prove the statement by induction on the structure of M

- Let $M = y$ then by lemma 2.5, $\mathcal{R}^{\beta\eta}_M = \varnothing$.
- Let $M = \lambda y.N$. Then by lemma 2.5:
  - Either $p = 0$ if $M \in \mathcal{R}^{\beta\eta}$. Then, $|\langle M[x := c(cx)], 0\rangle|^c = 0 = |\langle M, 0\rangle|^c$.
  - Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_N$. Then $|\langle M[x := c(cx)], p\rangle|^c = 1.|\langle N[x := c(cx)], p'\rangle|^c =^{IH} 1.|\langle N, p'\rangle|^c = |\langle M, p\rangle|^c$ such that $y \notin \{x, c\}$.
- Let $M = M_1M_2$. Then by lemma 2.5:
  - Either $p = 0$ if $M \in \mathcal{R}^{\beta\eta}$. Then, $|\langle M[x := c(cx)], 0\rangle|^c = 0 = |\langle M, 0\rangle|^c$.
  - Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{M_1}$. Then $|\langle M[x := c(cx)], p\rangle|^c = 1.|\langle M_1[x := c(cx)], p'\rangle|^c =^{IH} 1.|\langle M_1, p'\rangle|^c = |\langle M, p\rangle|^c$.
  - Or $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{M_2}$.
    - If $M_1 = c$ then $M_1[x := c(cx)] = c$ and $|\langle M[x := c(cx)], p\rangle|^c = |\langle M_2[x := c(cx)], p'\rangle|^c =^{IH} |\langle M_2, p'\rangle|^c = |\langle M, p\rangle|^c$.
    - If $M_1 \neq c$ then $M_1[x := c(cx)] \neq c$ and $|\langle M[x := c(cx)], p\rangle|^c = 2.|\langle M_2[x := c(cx)], p'\rangle|^c =^{IH} 2.|\langle M_2, p'\rangle|^c = |\langle M, p\rangle|^c$.

$\square$

Lemma 2.19. We prove this lemma by induction on the structure of M.

- Let $M \in \mathcal{V} \setminus \{c\}$ then $|M|^c = M$ and $\text{fv}(M) \setminus \{c\} = \{M\} = \text{fv}(|M|^c)$.

- Let $M = \lambda y.P \in \Lambda\text{I}_c$ such that $P \in \Lambda\text{I}_c$ and $y \neq c$. Then $|M|^c = \lambda y.|P|^c$ and $\text{fv}(M) \setminus \{c\} = \text{fv}(P) \setminus \{y, c\} =^{IH} \text{fv}(|P|^c) \setminus \{y\} = \text{fv}(|M|^c)$.

- Let $M = \lambda y.P[y := c(cy)] \in \Lambda\eta_c$ such that $P \in \Lambda\eta_c$ and $y \neq c$. Then $|M|^c = \lambda y.|P[y := c(cy)]|^c =^{2.17} \lambda y.|P|^c$ and $\text{fv}(M) \setminus \{c\} = \text{fv}(P[y := c(cy)]) \setminus \{c, y\} = \text{fv}(P) \setminus \{c, y\} =^{IH} \text{fv}(|P|^c) \setminus \{y\} = \text{fv}(|M|^c)$.

- Let $M = \lambda y.Py \in \Lambda\eta_c$ such that $Py \in \Lambda\eta_c$, $y \notin \text{fv}(P) \cup \{c\}$ and $c \neq N$. Then $|M|^c = \lambda y.|Py|^c$ and $\text{fv}(M) \setminus \{c\} = \text{fv}(Py) \setminus \{c, y\} =^{IH} \text{fv}(|Py|^c) \setminus \{y\} = \text{fv}(|M|^c)$.

- Let $M = cPQ \in \mathcal{M}_c$ such that $P, Q \in \mathcal{M}_c$. Then $|M|^c = |P|^c|Q|^c$ and $\text{fv}(M) \setminus \{c\} = (\text{fv}(P) \cup \text{fv}(Q)) \setminus \{c\} = (\text{fv}(P) \setminus \{c\}) \cup (\text{fv}(Q) \setminus \{c\}) =^{IH} \text{fv}(|P|^c) \cup \text{fv}(|Q|^c) = \text{fv}(|M|^c)$.

- Let $M = (\lambda y.P)Q \in \mathcal{M}_c$ such that $\lambda y.P, Q \in \mathcal{M}_c$. Then $|M|^c = |\lambda y.P|^c|Q|^c$ and $\text{fv}(M) \setminus \{c\} = (\text{fv}(\lambda y.P) \cup \text{fv}(Q)) \setminus \{c\} = (\text{fv}(\lambda y.P) \setminus \{c\}) \cup (\text{fv}(Q) \setminus \{c\}) =^{IH} \text{fv}(|\lambda y.P|^c) \cup \text{fv}(|Q|^c) = \text{fv}(|M|^c)$.

- Let $M = cP \in \Lambda\eta_c$ such that $N \in \Lambda\eta_c$. Then $|M|^c = |P|^c$ and $\text{fv}(M) \setminus \{c\} = \text{fv}(P) \setminus \{c\} =^{IH} \text{fv}(|P|^c) = \text{fv}(|M|^c)$. □

Lemma 2.20. We prove this lemma by induction on the structure of M.

- Let $M \in \mathcal{V} \setminus \{c\}$.
  - Either $M = x$ then $|M[x := N]|^c = |N|^c = M[x := |N|^c] = |M|^c[x := |N|^c]$.
  - Or $M = y \neq x$ then $|M[x := N]|^c = |M|^c = M = M[x := |N|^c] = |M|^c[x := |N|^c]$.

- Let $M = \lambda y.P \in \Lambda\text{I}_c$ such that $P \in \Lambda\text{I}_c$ and $y \neq c$. Then $|M[x := N]|^c = \lambda y.|P[x := N]|^c =^{IH} \lambda y.|P|^c[x := |N|^c] = |M|^c[x := |N|^c]$, where $y \notin \text{fv}(N) \cup \{x\}$ and so by lemma 2.19, $y \notin \text{fv}(|N|^c)$.

- Let $M = \lambda y.P[y := c(cy)] \in \Lambda\eta_c$ such that $P \in \Lambda\eta_c$ and $y \neq c$. Then $|M[x := N]|^c = \lambda y.|P[y := c(cy)][x := N]|^c = \lambda y.|P[x := N][y := c(cy)]|^c =^{2.17} \lambda y.|P[x := N]|^c =^{IH} \lambda y.|P|^c[x := |N|^c] =^{2.17} \lambda y.|P[y := c(cy)]|^c[x := |N|^c] = |M|^c[x := |N|^c]$, where $y \notin \text{fv}(N) \cup \{x\}$ and so by lemma 2.19, $y \notin \text{fv}(|N|^c)$.

- Let $M = \lambda y.Py \in \Lambda\eta_c$ such that $Py \in \Lambda\eta_c$, $y \notin \text{fv}(P) \cup \{c\}$ and $c \neq P$. $|M[x := N]|^c = \lambda y.|(Py)[x := N]|^c =^{IH} \lambda y.|Py|^c[x := |N|^c] = |M|^c[x := |N|^c]$, where $y \notin \text{fv}(N) \cup \{x\}$ and so by lemma 2.19, $y \notin \text{fv}(|N|^c)$.

- Let $M = cPQ \in \mathcal{M}_c$ such that $P, Q \in \mathcal{M}_c$. $|M[x := N]|^c = |P[x := N]|^c|Q[x := N]|^c =^{IH} |P|^c[x := |N|^c]|Q|^c[x := |N|^c] = (|P|^c|Q|^c)[x := |N|^c] = |M|^c[x := |N|^c]$.

- Let $M = (\lambda y.P)Q \in \mathcal{M}_c$ such that $\lambda y.P, Q \in \mathcal{M}_c$. $|M[x := N]|^c = |(\lambda y.P)[x := N]|^c|Q[x := N]|^c =^{IH} |\lambda y.P|^c[x := |N|^c]|Q|^c[x := |N|^c] = (|\lambda y.P|^c|Q|^c)[x := |N|^c] = |M|^c[x := |N|^c]$.
- Let $M = cP \in \Lambda\eta_c$ such that $N \in \Lambda\eta_c$. $|M[x := N]|^c = |P[x := N]|^c =^{IH} |P|^c[x := |N|^c] = |M|^c[x := |N|^c]$. $\square$

Lemma 2.21. We prove the lemma by induction on the structure of M.

- Let $M \in \mathcal{V} \setminus \{c\}$ then $|M|^c = M \in \mathcal{V} \setminus \{c\} \subseteq \Lambda\mathrm{I}$.
- let $M = \lambda x.N$ such that $N \in \Lambda\mathrm{I}_c$ and $x \in \mathrm{fv}(N)$ and $x \neq c$. Then $|M|^c = \lambda x.|N|^c$ and by IH $|N|^c \in \Lambda\mathrm{I}$. Since $x \in \mathrm{fv}(N)$, by lemma 2.19, $x \in \mathrm{fv}(|N|^c)$, so $|M|^c \in \Lambda\mathrm{I}$.
- Let $M = cPQ$ such that $P, Q \in \Lambda\mathrm{I}_c$ then $|M|^c = |P|^c|Q|^c$ and by IH, $|P|^c, |Q|^c \in \Lambda\mathrm{I}$, hence $|M|^c \in \Lambda\mathrm{I}$.
- Let $M = (\lambda x.P)Q$ such that $\lambda x.P, Q \in \Lambda\mathrm{I}_c$ then $|M|^c = |\lambda x.P|^c|Q|^c$ and by IH, $|\lambda x.P|^c, |Q|^c \in \Lambda\mathrm{I}$, hence $|M|^c \in \Lambda\mathrm{I}$. $\square$

Lemma 2.22. Let $p \in \mathcal{R}^r_M$, then by definition, $M|_p \in \mathcal{R}^r$. We prove the result by induction on the structure of p.

- Let $p = 0$.
  - Let $r = \beta I$ then $M = (\lambda x.M_1)M_2$ such that $x \in \mathrm{fv}(M_1)$ and $\lambda x.M_1, M_2 \in \Lambda\mathrm{I}_c$ and $M' = M_1[x := M_2]$. By definition $M_1 \in \Lambda\mathrm{I}_c$, $x \in \mathrm{fv}(M_1)$ and $x \neq c$. Then $|M|^c = (\lambda x.|M_1|^c)|M_2|^c$ and $|M'|^c = |M_1[x := M_2]|^c =^{2.20} |M_1|^c[x := |M_2|^c]$. By lemma 2.19, $x \in \mathrm{fv}(|M_1|^c)$. So, $|M|^c \xrightarrow{0}_{\beta I} |M'|^c$ and $|\langle M, 0\rangle|^c = 0$.
  - Let $r = \beta\eta$.
    - Either $M = (\lambda x.M_1)M_2$ such that $\lambda x.M_1, M_2 \in \Lambda\eta_c$ and $M' = M_1[x := M_2]$. By lemma 2.4, $M_1 \in \Lambda\mathrm{I}_c$ and $x \neq c$. Then $|M|^c = (\lambda x.|M_1|^c)|M_2|^c$ and $|M'|^c = |M_1[x := M_2]|^c =^{2.20} |M_1|^c[x := |M_2|^c]$. So, $|M|^c \xrightarrow{0}_{\beta} |M'|^c$ and $|\langle M, 0\rangle|^c = 0$.
    - Or $M = \lambda x.M'x$ such that $M'x \in \Lambda\eta_c$, $x \notin \mathrm{fv}(M')$, $x \neq c$ and $M' \neq c$. Then $|M|^c = \lambda x.|M'|^c x$. By lemma 2.19, $x \in \mathrm{fv}(|M'|^c)$. So, $|M|^c \xrightarrow{0}_{\beta} |M'|^c$ and $|\langle M, 0\rangle|^c = 0$.
- Let $p = 1.p'$.
  - Either $M = \lambda x.M_1$ and $M' = \lambda x.M_1'$ such that $M_1 \xrightarrow{p'}_r M_1'$. By lemma 2.5, $p' \in \mathcal{R}^r_{M_1}$. By lemma 2.4, $M_1 \in \mathcal{M}_c$ and $x \neq c$. By IH, $|M_1|^c \xrightarrow{p''}_r |M_1'|^c$ such that $p'' = |\langle M_1, p'\rangle|^c$. So $|M|^c \xrightarrow{1.p''}_r |M'|^c$ and $1.p'' = |\langle M, p\rangle|^c$.

- Or $M = M_1M_2$ and $M' = M'_1M_2$ such that $M_1 \xrightarrow{p'}_r M'_1$. By lemma 2.5, $p' \in \mathcal{R}^r_{M_1}$. By lemma 2.5, $M_1 \neq c$. By lemma 2.4.5:
  * Either $M_1 = cM_0$ where $M_0 \in \mathcal{M}_c$. By lemma 2.5, $p' = 2.p'_0$ such that $p'_0 \in \mathcal{R}^r_{M_0}$. So by definition $M'_1 = cM'_0$ such that $M_0 \xrightarrow{p'_0}_r M'_0$. By IH, $|M_0|^c \xrightarrow{p''_0}_r |M'_0|^c$ such that $p''_0 = |\langle M_0, p'_0\rangle|^c$. Hence $|M|^c \xrightarrow{1.p''_0}_r |M'|^c$ and $|\langle M, p\rangle|^c = |\langle cM_0M_2, 1.2.p'_0\rangle|^c = 1.|\langle cM_0, 2.p'_0\rangle|^c = 1.|\langle M_0, p'_0\rangle|^c = 1.p''_0$
  * Or $M_1 = \lambda x.M_0 \in \mathcal{M}_c$. By IH, $|M_1|^c \xrightarrow{p''}_r |M'_1|^c$ such that $p'' = |\langle M_1, p'\rangle|^c$. By lemma 2.10, $M'_1 \in \mathcal{M}_c$ and by lemma 2.4.3, $M'_1 \neq c$. So, $|M|^c \xrightarrow{1.p''}_r |M'|^c$ and $|\langle M, p\rangle|^c = 1.|\langle M_1, p'\rangle|^c = 1.p''$.

- Let $p = 2.p'$ then $M = M_1M_2$ and $M' = M_1M'_2$ such that $M_2 \xrightarrow{p'}_r M'_2$. By lemma 2.5, $p' \in \mathcal{R}^r_{M_2}$. By lemma 2.4.5, $M_2 \in \mathcal{M}_c$. By IH, $|M_2|^c \xrightarrow{p''}_r |M'_2|^c$ such that $p'' = |\langle M_2, p'\rangle|^c$.
  - If $M_1 = c$ then $|M|^c \xrightarrow{p''}_r |M'|^c$ and $|\langle M, p\rangle|^c = |\langle M_2, p'\rangle|^c = p''$.
  - Otherwise $|M|^c \xrightarrow{2.p''}_r |M'|^c$ and $|\langle M, p\rangle|^c = 2.|\langle M_2, p'\rangle|^c = 2.p''$.

$\square$

Lemma 2.23. The proof is by induction on the structure of M_1.

- Let $M_1 \in \mathcal{V} \setminus \{c\}$. Then $M_1 = |M_1|^c = |M_2|^c$. By lemma 2.15, $M_2 = c^n(M_1)$.
  - Either $M_1 = x$, then $M_1[x := N_1] = N_1$ and $M_2[x := N_2] = c^n(N_2)$. By hypothesis $|\langle N_1, \mathcal{R}^r_{N_1}\rangle|^c \subseteq |\langle N_2, \mathcal{R}^r_{N_2}\rangle|^c =^{2.13} |\langle c^n(N_2), \mathcal{R}^r_{c^n(N_2)}\rangle|^c$
  - Or $M_1 = y \neq x$ then $M_1[x := N_1] = y$ and $M_2[x := N_2] = c^n(y)$. We conclude using lemma 2.13.

- Let $M_1 = \lambda y.M'_1 \in \Lambda\mathrm{I}_c$ such that $y \in \mathrm{fv}(M'_1)$, $y \neq c$ and $M'_1 \in \Lambda\mathrm{I}_c$ then $|M_1|^c = \lambda y.M'_1 = |M_2|^c$. By lemma 2.15 and because $M_2 \in \Lambda\mathrm{I}_c$, $M_2 = \lambda y.M'_2$, $y \in \mathrm{fv}(M'_2)$, $M'_2 \in \Lambda\mathrm{I}_c$ and $|M'_2|^c = |M'_1|^c$. By lemma 2.5, $\mathcal{R}^{\beta I}_{M_1} = \{1.p \mid p \in \mathcal{R}^{\beta I}_{M'_1}\}$ and $\mathcal{R}^{\beta I}_{M_2} = \{1.p \mid p \in \mathcal{R}^{\beta I}_{M'_2}\}$. So, $|\langle M_1, \mathcal{R}^{\beta I}_{M_1}\rangle|^c = \{1.p \mid p \in |\langle M'_1, \mathcal{R}^{\beta I}_{M'_1}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^{\beta I}_{M_2}\rangle|^c = \{1.p \mid p \in |\langle M'_2, \mathcal{R}^{\beta I}_{M'_2}\rangle|^c\}$. Let $p \in |\langle M'_1, \mathcal{R}^{\beta I}_{M'_1}\rangle|^c$, then $1.p \in |\langle M_1, \mathcal{R}^{\beta I}_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^{\beta I}_{M_2}\rangle|^c$. So $p \in |\langle M'_2, \mathcal{R}^{\beta I}_{M'_2}\rangle|^c$, i.e. $|\langle M'_1, \mathcal{R}^{\beta I}_{M'_1}\rangle|^c \subseteq |\langle M'_2, \mathcal{R}^{\beta I}_{M'_2}\rangle|^c$. By IH, $|\langle M'_1[x := N_1], \mathcal{R}^{\beta I}_{M'_1[x:=N_1]}\rangle|^c \subseteq |\langle M'_2[x := N_2], \mathcal{R}^{\beta I}_{M'_2[x:=N_2]}\rangle|^c$.

  Since $M_1[x := N_1] = \lambda y.M'_1[x := N_1]$ and $M_2[x := N_2] = \lambda y.M'_2[x := N_2]$ where $y \notin \mathrm{fv}(N_1) \cup \mathrm{fv}(N_2)$, by lemma 2.5, $\mathcal{R}^{\beta I}_{M_1[x:=N_1]} = \{1.p \mid p \in$

$\mathcal{R}^{\beta I}_{M_1'[x:=N_1]}\}$ and $\mathcal{R}^{\beta I}_{M_2[x:=N_2]} = \{1.p \mid p \in \mathcal{R}^{\beta I}_{M_2'[x:=N_2]}\}$. So $|\langle M_1[x := N_1], \mathcal{R}^{\beta I}_{M_1[x:=N_1]}\rangle|^c = \{1.p \mid p \in |\langle M_1'[x := N_1], \mathcal{R}^{\beta I}_{M_1'[x:=N_1]}\rangle|^c\}$ and $|\langle M_2[x := N_2], \mathcal{R}^{\beta I}_{M_2[x:=N_2]}\rangle|^c = \{1.p \mid p \in |\langle M_2'[x := N_2], \mathcal{R}^{\beta I}_{M_2'[x:=N_2]}\rangle|^c\}$. Let $p \in |\langle M_1[x := N_1], \mathcal{R}^{\beta I}_{M_1[x:=N_1]}\rangle|^c$ then $p = 1.p'$ such that $p' \in |\langle M_1'[x := N_1], \mathcal{R}^{\beta I}_{M_1'[x:=N_1]}\rangle|^c \subseteq |\langle M_2'[x := N_2], \mathcal{R}^{\beta I}_{M_2'[x:=N_2]}\rangle|^c$. So $p \in |\langle M_2[x := N_2], \mathcal{R}^{\beta I}_{M_2[x:=N_2]}\rangle|^c$.

- Let $M_1 = \lambda y.M_1'[y := c(cy)] \in \Lambda\eta_c$ such that $M_1' \in \Lambda\eta_c$ and $y \neq c$, then $|M_1|^c =^{2.17} \lambda y.|M_1'|^c$. Because $|M_2|^c = \lambda y.|M_1'|^c$, then by lemma 2.15, $M_2 = c^n(\lambda y.P)$ such that $|P|^c = |M_1'|^c$. By lemma 2.4.6, $\lambda y.P \in \Lambda\eta_c$. By lemma 2.4.11a, $P \in \Lambda\eta_c$. We prove the lemma by case on $\lambda y.P$.
  - Either $\lambda y.P = \lambda y.M_2'[y := c(cy)]$ such that $M_2' \in \Lambda\eta_c$. Hence $|M_2'|^c =^{2.17} |M_2'[y := c(cy)]|^c = |M_1'|^c$. We also have $\mathcal{R}^{\beta\eta}_{M_1} =^{2.7.3} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_1'[y:=c(cy)]}\} =^{2.7.4} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_1'}\}$ and $\mathcal{R}^{\beta\eta}_{\lambda y.P} =^{2.7.3} \{1.p \in \mathcal{R}^{\beta\eta}_{M_2'[y:=c(cy)]}\} =^{2.7.4} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_2'}\}$. So $|\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c =^{2.18} \{1.p \mid p \in |\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c =^{2.13} |\langle \lambda y.P, \mathcal{R}^{\beta\eta}_{\lambda y.P}\rangle|^c =^{2.18} \{1.p \mid p \in |\langle M_2', \mathcal{R}^{\beta\eta}_{M_2'}\rangle|^c\}$. Let $p \in |\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c$, so $p \in |\langle M_2', \mathcal{R}^{\beta\eta}_{M_2'}\rangle|^c$, i.e. $|\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c \subseteq |\langle M_2', \mathcal{R}^{\beta\eta}_{M_2'}\rangle|^c$. By IH, $|\langle M_1'[x := N_1], \mathcal{R}^{\beta\eta}_{M_1'[x:=N_1]}\rangle|^c \subseteq |\langle M_2'[x := N_2], \mathcal{R}^{\beta\eta}_{M_2'[x:=N_2]}\rangle|^c$. Because $M_1[x := N_1] = \lambda y.M_1'[y := c(cy)][x := N_1] = \lambda y.M_1'[x := N_1][y := c(cy)]$ and $(\lambda y.P)[x := N_2] = \lambda y.M_2'[y := c(cy)][x := N_2] = \lambda y.M_2'[x := N_2][y := c(cy)]$ such that $y \notin \mathrm{fv}(N_1) \cup \mathrm{fv}(N_2) \cup \{x\}$, we obtain $\mathcal{R}^{\beta\eta}_{M_1[x:=N_1]} =^{2.7.3} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_1'[x:=N_1][y:=c(cy)]}\} =^{2.7.4} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_1'[x:=N_1]}\}$ and $\mathcal{R}^{\beta\eta}_{(\lambda y.P)[x:=N_2]} =^{2.7.3} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_2'[x:=N_2][y:=c(cy)]}\} =^{2.7.4} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_2'[x:=N_2]}\}$. So $|\langle M_1[x := N_1], \mathcal{R}^{\beta\eta}_{M_1[x:=N_1]}\rangle|^c =^{2.18} \{1.p \mid p \in |\langle M_1'[x := N_1], \mathcal{R}^{\beta\eta}_{M_1'[x:=N_1]}\rangle|^c\}$ and $|\langle M_2[x := N_2], \mathcal{R}^{\beta\eta}_{M_2[x:=N_2]}\rangle|^c =^{2.13} |\langle(\lambda y.P)[x := N_2], \mathcal{R}^{\beta\eta}_{(\lambda y.P)[x:=N_2]}\rangle|^c =^{2.18} \{1.p \mid p \in |\langle M_2'[x := N_2], \mathcal{R}^{\beta\eta}_{M_2'[x:=N_2]}\rangle|^c\}$. Let $p \in |\langle M_1[x := N_1], \mathcal{R}^{\beta I}_{M_1[x:=N_1]}\rangle|^c$ then $p = 1.p'$ such that $p' \in |\langle M_1'[x := N_1], \mathcal{R}^{\beta I}_{M_1'[x:=N_1]}\rangle|^c \subseteq |\langle M_2'[x := N_2], \mathcal{R}^{\beta I}_{M_2'[x:=N_2]}\rangle|^c$. Hence, $p \in |\langle M_2[x := N_2], \mathcal{R}^{\beta I}_{M_2[x:=N_2]}\rangle|^c$.
  - Let $\lambda y.P = \lambda y.M_2'y$ such that $P = M_2'y \in \Lambda\eta_c$, $y \notin \mathrm{fv}(M_2')$ and $M_2' \neq c$. So we have $|M_2'y|^c = |M_1'|^c$. We already showed that $\mathcal{R}^{\beta\eta}_{M_1} = \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_1'}\}$. Since $\lambda y.P \in \mathcal{R}^{\beta\eta}$, by lemma 2.5, $\mathcal{R}^{\beta\eta}_{\lambda y.P} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_2'y}\}$. So $|\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c =^{2.18} \{1.p \mid p \in |\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c =^{2.13} |\langle \lambda y.P, \mathcal{R}^{\beta\eta}_{\lambda y.P}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle M_2'y, \mathcal{R}^{\beta\eta}_{M_2'y}\rangle|^c\}$. Let $p \in |\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c$, so $p \in |\langle M_2'y, \mathcal{R}^{\beta\eta}_{M_2'y}\rangle|^c$, i.e. $|\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c \subseteq$

$|\langle M_2'y, \mathcal{R}^{\beta\eta}_{M_2'y}\rangle|^c$. By IH, $|\langle M_1'[x := N_1], \mathcal{R}^{\beta\eta}_{M_1'[x:=N_1]}\rangle|^c = |\langle (M_2'y)[x := N_2], \mathcal{R}^{\beta\eta}_{(M_2'y)[x:=N_2]}\rangle|^c$.

Because $M_1[x := N_1] = \lambda y.M_1'[y := c(cy)][x := N_1] = \lambda y.M_1'[x := N_1][y := c(cy)]$, $(\lambda y.P)[x := N_2] = \lambda y.(M_2'y)[x := N_2] = \lambda y.M_2'[x := N_2]y$ such that $y \notin \text{fv}(N_1) \cup \text{fv}(N_2) \cup \{x\}$, we obtain $(\lambda y.P)[x := N_2] \in \mathcal{R}^{\beta\eta}$, $\mathcal{R}^{\beta\eta}_{M_1[x:=N_1]} =^{2.7.3} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_1'[x:=N_1][y:=c(cy)]}\} =^{2.7.4} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_1'[x:=N_1]}\}$ and $\mathcal{R}^{\beta\eta}_{(\lambda y.P)[x:=N_2]} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{(M_2'y)[x:=N_2]}\}$. So $|\langle M_1[x := N_1], \mathcal{R}^{\beta\eta}_{M_1[x:=N_1]}\rangle|^c =^{2.18} \{1.p \mid p \in |\langle M_1'[x := N_1], \mathcal{R}^{\beta\eta}_{M_1'[x:=N_1]}\rangle|^c\}$ and $|\langle M_2[x := N_2], \mathcal{R}^{\beta\eta}_{M_2[x:=N_2]}\rangle|^c =^{2.13} |\langle (\lambda y.P)[x := N_2], \mathcal{R}^{\beta\eta}_{(\lambda y.P)[x:=N_2]}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle (M_2'y)[x := N_2], \mathcal{R}^{\beta\eta}_{(M_2'y)[x:=N_2]}\rangle|^c\}$. Let $p \in |\langle M_1[x := N_1], \mathcal{R}^{\beta I}_{M_1[x:=N_1]}\rangle|^c$ then $p = 1.p'$ such that $p' \in |\langle M_1'[x := N_1], \mathcal{R}^{\beta I}_{M_1'[x:=N_1]}\rangle|^c \subseteq |\langle (M_2'y)[x := N_2], \mathcal{R}^{\beta I}_{(M_2'y)[x:=N_2]}\rangle|^c$. So $p \in |\langle M_2[x := N_2], \mathcal{R}^{\beta I}_{M_2[x:=N_2]}\rangle|^c$.

- Let $M_1 = \lambda y.M_1'y \in \Lambda\eta_c$ such that $M_1'y \in \Lambda\eta_c$, $M_1' \neq c$ and $y \notin \text{fv}(M_1') \cup \{c\}$, then $|M_1|^c = \lambda y.|M_1'y|^c$. Because $|M_2|^c = \lambda y.|M_1'y|^c$, then by lemma 2.15, $M_2 = c^n(\lambda y.P)$ such that $|P|^c = |M_1'y|^c$. By lemma 2.4.6, $\lambda y.P \in \Lambda\eta_c$. By lemma 2.4.11a, $P \in \Lambda\eta_c$. We prove the lemma by case on $\lambda y.P$.
  - Either $\lambda y.P = \lambda y.M_2'[y := c(cy)]$ such that $M_2' \in \Lambda\eta_c$. Since $M_1 \in \mathcal{R}^{\beta\eta}$, $\mathcal{R}^{\beta\eta}_{M_1} =^{2.5} \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_1'y}\}$. Moreover, $\mathcal{R}^{\beta\eta}_{\lambda y.P} =^{2.7.3} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_2'[y:=c(cy)]}\}$, so $|\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle M_1'y, \mathcal{R}^{\beta\eta}_{M_1'y}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c =^{2.13} |\langle \lambda y.P, \mathcal{R}^{\beta\eta}_{\lambda y.P}\rangle|^c = \{1.p \mid p \in |\langle M_2'[y := c(cy)], \mathcal{R}^{\beta\eta}_{M_2'[y:=c(cy)]}\rangle|^c\}$. We have $0 \in |\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c$ but $0 \notin |\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c$.
  - Or $\lambda y.P = \lambda y.M_2'y$ such that $M_2'y \in \Lambda\eta_c$, $y \notin \text{fv}(M_2') \cup \{x\}$ and $M_2' \neq c$. So we have $|M_2'y|^c = |M_1'y|^c$. Because $M_1, \lambda y.P \in \mathcal{R}^{\beta\eta}$, by lemma 2.5, $\mathcal{R}^{\beta\eta}_{M_1} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_1'y}\}$ and $\mathcal{R}^{\beta\eta}_{\lambda y.P} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_2'y}\}$. So $|\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle M_1'y, \mathcal{R}^{\beta\eta}_{M_1'y}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c =^{2.13} |\langle \lambda y.P, \mathcal{R}^{\beta\eta}_{\lambda y.P}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle M_2'y, \mathcal{R}^{\beta\eta}_{M_2'y}\rangle|^c\}$. Let $p \in |\langle M_1'y, \mathcal{R}^{\beta\eta}_{M_1'y}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c$, so $p \in |\langle M_2'y, \mathcal{R}^{\beta\eta}_{M_2'y}\rangle|^c$, i.e. $|\langle M_1'y, \mathcal{R}^{\beta\eta}_{M_1'y}\rangle|^c \subseteq |\langle M_2'y, \mathcal{R}^{\beta\eta}_{M_2'y}\rangle|^c$. By IH, $|\langle (M_1'y)[x := N_1], \mathcal{R}^{\beta\eta}_{(M_1'y)[x:=N_1]}\rangle|^c = |\langle (M_2'y)[x := N_2], \mathcal{R}^{\beta\eta}_{(M_2'y)[x:=N_2]}\rangle|^c$. Because $M_1[x := N_1] = \lambda y.(M_1'y)[x := N_1] = \lambda y.M_1'[x := N_1]y$, $(\lambda y.P)[x := N_2] = \lambda y.(M_2'y)[x := N_2] = \lambda y.M_2'[x := N_2]y$ and $y \notin \text{fv}(N_1) \cup \text{fv}(N_2)$ such that $y \notin \text{fv}(N_1) \cup \text{fv}(N_2) \cup \{x\}$, we have $M_1[x := N_1], (\lambda y.P)[x := N_2] \in \mathcal{R}^{\beta\eta}$, $\mathcal{R}^{\beta\eta}_{M_1[x:=N_1]} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{(M_1'y)[x:=N_1]}\}$ and $\mathcal{R}^{\beta\eta}_{M_2[x:=N_2]} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{(M_2'y)[x:=N_2]}\}$.

So $|\langle M_1[x := N_1], \mathcal{R}^{\beta\eta}_{M_1[x:=N_1]}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle (M'_1 y)[x := N_1], \mathcal{R}^{\beta\eta}_{(M'_1 y)[x:=N_1]}\rangle|^c\}$ and $|\langle M_2[x := N_2], \mathcal{R}^{\beta\eta}_{M_2[x:=N_2]}\rangle|^c =^{2.13} |\langle (\lambda y.P)[x := N_2], \mathcal{R}^{\beta\eta}_{(\lambda y.P)[x:=N_2]}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle (M'_2 y)[x := N_2], \mathcal{R}^{\beta\eta}_{(M'_2 y)[x:=N_2]}\rangle|^c\}$. Let $p \in |\langle M_1[x := N_1], \mathcal{R}^{\beta I}_{M_1[x:=N_1]}\rangle|^c$ then either $p = 0 \in |\langle M_2[x := N_2], \mathcal{R}^{\beta\eta}_{M_2[x:=N_2]}\rangle|^c$ or $p = 1.p'$ such that $p' \in |\langle (M'_1 y)[x := N_1], \mathcal{R}^{\beta I}_{(M'_1 y)[x:=N_1]}\rangle|^c \subseteq |\langle (M'_2 y)[x := N_2], \mathcal{R}^{\beta I}_{(M'_2 y)[x:=N_2]}\rangle|^c$. So $p \in |\langle M_2[x := N_2], \mathcal{R}^{\beta I}_{M_2[x:=N_2]}\rangle|^c$.

- Let $M_1 = cP_1Q_1 \in \mathcal{M}_c$ such that $P_1, Q_2 \in \mathcal{M}_c$ then $|M_1|^c = |P_1|^c|Q_1|^c = |M_2|^c$. Note that $M_1 \notin \mathcal{R}^r$. Because $|M_2|^c = |P_1|^c|Q_1|^c$, then by lemma 2.15, $M_2 = c^n(PQ)$ such that $P \neq c$, $|P|^c = |P_1|^c$ and $|Q|^c = |Q_1|^c$. By lemma 2.4.6, $PQ \in \mathcal{M}_c$. We prove the lemma by case on PQ.
  - Either $P, Q \in \mathcal{M}_c$ and P is a λ-abstraction $\lambda y.P'$. Because $PQ \in \mathcal{M}_c$, by lemma 2.8, $PQ = (\lambda y.P')Q \in \mathcal{R}^r$. By lemma 2.5, $\mathcal{R}^r_{M_1} = \{1.2.p \mid p \in \mathcal{R}^r_{P_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{PQ} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_P\} \cup \{2.p \mid p \in \mathcal{R}^r_Q\}$. So $|\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c = \{1.p \mid p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c =^{2.13} |\langle PQ, \mathcal{R}^r_{PQ}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle P, \mathcal{R}^r_P\rangle|^c\} \cup \{2.p \mid p \in |\langle Q, \mathcal{R}^r_Q\rangle|^c\}$. Let $p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. So $p \in |\langle P, \mathcal{R}^r_P\rangle|^c$, i.e. $|\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c \subseteq |\langle P, \mathcal{R}^r_P\rangle|^c$. Let $p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c$ then $2.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. So $p \in |\langle Q, \mathcal{R}^r_Q\rangle|^c$, i.e. $|\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c \subseteq |\langle Q, \mathcal{R}^r_Q\rangle|^c$. By IH, $|\langle P_1[x := N_1], \mathcal{R}^r_{P_1[x:=N_1]}\rangle|^c \subseteq |\langle P[x := N_2], \mathcal{R}^r_{P[x:=N_2]}\rangle|^c$ and $|\langle Q_1[x := N_1], \mathcal{R}^r_{Q_1[x:=N_1]}\rangle|^c \subseteq |\langle Q[x := N_2], \mathcal{R}^r_{Q[x:=N_2]}\rangle|^c$.

    Because $M_1[x := N_1] = cP_1[x := N_1]Q_1[x := N_1]$ and $(PQ)[x := N_2] = (\lambda y.P'[x := N_2])Q[x := N_2] \in^{2.4.9} \mathcal{M}_c$ such that $y \notin \mathrm{fv}(N_2)$, we obtain $M_1[x := N_1] \notin \mathcal{R}^r$ and $(PQ)[x := N_2] \in^{2.8} \mathcal{R}^r$. So by lemma 2.5 we have $\mathcal{R}^r_{M_1[x:=N_1]} = \{1.2.p \mid p \in \mathcal{R}^r_{P_1[x:=N_1]}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1[x:=N_1]}\}$ and $\mathcal{R}^r_{(PQ)[x:=N_2]} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{P[x:=N_2]}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q[x:=N_2]}\}$. So $|\langle M_1[x := N_1], \mathcal{R}^r_{M_1[x:=N_1]}\rangle|^c = \{1.p \mid p \in |\langle P_1[x := N_1], \mathcal{R}^r_{P_1[x:=N_1]}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1[x := N_1], \mathcal{R}^r_{Q_1[x:=N_1]}\rangle|^c\}$ and $|\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c =^{2.13} |\langle (PQ)[x := N_2], \mathcal{R}^r_{(PQ)[x:=N_2]}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle P[x := N_2], \mathcal{R}^r_{P[x:=N_2]}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q[x := N_2], \mathcal{R}^r_{Q[x:=N_2]}\rangle|^c\}$. Let $p \in |\langle M_1[x := N_1], \mathcal{R}^r_{M_1[x:=N_1]}\rangle|^c$ then either $p = 1.p'$ such that $p' \in |\langle P_1[x := N_1], \mathcal{R}^r_{P_1[x:=N_1]}\rangle|^c \subseteq |\langle P[x := N_2], \mathcal{R}^r_{P[x:=N_2]}\rangle|^c$. So $p \in |\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c$. Or $p = 2.p'$ such that $p' \in |\langle Q_1[x := N_1], \mathcal{R}^r_{Q_1[x:=N_1]}\rangle|^c \subseteq |\langle Q[x := N_2], \mathcal{R}^r_{Q[x:=N_2]}\rangle|^c$. So $p \in |\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c$.
  - Or $P = cP'$ such that $P', Q \in \mathcal{M}_c$, then $|P|^c = |P'|^c = |P_1|^c$. Since $M_1, PQ \notin \mathcal{R}^r$, by lemma 2.5, $\mathcal{R}^r_{M_1} = \{1.2.p \mid p \in \mathcal{R}^r_{P_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{PQ} = \{1.2.p \mid p \in \mathcal{R}^r_{P'}\} \cup \{2.p \mid p \in \mathcal{R}^r_Q\}$. So $|\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c = \{1.p \mid p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c =^{2.13} |\langle PQ, \mathcal{R}^r_{PQ}\rangle|^c = \{1.p \mid p \in$

$|\langle P', \mathcal{R}^r_{P'}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q, \mathcal{R}^r_Q\rangle|^c\}$. Let $p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. So $p \in |\langle P', \mathcal{R}^r_{P'}\rangle|^c$, i.e. $|\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c \subseteq |\langle P', \mathcal{R}^r_{P'}\rangle|^c$. Let $p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c$ then $2.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. So $p \in |\langle Q, \mathcal{R}^r_Q\rangle|^c$, i.e. $|\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c \subseteq |\langle Q, \mathcal{R}^r_Q\rangle|^c$. By IH, $|\langle P_1[x := N_1], \mathcal{R}^r_{P_1[x:=N_1]}\rangle|^c \subseteq |\langle P'[x := N_2], \mathcal{R}^r_{P'[x:=N_2]}\rangle|^c$ and $|\langle Q_1[x := N_1], \mathcal{R}^r_{Q_1[x:=N_1]}\rangle|^c \subseteq |\langle Q[x := N_2], \mathcal{R}^r_{Q[x:=N_2]}\rangle|^c$.

Because $M_1[x := N_1] = cP_1[x := N_1]Q_1[x := N_1]$ and $(PQ)[x := N_2] = cP'[x := N_2]Q[x := N_2]$, we obtain $M_1[x := N_1], (PQ)[x := N_2] \notin \mathcal{R}^r$. So by lemma 2.5 we have $\mathcal{R}^r_{M_1[x:=N_1]} = \{1.2.p \mid p \in \mathcal{R}^r_{P_1[x:=N_1]}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1[x:=N_1]}\}$ and $\mathcal{R}^r_{(PQ)[x:=N_2]} = \{1.2.p \mid p \in \mathcal{R}^r_{P'[x:=N_2]}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q[x:=N_2]}\}$. So $|\langle M_1[x := N_1], \mathcal{R}^r_{M_1[x:=N_1]}\rangle|^c = \{1.p \mid p \in |\langle P_1[x := N_1], \mathcal{R}^r_{P_1[x:=N_1]}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1[x := N_1], \mathcal{R}^r_{Q_1[x:=N_1]}\rangle|^c\}$ and $|\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c =^{2.13} |\langle (PQ)[x := N_2], \mathcal{R}^r_{(PQ)[x:=N_2]}\rangle|^c = \{1.p \mid p \in |\langle P'[x := N_2], \mathcal{R}^r_{P'[x:=N_2]}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q[x := N_2], \mathcal{R}^r_{Q[x:=N_2]}\rangle|^c\}$. Let $p \in |\langle M_1[x := N_1], \mathcal{R}^r_{M_1[x:=N_1]}\rangle|^c$ then either $p = 1.p'$ such that $p' \in |\langle P_1[x := N_1], \mathcal{R}^r_{P_1[x:=N_1]}\rangle|^c \subseteq |\langle P'[x := N_2], \mathcal{R}^r_{P'[x:=N_2]}\rangle|^c$. So $p \in |\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c$. Or $p = 2.p'$ such that $p' \in |\langle Q_1[x := N_1], \mathcal{R}^r_{Q_1[x:=N_1]}\rangle|^c \subseteq |\langle Q[x := N_2], \mathcal{R}^r_{Q[x:=N_2]}\rangle|^c$. So $p \in |\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c$.

- Let $M_1 = P_1Q_1 \in \mathcal{M}_c$ such that $P_1, Q_1 \in \mathcal{M}_c$ and P_1 is a λ-abstraction $\lambda y.P_0$. Then $|M_1|^c = |P_1|^c|Q_1|^c$. Note that because $M_1 \in \mathcal{M}_c$ then by lemma 2.8, $M_1 \in \mathcal{R}^r$. So by lemma 2.5, $0 \in \mathcal{R}^r_{M_1}$, so $0 \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c$. Because $|M_2|^c = |P_1|^c|Q_1|^c$, then by lemma 2.15, $M_2 = c^n(PQ)$ such that $P \neq c$, $|P|^c = |P_1|^c$ and $|Q|^c = |Q_1|^c$. By lemma 2.4.6, $PQ \in \mathcal{M}_c$. We prove the lemma by case on PQ.
  - Either $P = cP'$ such that $P', Q \in \mathcal{M}_c$, so $PQ \notin \mathcal{R}^r$. Hence, by lemma 2.5, $\mathcal{R}^r_{PQ} = \{1.2.p \mid p \in \mathcal{R}^r_{P'}\} \cup \{2.p \mid p \in \mathcal{R}^r_Q\}$. So $|\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c =^{2.13} |\langle PQ, \mathcal{R}^r_{PQ}\rangle|^c = \{1.p \mid p \in |\langle P', \mathcal{R}^r_{P'}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q, \mathcal{R}^r_Q\rangle|^c\}$. Hence $0 \notin |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$.
  - Or $P, Q \in \mathcal{M}_c$ and P is a λ-abstraction $\lambda y.P'$. Because $PQ = (\lambda y.P')Q \in \mathcal{M}_c$ then by lemma 2.8, $PQ \in \mathcal{R}^r$. By lemma 2.5, $\mathcal{R}^r_{M_1} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{P_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{PQ} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_P\} \cup \{2.p \in \mathcal{R}^r_Q\}$. So, $|\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c =^{2.13} |\langle PQ, \mathcal{R}^r_{PQ}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle P, \mathcal{R}^r_P\rangle|^c\} \cup \{2.p \mid p \in |\langle Q, \mathcal{R}^r_Q\rangle|^c\}$. Let $p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. So $p \in |\langle P, \mathcal{R}^r_P\rangle|^c$, i.e. $|\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c \subseteq |\langle P, \mathcal{R}^r_P\rangle|^c$. let $p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c$ then $2.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. So, $p \in |\langle Q, \mathcal{R}^r_Q\rangle|^c$, i.e. $|\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c \subseteq |\langle Q, \mathcal{R}^r_Q\rangle|^c$. By IH, $|\langle P_1[x := N_1], \mathcal{R}^r_{P_1[x:=N_1]}\rangle|^c \subseteq |\langle P[x := N_2], \mathcal{R}^r_{P[x:=N_2]}\rangle|^c$ and $|\langle Q_1[x := N_1], \mathcal{R}^r_{Q_1[x:=N_1]}\rangle|^c \subseteq |\langle Q[x := N_2], \mathcal{R}^r_{Q[x:=N_2]}\rangle|^c$.

  By lemma 2.4.9, $M_1[x := N_1] \in \mathcal{M}_c$ and by lemma 2.8, $M_1[x :=$

$N_1] = (\lambda y.P_0[x := N_1])Q_1[x := N_1] \in \mathcal{R}^r$. By lemma 2.4.9, $(PQ)[x := N_2] \in \mathcal{M}_c$ and by lemma 2.8, $(PQ)[x := N_2] = (\lambda y.P'[x := N_2])Q[x := N_2] \in \mathcal{R}^r$. So by lemma 2.5 we have $\mathcal{R}^r_{M_1[x:=N_1]} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{P_1[x:=N_1]}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1[x:=N_1]}\}$ and $\mathcal{R}^r_{(PQ)[x:=N_2]} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{P[x:=N_2]}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q[x:=N_2]}\}$. So $|\langle M_1[x := N_1], \mathcal{R}^r_{M_1[x:=N_1]}\rangle|^c = \{0\}\{1.p \mid p \in |\langle P_1[x := N_1], \mathcal{R}^r_{P_1[x:=N_1]}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1[x := N_1], \mathcal{R}^r_{Q_1[x:=N_1]}\rangle|^c\}$ and $|\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c =^{2.13} |\langle (PQ)[x := N_2], \mathcal{R}^r_{(PQ)[x:=N_2]}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle P[x := N_2], \mathcal{R}^r_{P[x:=N_2]}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q[x := N_2], \mathcal{R}^r_{Q[x:=N_2]}\rangle|^c\}$. Let $p \in |\langle M_1[x := N_1], \mathcal{R}^r_{M_1[x:=N_1]}\rangle|^c$ then either $p = 0 \in |\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c$. Or $p = 1.p'$ such that $p' \in |\langle P_1[x := N_1], \mathcal{R}^r_{P_1[x:=N_1]}\rangle|^c \subseteq |\langle P[x := N_2], \mathcal{R}^r_{P[x:=N_2]}\rangle|^c$. So $p \in |\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c$. Or $p = 2.p'$ such that $p' \in |\langle Q_1[x := N_1], \mathcal{R}^r_{Q_1[x:=N_1]}\rangle|^c \subseteq |\langle Q[x := N_2], \mathcal{R}^r_{Q[x:=N_2]}\rangle|^c$. So $p \in |\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c$.

- Let $M_1 = cM_1' \in \Lambda\eta_c$ such that $M_1' \in \Lambda\eta_c$. So $|M_1'|^c = |M_1|^c$. By lemm 2.13, $|\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c = |\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c$. By IH, $|\langle M_1'[x := N_1], \mathcal{R}^r_{M_1'[x:=N_1]}\rangle|^c \subseteq |\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c$. Since $M_1[x := N_1] = cM_1'[x := N_1]$ then by lemm 2.13, $|\langle M_1[x := N_1], \mathcal{R}^{\beta\eta}_{M_1[x:=N_1]}\rangle|^c = |\langle M_1'[x := N_1], \mathcal{R}^{\beta\eta}_{M_1'[x:=N_1]}\rangle|^c$. So $|\langle M_1[x := N_1], \mathcal{R}^r_{M_1[x:=N_1]}\rangle|^c \subseteq |\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c$. □

Lemma 2.24. By lemma 8, $p_1 \in \mathcal{R}^r_{M_1}$ and $p_2 \in \mathcal{R}^r_{M_2}$. We prove this lemma by induction on the structure of M_1.

1. Let $M_1 \in \mathcal{V} \setminus \{c\}$ then nothing to prove since M_1 does not reduce.

2. Let $M_1 = \lambda x.N_1 \in \Lambda\mathrm{I}_c$ such that $x \neq c$. So $|M_1|^c = \lambda x.|N_1|^c = |M_2|^c$. By lemma 2.15, because $M_2 \in \Lambda\mathrm{I}_c$ and by lemma 2.4, $M_2 = \lambda x.N_2$ and $|N_2|^c = |N_1|^c$. So $N_2 \in \Lambda\mathrm{I}_c$. Since $M_1, M_2 \notin \mathcal{R}^{\beta I}$, by lemma 2.5, $\mathcal{R}^{\beta I}_{M_1} = \{1.p \mid p \in \mathcal{R}^{\beta I}_{N_1}\}$ and $\mathcal{R}^{\beta I}_{M_2} = \{1.p \mid p\mathcal{R}^{\beta I}_{N_2}\}$ so $|\langle M_1, \mathcal{R}^{\beta I}_{M_1}\rangle|^c = \{1.p \mid p \in |\langle N_1, \mathcal{R}^{\beta I}_{N_1}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^{\beta I}_{M_2}\rangle|^c = \{1.p \mid p \in |\langle N_2, \mathcal{R}^{\beta I}_{N_2}\rangle|^c\}$. Let $p \in |\langle N_1, \mathcal{R}^{\beta I}_{N_1}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^{\beta I}_{M_1}\rangle|^c$, so by hypothesis, $1.p \in |\langle M_2, \mathcal{R}^{\beta I}_{M_2}\rangle|^c$. Hence, $p \in |\langle N_2, \mathcal{R}^{\beta I}_{N_2}\rangle|^c$, i.e. $|\langle N_1, \mathcal{R}^{\beta I}_{N_1}\rangle|^c \subseteq |\langle N_2, \mathcal{R}^{\beta I}_{N_2}\rangle|^c$. Since $p_1 \in \mathcal{R}^{\beta I}_{M_1}$, $p_1 = 1.p'_1$ such that $p'_1 \in \mathcal{R}^{\beta I}_{N_1}$. Since $p_2 \in \mathcal{R}^{\beta I}_{M_2}$, $p_2 = 1.p'_2$ such that $p'_2 \in \mathcal{R}^{\beta I}_{N_2}$. Since $|\langle M_1, p\rangle|^c = |\langle M_2, p\rangle|^c$ then $|\langle N_1, p'_1\rangle|^c = |\langle N_2, p'_2\rangle|^c$. Hence, $M_1 = \lambda x.N_1 \xrightarrow{p_1}_{\beta I} \lambda x.N'_1 = M'_1$ such that $N_1 \xrightarrow{p'_1}_{\beta I} N'_1$ and $M_2 = \lambda x.N_2 \xrightarrow{p_2}_{\beta I} \lambda x.N'_2 = M'_2$ such that $N_2 \xrightarrow{p'_2}_{\beta I} N'_2$. By IH, $|\langle N'_1, \mathcal{R}^{\beta I}_{N'_1}\rangle|^c \subseteq |\langle N'_2, \mathcal{R}^{\beta I}_{N'_2}\rangle|^c$. By lemma 2.5, $\mathcal{R}^{\beta I}_{M'_1} = \{1.p \mid p \in \mathcal{R}^{\beta I}_{N'_1}\}$ and $\mathcal{R}^{\beta I}_{M'_2} = \{1.p \mid p \in \mathcal{R}^{\beta I}_{N'_2}\}$, so $|\langle M'_1, \mathcal{R}^{\beta I}_{M'_1}\rangle|^c = \{1.p \mid p \in |\langle N'_1, \mathcal{R}^{\beta I}_{N'_1}\rangle|^c\}$ and $|\langle M'_2, \mathcal{R}^{\beta I}_{M'_2}\rangle|^c = \{1.p \mid p \in |\langle N'_2, \mathcal{R}^{\beta I}_{N'_2}\rangle|^c\}$. Let $p \in |\langle M'_1, \mathcal{R}^{\beta I}_{M'_1}\rangle|^c$, then $p = 1.p'$ such that $p' \in |\langle N'_1, \mathcal{R}^{\beta I}_{N'_1}\rangle|^c \subseteq |\langle N'_2, \mathcal{R}^{\beta I}_{N'_2}\rangle|^c$, so $p \in |\langle M'_2, \mathcal{R}^{\beta I}_{M'_2}\rangle|^c$.

3. Let $M_1 = \lambda x.N_1[x := c(cx)] \in \Lambda\eta_c$ such that $N_1 \in \Lambda\eta_c$ and $x \neq c$ then $|M_1|^c = \lambda x.|N_1[x := c(cx)]|^c =^{2.17} \lambda x.|N_1|^c$. Because $|M_2|^c = \lambda x.|N_1|^c$, then by lemma 2.15, $M_2 = c^n(\lambda x.P)$ such that $|P|^c = |N_1|^c$. By lemma 2.4.6, $\lambda x.P \in \Lambda\eta_c$. We prove the lemma by case on $\lambda x.P$.

   - Either $\lambda x.P = \lambda x.N_2[x := c(cx)]$ such that $N_2 \in \Lambda\eta_c$. Then, $|N_1|^c = |P|^c = |N_2[x := c(cx)]|^c =^{2.17} |N_2|^c$ and $\mathcal{R}^{\beta\eta}_{M_1} =^{2.7.3} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1[x:=c(cx)]}\} =^{2.7.4} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1}\}$ and $\mathcal{R}^{\beta\eta}_{\lambda x.P} =^{2.7.3} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2[x:=c(cx)]}\} =^{2.7.4} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2}\}$. So, $|\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c =^{2.18} \{1.p \mid p \in |\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c =^{2.13} |\langle \lambda x.P, \mathcal{R}^{\beta\eta}_{\lambda x.P}\rangle|^c =^{2.18} \{1.p \mid p \in |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c\}$. Let $p \in |\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c$, so $p \in |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c$, i.e. $|\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c \subseteq |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c$. Because $p_1 \in \mathcal{R}^{\beta\eta}_{M_1}$, we obtain $p_1 = 1.p'_1$ such that $p'_1 \in \mathcal{R}^{\beta\eta}_{N_1}$. Because $p_2 \in \mathcal{R}^{\beta\eta}_{M_2}$ and by lemma 2.7.5 we obtain $p_2 = 2^n.1.p'_2$ such that $p'_2 \in \mathcal{R}^{\beta\eta}_{N_2}$. Because $1.|\langle N_1, p'_1\rangle|^c =^{2.18} |\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c =^{2.18,2.14} 1.|\langle N_2, p'_2\rangle|^c$, we obtain $|\langle N_1, p'_1\rangle|^c = |\langle N_2, p'_2\rangle|^c$. So $M_1 = \lambda x.N_1[x := c(cx)] \xrightarrow{p_1}_{\beta\eta} \lambda x.P_1 = M'_1$ and $M_2 = c^n(\lambda x.N_2[x := c(cx)]) \xrightarrow{p_2}_{\beta\eta} c^n(\lambda x.P_2) = M'_2$ such that $N_1[x := c(cx)] \xrightarrow{p'_1}_{\beta\eta} P_1$ and $N_2[x := c(cx)] \xrightarrow{p'_2}_{\beta\eta} P_2$. By lemma 2.4.12a, $P_1 = N'_1[x := c(cx)]$, $P_2 =$

$N_2'[x := c(cx)]$, $N_1 \xrightarrow{p_1'}_{\beta\eta} N_1'$ and $N_2 \xrightarrow{p_2'}_{\beta\eta} N_2'$. By IH, $|\langle N_1', \mathcal{R}^{\beta\eta}_{N_1'}\rangle|^c \subseteq |\langle N_2', \mathcal{R}^{\beta\eta}_{N_2'}\rangle|^c$. Hence, $\mathcal{R}^{\beta\eta}_{M_1'} =^{2.7.3} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1'[x:=c(cx)]}\} =^{2.7.4} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1'}\}$ and $\mathcal{R}^{\beta\eta}_{\lambda x.P_2} =^{2.7.3} \{1.p \in \mathcal{R}^{\beta\eta}_{N_2'[x:=c(cx)]}\} =^{2.7.4} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2'}\}$. So, $|\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c =^{2.18} \{1.p \mid p \in |\langle N_1', \mathcal{R}^{\beta\eta}_{N_1'}\rangle|^c\}$ and $|\langle M_2', \mathcal{R}^{\beta\eta}_{M_2'}\rangle|^c =^{2.13} |\langle \lambda x.P_2, \mathcal{R}^{\beta\eta}_{\lambda x.P_2}\rangle|^c =^{2.18} \{1.p \mid p \in |\langle N_2', \mathcal{R}^{\beta\eta}_{N_2'}\rangle|^c\}$. Let $p \in |\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c$ then $p = 1.p'$ such that $p' \in |\langle N_1', \mathcal{R}^{\beta\eta}_{N_1'}\rangle|^c \subseteq |\langle N_2', \mathcal{R}^{\beta\eta}_{N_2'}\rangle|^c$, so $p \in |\langle M_2', \mathcal{R}^{\beta\eta}_{M_2'}\rangle|^c$, i.e. $|\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c \subseteq |\langle M_2', \mathcal{R}^{\beta\eta}_{M_2'}\rangle|^c$.

- Let $\lambda x.P = \lambda x.N_2x$ such that $N_2x \in \Lambda\eta_c$, $x \notin \mathrm{fv}(N_2)$ and $N_2 \neq c$, then $\lambda x.P \in \mathcal{R}^{\beta\eta}$, $\mathcal{R}^{\beta\eta}_{M_1} =^{2.7.3} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1[x:=c(cx)]}\} =^{2.7.4} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1}\}$ and $\mathcal{R}^{\beta\eta}_{\lambda x.P} =^{2.5} \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2x}\}$. By lemma 2.7.5, $\mathcal{R}^{\beta\eta}_{\lambda x.P} =^{2.5} \{2^n.0\} \cup \{2^n.1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2x}\}$. So, $|\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c =^{2.18} \{1.p \mid p \in |\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c =^{2.13} |\langle \lambda x.P, \mathcal{R}^{\beta\eta}_{\lambda x.P}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle N_2x, \mathcal{R}^{\beta\eta}_{N_2x}\rangle|^c\}$. Let $p \in |\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c$, so $p \in |\langle N_2x, \mathcal{R}^{\beta\eta}_{N_2x}\rangle|^c$, i.e. $|\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c \subseteq |\langle N_2x, \mathcal{R}^{\beta\eta}_{N_2x}\rangle|^c$. Since $p_1 \in \mathcal{R}^{\beta\eta}_{M_1}$, $p_1 = 1.p_1'$ such that $p_1' \in \mathcal{R}^{\beta\eta}_{N_1}$. Because $p_2 \in \mathcal{R}^{\beta\eta}_{M_2}$ and $1.|\langle N_1, p_1'\rangle|^c =^{2.18} |\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c$, then $p_2 = 2^n.1.p_2'$ such that $p_2' \in \mathcal{R}^{\beta\eta}_{N_2x}$. Because $1.|\langle N_1, p_1'\rangle|^c =^{2.18} |\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c =^{2.14} |\langle \lambda x.N_2x, 1.p_2'\rangle|^c = 1.|\langle N_2x, p_2'\rangle|^c$ then $|\langle N_1, p_1'\rangle|^c = |\langle N_2x, p_2'\rangle|^c$. So $M_1 = \lambda x.N_1[x := c(cx)] \xrightarrow{p_1}_{\beta\eta} \lambda x.P_1 = M_1'$ and $M_2 = c^n(\lambda x.N_2x) \xrightarrow{p_2}_{\beta\eta} c^n(\lambda x.N_2') = M_2'$ such that $N_1[x := c(cx)] \xrightarrow{p_1'}_{\beta\eta} P_1$ and $N_2x \xrightarrow{p_2'}_{\beta\eta} N_2'$. By lemma 2.4.12a, $P_1 = N_1'[x := c(cx)]$, and $N_1 \xrightarrow{p_1'}_{\beta\eta} N_1'$. By IH, $|\langle N_1', \mathcal{R}^{\beta\eta}_{N_1'}\rangle|^c \subseteq |\langle N_2', \mathcal{R}^{\beta\eta}_{N_2'}\rangle|^c$. Moreover, $\mathcal{R}^{\beta\eta}_{M_1'} =^{2.7.3} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1'[x:=c(cx)]}\} =^{2.7.4} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1'}\}$ and $\mathcal{R}^{\beta\eta}_{\lambda x.N_2'} \setminus \{0\} =^{2.5} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2'}\}$. So, $|\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c =^{2.18} \{1.p \mid p \in |\langle N_1', \mathcal{R}^{\beta\eta}_{N_1'}\rangle|^c\}$ and $|\langle M_2', \mathcal{R}^{\beta\eta}_{M_2'}\rangle|^c \setminus \{0\} =^{2.13} |\langle \lambda x.N_2', \mathcal{R}^{\beta\eta}_{\lambda x.N_2'}\rangle|^c \setminus \{0\} = \{1.p \in |\langle N_2', \mathcal{R}^{\beta\eta}_{N_2'}\rangle|^c\}$. Let $p \in |\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c$ then $p = 1.p'$ such that $p' \in |\langle N_1', \mathcal{R}^{\beta\eta}_{N_1'}\rangle|^c \subseteq |\langle N_2', \mathcal{R}^{\beta\eta}_{N_2'}\rangle|^c$, so $p \in |\langle M_2', \mathcal{R}^{\beta\eta}_{M_2'}\rangle|^c \setminus \{0\}$, i.e. $|\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c \subseteq |\langle M_2', \mathcal{R}^{\beta\eta}_{M_2'}\rangle|^c$.

4. Let $M_1 = \lambda x.N_1x \in \Lambda\eta_c$ such that $N_1x \in \Lambda\eta_c$, $x \notin \mathrm{fv}(N_1) \cup \{c\}$ and $N_1 \neq c$, then $M_1 \in \mathcal{R}^{\beta\eta}$ and $|M_1|^c = \lambda x.|N_1x|^c = \lambda x.|N_1|^c x$. Because $|M_2|^c = \lambda x.|N_1|^c x$, then by lemma 2.15, $M_2 = c^n(\lambda x.P)$ such that $|P|^c = |N_1|^c x$. By lemma 2.4.6, $\lambda x.P \in \Lambda\eta_c$. We prove the lemma by case on $\lambda x.P$.

   (a) Let $\lambda x.P = \lambda x.N_2[x := c(cx)]$ such that $N_2 \in \Lambda\eta_c$ then $\mathcal{R}^{\beta\eta}_{M_1} =^{2.5} \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1x}\}$ and $\mathcal{R}^{\beta\eta}_{\lambda x.P} =^{2.7.3} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2[x:=c(cx)]}\} =^{2.7.4} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2}\}$. So, $|\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle N_1x, \mathcal{R}^{\beta\eta}_{N_1x}\rangle|^c\}$

and $|\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c =^{2.13} |\langle \lambda x.P, \mathcal{R}^{\beta\eta}_{\lambda x.P}\rangle|^c =^{2.18} \{1.p \mid p \in |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c\}$.
Hence, $0 \in |\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c$ but $0 \notin |\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c$.

(b) Let $\lambda x.P = \lambda x.N_2 x$ such that $N_2 x \in \Lambda\eta_c$, $x \notin \mathrm{fv}(N_2)$ and $N_2 \neq c$, then $M_2 \in \mathcal{R}^{\beta\eta}$. Since $|M_2|^c = \lambda x.|N_2 x|^c = \lambda x.|N_2|^c x$, $|N_1 x|^c = |N_2 x|^c$ and $|N_1|^c = |N_2|^c$. Moreover, $\mathcal{R}^{\beta\eta}_{M_1} =^{2.5} \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1 x}\}$, $\mathcal{R}^{\beta\eta}_{\lambda x.P} =^{2.5} \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2 x}\}$ and $\mathcal{R}^{\beta\eta}_{M_2} =^{2.7.5} \{2^n.p \mid p \in \mathcal{R}^{\beta\eta}_{\lambda x.P}\} =^{2.5} \{2^n.0\} \cup \{2^n.1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2 x}\}$. So, $|\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle N_1 x, \mathcal{R}^{\beta\eta}_{N_1 x}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c =^{2.13} |\langle \lambda x.P, \mathcal{R}^{\beta\eta}_{\lambda x.P}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle N_2 x, \mathcal{R}^{\beta\eta}_{N_2 x}\rangle|^c\}$. Let $p \in |\langle N_1 x, \mathcal{R}^{\beta\eta}_{N_1 x}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c$, so $p \in |\langle N_2 x, \mathcal{R}^{\beta\eta}_{N_2 x}\rangle|^c$, i.e. $|\langle N_1 x, \mathcal{R}^{\beta\eta}_{N_1 x}\rangle|^c \subseteq |\langle N_2 x, \mathcal{R}^{\beta\eta}_{N_2 x}\rangle|^c$. Moreover, $\mathcal{R}^{\beta\eta}_{N_1 x} \setminus \{0\} =^{2.5} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1}\}$ and $\mathcal{R}^{\beta\eta}_{N_2 x} \setminus \{0\} =^{2.5} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2}\}$, so $|\langle N_1 x, \mathcal{R}^{\beta\eta}_{N_1 x}\rangle|^c \setminus \{0\} = \{1.p \mid p \in |\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c\}$ and $|\langle N_2 x, \mathcal{R}^{\beta\eta}_{N_2 x}\rangle|^c \setminus \{0\} = \{1.p \mid p \in |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c\}$. Let $p \in |\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c$ then $1.p \in |\langle N_1 x, \mathcal{R}^{\beta\eta}_{N_1 x}\rangle|^c \setminus \{0\} \subseteq |\langle N_1 x, \mathcal{R}^{\beta\eta}_{N_1 x}\rangle|^c \subseteq |\langle N_2 x, \mathcal{R}^{\beta\eta}_{N_2 x}\rangle|^c$, so $p \in |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c$, i.e. $|\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c \subseteq |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c$. Since $p_1 \in \mathcal{R}^{\beta\eta}_{M_1}$:

- Either $p_1 = 0$. Because $p_2 \in \mathcal{R}^{\beta\eta}_{M_2}$ and $|\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c$, we obtain $p_2 = 2^n.0$. So $M_1 \xrightarrow{0}_{\beta\eta} N_1$ and $M_2 = c^n(\lambda x.N_2 x) \xrightarrow{p_2}_{\beta\eta} c^n(N_2)$. It is done since $|\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c \subseteq |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c =^{2.13} |\langle c^n(N_2), \mathcal{R}^{\beta\eta}_{c^n(N_2)}\rangle|^c$.
- Or $p_1 = 1.p_1'$ such that $p_1' \in \mathcal{R}^{\beta\eta}_{N_1 x}$. Becasue $p_2 \in \mathcal{R}^{\beta\eta}_{M_2}$ and $|\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c$, we obtain $p_2 = 2^n.1.p_2'$ such that $p_2' \in \mathcal{R}^{\beta\eta}_{N_2 x}$. Becasue $1.|\langle N_1 x, p_1'\rangle|^c = |\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c =^{2.14} |\langle \lambda x.N_2 x, 1.p_2'\rangle|^c = 1.|\langle N_2 x, p_2'\rangle|^c$, we obtain $|\langle N_1 x, p_1'\rangle|^c = |\langle N_2 x, p_2'\rangle|^c$. So $M_1 = \lambda x.N_1 x \xrightarrow{p_1}_{\beta\eta} \lambda x.N_1' = M_1'$ and $M_2 = c^n(\lambda x.N_2 x) \xrightarrow{p_2}_{\beta\eta} c^n(\lambda x.N_2') = M_2'$ such that $N_1 x \xrightarrow{p_1'}_{\beta\eta} N_1'$ and $N_2 x \xrightarrow{p_2'}_{\beta\eta} N_2'$. By IH, $|\langle N_1', \mathcal{R}^{\beta\eta}_{N_1'}\rangle|^c \subseteq |\langle N_2', \mathcal{R}^{\beta\eta}_{N_2'}\rangle|^c$.
  - Either $N_1 x \in \mathcal{R}^{\beta\eta}$, so $N_1 = \lambda y.P_1$ and by lemma 2.5, $\mathcal{R}^{\beta\eta}_{N_1 x} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1}\}$. Because $|\langle N_1 x, \mathcal{R}^{\beta\eta}_{N_1 x}\rangle|^c \subseteq |\langle N_2 x, \mathcal{R}^{\beta\eta}_{N_2 x}\rangle|^c$, we obtain $0 \in |\langle N_2 x, \mathcal{R}^{\beta\eta}_{N_2 x}\rangle|^c$. Hence, $0 \in \mathcal{R}^{\beta\eta}_{N_2 x}$ and by lemma 2.5, $\mathcal{R}^{\beta\eta}_{N_2 x} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2}\}$. Hence, $N_2 x \in \mathcal{R}^{\beta\eta}$ and by lemma 2.15, $N_2 = \lambda y.P_2$ such that $|P_1|^c = |P_2|^c$.
    * Either $p_1' = 0$. Because $|\langle N_1 x, p_1'\rangle|^c = |\langle N_2 x, p_2'\rangle|^c$, we obtain $p_2' = 0$. So $M_1 = \lambda x.(\lambda y.P_1)x \xrightarrow{p_1}_{\beta\eta} \lambda x.P_1[y := x] = M_1'$ and $M_2 = c^n(\lambda x.(\lambda y.P_2)x) \xrightarrow{p_2}_{\beta\eta} c^n(\lambda x.P_2[y := x]) = M_2'$. Because $x \notin \mathrm{fv}(N_1) \cup \mathrm{fv}(N_2)$, we obtain $M_1' = N_1$ and $M_2' = c^n(N_2)$. It is done since $|\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c \subseteq |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c =^{2.13} |\langle c^n(N_2), \mathcal{R}^{\beta\eta}_{c^n(N_2)}\rangle|^c$.
    * Let $p_1' = 1.p_1''$ such that $p_1'' \in \mathcal{R}^{\beta\eta}_{N_1}$. Because $|\langle N_1 x, p_1'\rangle|^c =$

$|\langle N_2x, p_2'\rangle|^c$, we obtain $p_2' = 1.p_2''$ such that $p_2'' \in \mathcal{R}^{\beta\eta}_{N_2}$. So $M_1 = \lambda x.N_1x \xrightarrow{p_1}_{\beta\eta} \lambda x.N_1''x = M_1'$ and $M_2 = c^n(\lambda x.N_2x) \xrightarrow{p_2}_{\beta\eta} c^n(\lambda x.N_2''x) = M_2'$ such that $N_1 \xrightarrow{p_1''}_{\beta\eta} N_1''$ and $N_2 \xrightarrow{p_2''}_{\beta\eta} N_2''$. because $x \notin \text{fv}(N_1) \cup \text{fv}(N_2)$, by lemma 2.2.3, we obtain $x \notin \text{fv}(N_1'') \cup \text{fv}(N_2'')$. So, $M_1', \lambda x.N_2''x \in \mathcal{R}^{\beta\eta}$ and by lemma 2.5, $\mathcal{R}^{\beta\eta}_{M_1'} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1'}\}$ and $\mathcal{R}^{\beta\eta}_{\lambda x.N_2''x} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2'}\}$. Hence, $|\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c = \{0\} \cup \{\lambda x.C \mid C \in |\langle N_1', \mathcal{R}^{\beta\eta}_{N_1'}\rangle|^c\}$ and $|\langle M_2', \mathcal{R}^{\beta\eta}_{M_2'}\rangle|^c =^{2.13} |\langle \lambda x.N_2''x, \mathcal{R}^{\beta\eta}_{\lambda x.N_2''x}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle N_2', \mathcal{R}^{\beta\eta}_{N_2'}\rangle|^c\}$. Because $|\langle N_1', \mathcal{R}^{\beta\eta}_{N_1'}\rangle|^c \subseteq |\langle N_2', \mathcal{R}^{\beta\eta}_{N_2'}\rangle|^c$, we obtain $|\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c = \{0\}\cup\{1.p \mid p \in |\langle N_1', \mathcal{R}^{\beta\eta}_{N_1'}\rangle|^c\} \subseteq \{0\}\cup\{1.p \mid p \in |\langle N_2', \mathcal{R}^{\beta\eta}_{N_2'}\rangle|^c\} = |\langle M_2', \mathcal{R}^{\beta\eta}_{M_2'}\rangle|^c$.

– Else by lemma 2.5, $\mathcal{R}^{\beta\eta}_{N_1x} = \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1}\}$. Let $p_1' = 1.p_1''$ such that $p_1'' \in \mathcal{R}^{\beta\eta}_{N_1}$. Then, $p_2' = 1.p_2''$ such that $p_2'' \in \mathcal{R}^{\beta\eta}_{N_2}$. So $M_1 = \lambda x.N_1x \xrightarrow{p_1}_{\beta\eta} \lambda x.N_1''x = M_1'$ and $M_2 = c^n(\lambda x.N_2x) \xrightarrow{p_2}_{\beta\eta} c^n(\lambda x.N_2''x) = M_2'$ such that $N_1 \xrightarrow{p_1''}_{\beta\eta} N_1''$ and $N_2 \xrightarrow{p_2''}_{\beta\eta} N_2''$. Because $x \notin \text{fv}(N_1) \cup \text{fv}(N_2)$, by lemma 2.2.3 we obtain, $x \notin \text{fv}(N_1'')\cup\text{fv}(N_2'')$. So, $M_1', \lambda x.N_2'x \in \mathcal{R}^{\beta\eta}$ and by lemma 2.5, $\mathcal{R}^{\beta\eta}_{M_1'} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1'}\}$ and $\mathcal{R}^{\beta\eta}_{\lambda x.N_2'} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2'}\}$. Hence, $|\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle N_1', \mathcal{R}^{\beta\eta}_{N_1'}\rangle|^c\}$ and $|\langle M_2', \mathcal{R}^{\beta\eta}_{M_2'}\rangle|^c ==^{2.13} |\langle \lambda x.N_2', \mathcal{R}^{\beta\eta}_{\lambda x.N_2'}\rangle|^c = \{0\}\cup\{1.p \mid p \in |\langle N_2', \mathcal{R}^{\beta\eta}_{N_2'}\rangle|^c\}$. Because $|\langle N_1', \mathcal{R}^{\beta\eta}_{N_1'}\rangle|^c \subseteq |\langle N_2', \mathcal{R}^{\beta\eta}_{N_2'}\rangle|^c$, we obtain $|\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c = \{0\}\cup \{1.p \mid p \in |\langle N_1', \mathcal{R}^{\beta\eta}_{N_1'}\rangle|^c\} \subseteq \{0\} \cup \{1.p \mid p \in |\langle N_2', \mathcal{R}^{\beta\eta}_{N_2'}\rangle|^c\} = |\langle M_2', \mathcal{R}^{\beta\eta}_{M_2'}\rangle|^c$.

5. Let $M_1 = cP_1Q_1 \in \mathcal{M}_c$ such that $P_1, P_2 \in \mathcal{M}_c$. So $|M_1|^c = |P_1|^c|Q_1|^c = |M_2|^c$. We prove the statement by induction on the structure of M_2:

   - Let $M_2 \in \mathcal{V} \setminus \{c\}$ then $|M_2|^c = M_2 \neq |P_1|^c|Q_1|^c$.
   - Let $M_2 = \lambda x.N_2 \in \Lambda\mathrm{I}_c$ such that $N_2 \in \Lambda\mathrm{I}_c$ and $x \neq c$ then $|M_2|^c = \lambda x.|N_2|^c \neq |P_1|^c|Q_1|^c$.
   - Let $M_2 = \lambda x.N_2[x := c(cx)] \in \Lambda\eta_c$ such that $N_2 \in \Lambda\eta_c$ and $x \neq c$then $|M_2|^c = \lambda x.|N_2[x := c(cx)]|^c \neq |P_1|^c|Q_1|^c$.
   - Let $M_2 = \lambda x.N_2x \in \Lambda\eta_c$ such that $N_2x \in \Lambda\mathrm{I}_c$ and $x \notin \text{fv}(N_2) \cup \{c\}$ and $N_2 \neq c$ then $|M_2|^c = \lambda x.|N_2x|^c \neq |P_1|^c|Q_1|^c$.
   - Let $M_2 = cP_2Q_2 \in \mathcal{M}_c$ such that $P_2, Q_2 \in \mathcal{M}_c$, then $|cP_2|^c = |P_2|^c = |P_1|^c$ and $|Q_2|^c = |Q_1|^c$. Since $M_1, cP_2 \notin \mathcal{R}^r$, by lemma 2.5, $\mathcal{R}^r_{M_1} =$

$\{1.2.p \mid p \in \mathcal{R}^r_{P_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1}\}$. So, $|\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c = \{1.p \mid p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c\}$. Again by lemma 2.5, since $M_2 \notin \mathcal{R}^r$, $\mathcal{R}^r_{M_2} = \{1.2.p \mid p \in \mathcal{R}^r_{P_2}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_2}\}$. So, $|\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c = \{1.p \mid p \in |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c\}$. Let $p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. Hence, $p \in |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c$, i.e. $|\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c \subseteq |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c$. Let $p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c$ then $2.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. Hence, $p \in |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c$, i.e. $|\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c \subseteq |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c$. Since $p_1 \in \mathcal{R}^r_{M_1}$:

- Either $p_1 = 1.2.p'_1$ such that $p'_1 \in \mathcal{R}^r_{P_1}$ and so $1.|\langle P_1, p'_1\rangle|^c = |\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c$. Hence, because $p_2 \in \mathcal{R}^r_{M_2}$, we obtain $p_2 = 1.2.p'_2$ such that $|\langle P_1, p'_1\rangle|^c = |\langle P_2, p'_2\rangle|^c$ and $p'_2 \in \mathcal{R}^r_{P_2}$. Hence, $M_1 = cP_1Q_1 \xrightarrow{p_1}_r cP'_1Q_1 = M'_1$ and $M_2 = cP_2Q_2 \xrightarrow{p_2}_r cP'_2Q_2 = M'_2$ such that $P_1 \xrightarrow{p'_1}_r P'_1$ and $P_2 \xrightarrow{p'_2}_r P'_2$. By IH, $|\langle P'_1, \mathcal{R}^r_{P'_1}\rangle|^c \subseteq |\langle P'_2, \mathcal{R}^r_{P'_2}\rangle|^c$. By lemma 2.5, $\mathcal{R}^r_{M'_1} = \{1.2.p \mid p \in \mathcal{R}^r_{P'_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{M'_2} = \{1.2.p \mid p \in \mathcal{R}^r_{P'_2}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_2}\}$, so $|\langle M'_1, \mathcal{R}^r_{M'_1}\rangle|^c = \{1.p \mid p \in |\langle P'_1, \mathcal{R}^r_{P'_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c\}$ and $|\langle M'_2, \mathcal{R}^r_{M'_2}\rangle|^c = \{1.p \mid p \in |\langle P'_2, \mathcal{R}^r_{P'_2}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c\}$. Let $p \in |\langle M'_1, \mathcal{R}^r_{M'_1}\rangle|^c$. Either $p = 1.p'$ such that $p' \in |\langle P'_1, \mathcal{R}^r_{P'_1}\rangle|^c \subseteq |\langle P'_2, \mathcal{R}^r_{P'_2}\rangle|^c$. So $p \in |\langle M'_2, \mathcal{R}^r_{M'_2}\rangle|^c$. Or $p = 2.p$ such that $p' \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c \subseteq |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c$. So $p \in |\langle M'_2, \mathcal{R}^r_{M'_2}\rangle|^c$.
- Or $p_1 = 2.p'_1$ such that $p'_1 \in \mathcal{R}^r_{Q_1}$ and so $2.|\langle Q_1, p'_1\rangle|^c = |\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c$. Because $p_2 \in \mathcal{R}^r_{M_2}$, we obtain $p_2 = 2.p'_2$ such that $|\langle Q_1, p'_1\rangle|^c = |\langle Q_2, p'_2\rangle|^c$. Hence, $M_1 = cP_1Q_1 =\xrightarrow{p_1}_r cP_1Q'_1 = M'_1$ and $M_2 = cP_2Q_2 \xrightarrow{p_2}_r cP_2Q'_2 = M'_2$ such that $Q_1 \xrightarrow{p'_1}_r Q'_1$ and $Q_2 \xrightarrow{p'_2}_r Q'_2$. By IH, $|\langle Q'_1, \mathcal{R}^r_{Q'_1}\rangle|^c \subseteq |\langle Q'_2, \mathcal{R}^r_{Q'_2}\rangle|^c$. By lemma 2.5, $\mathcal{R}^r_{M'_1} = \{1.2.p \mid p \in \mathcal{R}^r_{P_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q'_1}\}$ and $\mathcal{R}^r_{M'_2} = \{1.2.p \mid p \in \mathcal{R}^r_{P_2}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q'_2}\}$, so $|\langle M'_1, \mathcal{R}^r_{M'_1}\rangle|^c = \{1.p \mid p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q'_1, \mathcal{R}^r_{Q'_1}\rangle|^c\}$ and $|\langle M'_2, \mathcal{R}^r_{M'_2}\rangle|^c = \{1.p \mid p \in |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q'_2, \mathcal{R}^r_{Q'_2}\rangle|^c\}$. Let $p \in |\langle M'_1, \mathcal{R}^r_{M'_1}\rangle|^c$. Either $p = 1.p'$ such that $p' \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c \subseteq |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c$. So $p \in |\langle M'_2, \mathcal{R}^r_{M'_2}\rangle|^c$. Or $p = 2.p'$ such that $p' \in |\langle Q'_1, \mathcal{R}^r_{Q'_1}\rangle|^c \subseteq |\langle Q'_2, \mathcal{R}^r_{Q'_2}\rangle|^c$. So $p \in |\langle M'_2, \mathcal{R}^r_{M'_2}\rangle|^c$.

- Let $M_2 = P_2Q_2 \in \mathcal{M}_c$ such that $P_2, Q_2 \in \mathcal{M}_c$ and P_2 is a λ-abstraction. Then $|P_2|^c = |P_1|^c$ and $|Q_2|^c = |Q_1|^c$. Since $M_1 \notin \mathcal{R}^r$, by lemma 2.5, $\mathcal{R}^r_{M_1} = \{1.2.p \mid p \in \mathcal{R}^r_{P_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1}\}$. So, $|\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c = \{1.p \mid p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c\}$. Again by lemma 2.5, since $M_2 \in \mathcal{R}^r$ by lemma 2.8, $\mathcal{R}^r_{M_2} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{P_2}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_2}\}$. So, $|\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c = \{0\} \cup = \{1.p \mid p \in |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c\}$. Let $p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. Hence, $p \in$

$|\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c$, i.e. $|\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c \subseteq |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c$. Let $p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c$ then $2.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. Hence, $p \in |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c$, i.e. $|\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c \subseteq |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c$. Since $p_1 \in \mathcal{R}^r_{M_1}$:

- Either $p_1 = 1.2.p'_1$ such that $p'_1 \in \mathcal{R}^r_{P_1}$ and so $1.|\langle P_1, p'_1\rangle|^c = |\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c$. Because $p_2 \in \mathcal{R}^r_{M_2}$, we obtain $p_2 = 1.p'_2$ such that $|\langle P_1, p'_1\rangle|^c = |\langle P_2, p'_2\rangle|^c$ and $p'_2 \in \mathcal{R}^r_{P_2}$. Hence, $M_1 = cP_1Q_1 \xrightarrow{p_1}_r cP'_1Q_1 = M'_1$ and $M_2 = P_2Q_2 \xrightarrow{p_2}_r P'_2Q_2 = M'_2$ such that $P_1 \xrightarrow{p'_1}_r P'_1$ and $P_2 \xrightarrow{p'_2}_r P'_2$. By IH, $|\langle P'_1, \mathcal{R}^r_{P'_1}\rangle|^c \subseteq |\langle P'_2, \mathcal{R}^r_{P'_2}\rangle|^c$. Because $P_2 \in \mathcal{M}_c$, then by lemma 2.10, $P'_2 \in \mathcal{M}_c$. By lemma 2.4.3, $P'_2 \neq c$. By lemma 2.5, $\mathcal{R}^r_{M'_1} = \{1.2.p \mid p \in \mathcal{R}^r_{P'_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{M'_2} \setminus \{0\} = \{1.p \mid p \in \mathcal{R}^r_{P'_2}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_2}\}$, so $|\langle M'_1, \mathcal{R}^r_{M'_1}\rangle|^c = \{1.p \mid p \in |\langle P'_1, \mathcal{R}^r_{P'_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c\}$ and $|\langle M'_2, \mathcal{R}^r_{M'_2}\rangle|^c \setminus \{0\} = \{1.p \mid p \in |\langle P'_2, \mathcal{R}^r_{P'_2}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c\}$. Let $p \in |\langle M'_1, \mathcal{R}^r_{M'_1}\rangle|^c$. Either $p = 1.p'$ such that $p' \in |\langle P'_1, \mathcal{R}^r_{P'_1}\rangle|^c \subseteq |\langle P'_2, \mathcal{R}^r_{P'_2}\rangle|^c$. So $p \in |\langle M'_2, \mathcal{R}^r_{M'_2}\rangle|^c$. Or $p = 2.p'$ such that $p' \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c \subseteq |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c$. So $p \in |\langle M'_2, \mathcal{R}^r_{M'_2}\rangle|^c$.
- Or $p_1 = 2.p'_1$ such that $p'_1 \in \mathcal{R}^r_{Q_1}$ and so $2.|\langle Q_1, p'_1\rangle|^c = |\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c$. Because $p_2 \in \mathcal{R}^r_{M_2}$, we obtain $p_2 = 2.p'_2$ such that $|\langle Q_1, p'_1\rangle|^c = |\langle Q_2, p'_2\rangle|^c$. Hence, $M_1 = cP_1Q_1 \xrightarrow{p_1}_r cP_1Q'_1 = M'_1$ and $M_2 = P_2Q_2 \xrightarrow{p_2}_r P_2Q'_2 = M'_2$ such that $Q_1 \xrightarrow{p'_1}_r Q'_1$ and $Q_2 \xrightarrow{p'_2}_r Q'_2$. By IH, $|\langle Q'_1, \mathcal{R}^r_{Q'_1}\rangle|^c \subseteq |\langle Q'_2, \mathcal{R}^r_{Q'_2}\rangle|^c$. By lemma 2.5, $\mathcal{R}^r_{M'_1} = \{1.2.p \mid p \in \mathcal{R}^r_{P_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q'_1}\}$ and $\mathcal{R}^r_{M'_2} \setminus \{0\} = \{1.p \mid p \in \mathcal{R}^r_{P_2}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q'_2}\}$, so $|\langle M'_1, \mathcal{R}^r_{M'_1}\rangle|^c = \{1.p \mid p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q'_1, \mathcal{R}^r_{Q'_1}\rangle|^c\}$ and $|\langle M'_2, \mathcal{R}^r_{M'_2}\rangle|^c \setminus \{0\} = \{1.p \mid p \in |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q'_2, \mathcal{R}^r_{Q'_2}\rangle|^c\}$. Let $p \in |\langle M'_1, \mathcal{R}^r_{M'_1}\rangle|^c$. Either $p = 1.p'$ such that $p' \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c \subseteq |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c$. So $p \in |\langle M'_2, \mathcal{R}^r_{M'_2}\rangle|^c$. Or $p = 2.p'$ such that $p' \in |\langle Q'_1, \mathcal{R}^r_{Q'_1}\rangle|^c \subseteq |\langle Q'_2, \mathcal{R}^r_{Q'_2}\rangle|^c$. So $p \in |\langle M'_2, \mathcal{R}^r_{M'_2}\rangle|^c$.

- Let $M_2 = cN_2 \in \mathcal{M}_c = \Lambda\eta_c$ such that $N_2 \in \Lambda\eta_c$. So $|N_2|^c = |M_2|^c = |M_1|^c$. By lemma 2.7.5, $\mathcal{R}^{\beta\eta}_{M_2} = \{2.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2}\}$ and $|\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c =^{2.13} |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c$. Because $p_2 \in \mathcal{R}^{\beta\eta}_{M_2}$, we obtain $p_2 = 2.p'_2$ such that $p'_2 \in \mathcal{R}^{\beta\eta}_{N_2}$. So, $M_2 = cN_2 \xrightarrow{p_2}_{\beta\eta} cN'_2 = M'_2$ such that $N_2 \xrightarrow{p'_2}_{\beta\eta} N'_2$. Because $|\langle N_2, p'_2\rangle|^c =^{2.14} |\langle M_2, p_2\rangle|^c = |\langle M_1, p_1\rangle|^c$, by IH, $|\langle M'_1, \mathcal{R}^{\beta\eta}_{M'_1}\rangle|^c \subseteq |\langle N'_2, \mathcal{R}^{\beta\eta}_{N'_2}\rangle|^c =^{2.13} |\langle M'_2, \mathcal{R}^{\beta\eta}_{M'_2}\rangle|^c$.

6. Let $M_1 = (\lambda x.P_1)Q_1 \in \mathcal{M}_c$ such that $\lambda x.P_1, Q_1 \in \mathcal{M}_c$. By lemma 2.4.7, lemma 2.4.11a and lemma 2.4.8, $P_1 \in \mathcal{M}_c$ and $x \neq c$. So $|M_1|^c = |\lambda x.P_1|^c|Q_1|^c = |M_2|^c = (\lambda x.|P_1|^c)|Q_1|^c$. By lemma 2.8, $M_1 \in \mathcal{R}^r$, so by lemma 2.5, $\mathcal{R}^r_{M_1} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{\lambda x.P_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{M_1} \setminus \{1.0\} = \{0\} \cup \{1.1.p \mid p \in \mathcal{R}^r_{P_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1}\}$. So $|\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle \lambda x.P_1, \mathcal{R}^r_{\lambda x.P_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c\}$ and $|\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \setminus \{1.0\} = \{0\} \cup \{1.1.p \mid p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c\}$. We prove this statement by induction on the structure of M_2:

   - Let $M_2 \in \mathcal{V} \setminus \{c\}$ then $|M_2|^c = M_2 \neq |P_1|^c|Q_1|^c$.
   - Let $M_2 = \lambda x.N_2 \in \Lambda\mathrm{I}_c$ such that $N_2 \in \Lambda\mathrm{I}_c$ and $x \neq c$ then $|M_2|^c = \lambda x.|N_2|^c \neq |P_1|^c|Q_1|^c$.
   - Let $M_2 = \lambda x.N_2[x := c(cx)] \in \Lambda\eta_c$ such that $N_2 \in \Lambda\eta_c$ and $x \neq c$ then $|M_2|^c = \lambda x.|N_2[x := c(cx)]|^c \neq |P_1|^c|Q_1|^c$.
   - Let $M_2 = \lambda x.N_2x \in \Lambda\eta_c$ such that $N_2x \in \Lambda\eta_c$, $N_2 \neq c$ and $x \notin \mathrm{fv}(N_2) \cup \{c\}$ then $|M_2|^c = \lambda x.|N_2x|^c \neq |P_1|^c|Q_1|^c$.
   - Let $M_2 = cP_2Q_2 \in \mathcal{M}_c$ such that $P_2, Q_2 \in \mathcal{M}_c$. By lemma 2.5, $\mathcal{R}^r_{M_2} = \{1.2.p \mid p \in \mathcal{R}^r_{P_2}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_2}\}$, so $|\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c = \{1.p \mid p \in |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c\}$. Because $0 \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c$ and $0 \notin |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$, we obtain $|\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \not\subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$.
   - Let $M_2 = (\lambda x.P_2)Q_2 \in \mathcal{M}_c$ such that $\lambda x.P_2, Q_2 \in \mathcal{M}_c$, then $|P_1|^c = |P_2|^c$ and $|Q_1|^c = |Q_2|^c$. By lemma 2.4.7, lemma 2.4.11a and lemma 2.4.8, $P_2 \in \mathcal{M}_c$. By lemma 2.5, $\mathcal{R}^r_{M_2} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{\lambda x.P_2}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_2}\}$ and $\mathcal{R}^r_{M_2} \setminus \{1.0\} = \{0\} \cup \{1.1.p \mid p \in \mathcal{R}^r_{P_2}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_2}\}$. So $|\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle \lambda x.P_2, \mathcal{R}^r_{\lambda x.P_2}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c \setminus \{1.0\} = \{0\} \cup \{1.1.p \mid p \in |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c\}$. Let $p \in |\langle \lambda x.P_1, \mathcal{R}^r_{\lambda x.P_1}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. So $p \in |\langle \lambda x.P_2, \mathcal{R}^r_{\lambda x.P_2}\rangle|^c$, i.e. $|\langle \lambda x.P_1, \mathcal{R}^r_{\lambda x.P_1}\rangle|^c \subseteq |\langle \lambda x.P_2, \mathcal{R}^r_{\lambda x.P_2}\rangle|^c$. Let $p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c$ then $1.1.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. So $p \in |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c$, i.e. $|\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c \subseteq |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c$. Let $p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c$ then $2.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. So $p \in |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c$, i.e. $|\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c \subseteq |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c$. Since $p_1 \in \mathcal{R}^r_{M_1}$:
     - Either $p_1 = 0$. Because $p_2 \in \mathcal{R}^r_{M_2}$, we obtain $p_2 = 0$. Hence, $M_1 = (\lambda x.P_1)Q_1 \xrightarrow{0}_r P_1[x := Q_1] = M'_1$ and $M_2 = (\lambda x.P_2)Q_2 \xrightarrow{0}_r P_2[x := Q_2] = M'_2$. By lemma 2.23, $|\langle M'_1, \mathcal{R}^r_{M'_1}\rangle|^c \subseteq |\langle M'_2, \mathcal{R}^r_{M'_2}\rangle|^c$.
     - Or $p_1 = 1.p'_1$ such that $p'_1 \in \mathcal{R}^r_{\lambda x.P_1}$ and so $1.|\langle \lambda x.P_1, p'_1\rangle|^c = |\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c$. Because $p_2 \in \mathcal{R}^r_{M_2}$, we obtain $p_2 = 1.p'_2$ such that $|\langle \lambda x.P_1, p'_1\rangle|^c = |\langle \lambda x.P_2, p'_2\rangle|^c$ and $p'_2 \in \mathcal{R}^r_{\lambda x.P_2}$. By lemma 2.5:
       * Either $\lambda x.P_1 = \lambda x.N_1x \in \mathcal{R}^r$ such that $x \notin \mathrm{fv}(N_1)$, $\mathcal{M}_c = \Lambda\eta_c$ and $p'_1 = 0$. So, $|\langle \lambda x.P_2, p'_2\rangle|^c = 0$. Hence, $p'_2 = 0$

and $\lambda x.P_2 = \lambda x.N_2 x$ such that $x \notin \mathrm{fv}(N_2)$. Hence, $M_1 = (\lambda x.N_1 x)Q_1 \xrightarrow{p_1}_r N_1 Q_1 = M_1'$ and $M_2 = (\lambda x.N_2 x)Q_2 \xrightarrow{p_2}_r N_2 Q_2 = M_2'$ such that $\lambda x.N_1 x \xrightarrow{p_1'}_r N_1$ and $\lambda x.N_2 x \xrightarrow{p_2'}_r N_2$. By IH, $|\langle N_1, \mathcal{R}^r_{N_1}\rangle|^c \subseteq |\langle N_2, \mathcal{R}^r_{N_2}\rangle|^c$.

- If N_1 is a λ-abstraction then by lemma 2.8, $N_1 x \in \mathcal{R}^r$. So $1.1.0 \in \mathcal{R}^r_{M_1}$ and $|\langle M_2, 1.1.0\rangle|^c = 1.1.0 = |\langle M_1, 1.1.0\rangle|^c \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. Hence, $1.1.0 \in \mathcal{R}^r_{M_2}$. So N_2 is a λ-abstraction. So $\mathcal{R}^r_{M_1'} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{N_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{M_2'} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{N_2}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_2}\}$, so $|\langle M_1', \mathcal{R}^r_{M_1'}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle N_1, \mathcal{R}^r_{N_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c\}$ and $|\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle N_2, \mathcal{R}^r_{N_2}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c\}$. Let $p \in |\langle M_1', \mathcal{R}^r_{M_1'}\rangle|^c$. Either $p = 0 \in |\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c$. Or $p = 1.p'$ such that $p' \in |\langle N_1, \mathcal{R}^r_{N_1}\rangle|^c \subseteq |\langle N_2, \mathcal{R}^r_{N_2}\rangle|^c$. So $p \in |\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c$. Or $p = 2.p'$ such that $p' \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c \subseteq |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c$. So $p \in |\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c$.
- Otherwise $\mathcal{R}^r_{M_1'} = \{1.p \mid p \in \mathcal{R}^r_{N_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{M_2'} \setminus \{0\} = \{1.p \mid p \in \mathcal{R}^r_{N_2}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_2}\}$, so $|\langle M_1', \mathcal{R}^r_{M_1'}\rangle|^c = \{1.p \mid p \in |\langle N_1, \mathcal{R}^r_{N_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c\}$ and $|\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c \setminus \{0\} = \{1.p \mid p \in |\langle N_2, \mathcal{R}^r_{N_2}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c\}$. Let $p \in |\langle M_1', \mathcal{R}^r_{M_1'}\rangle|^c$. Either $p = 1.p'$ such that $p' \in |\langle N_1, \mathcal{R}^r_{N_1}\rangle|^c \subseteq |\langle N_2, \mathcal{R}^r_{N_2}\rangle|^c$. So $p \in |\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c$. Or $p = 2.p'$ such that $p' \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c \subseteq |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c$. So $p \in |\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c$.

* Or $p_1' = 1.p_1''$ such that $p_1'' \in \mathcal{R}^r_{P_1}$. So $p_2' = 1.p_2''$ such that $p_2'' \in \mathcal{R}^r_{P_2}$. Hence, $M_1 = (\lambda x.P_1)Q_1 \xrightarrow{p_1}_r (\lambda x.P_1')Q_1 = M_1'$ and $M_2 = (\lambda x.P_2)Q_2 \xrightarrow{p_2}_r (\lambda x.P_2')Q_2 = M_2'$ such that $\lambda x.P_1 \xrightarrow{p_1'}_r \lambda x.P_1'$ and $\lambda x.P_2 \xrightarrow{p_2'}_r \lambda x.P_2'$. By IH, $|\langle \lambda x.P_1', \mathcal{R}^r_{\lambda x.P_1'}\rangle|^c \subseteq |\langle \lambda x.P_2', \mathcal{R}^r_{\lambda x.P_2'}\rangle|^c$. Since $M_1, M_2 \in \mathcal{M}_c$, by lemma 2.10, $M_1', M_2' \in \mathcal{M}_c$. By lemma 2.5 and lemma 2.8, $\mathcal{R}^r_{M_1'} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{\lambda x.P_1'}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{M_2'} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{\lambda x.P_2'}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_2}\}$, so $|\langle M_1', \mathcal{R}^r_{M_1'}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle \lambda x.P_1', \mathcal{R}^r_{\lambda x.P_1'}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c\}$ and $|\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle \lambda x.P_2', \mathcal{R}^r_{\lambda x.P_2'}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c\}$. Let $p \in |\langle M_1', \mathcal{R}^r_{M_1'}\rangle|^c$. Either $p = 0$ then $p \in |\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c$. Or $p = 1.p'$ such that $p' \in |\langle \lambda x.P_1', \mathcal{R}^r_{\lambda x.P_1'}\rangle|^c \subseteq |\langle \lambda x.P_2', \mathcal{R}^r_{\lambda x.P_2'}\rangle|^c$. So $p \in |\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c$. Or $p = 2.p'$ such that $p' \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c \subseteq |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c$. So $p \in |\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c$.

– Or $p_1 = 2.p_1'$ such that $p_1' \in \mathcal{R}^r_{Q_1}$ and so $2.|\langle Q_1, p_1'\rangle|^c = |\langle M_1, p_1\rangle|^c =$

$|\langle M_2, p_2\rangle|^c$. Because $p_2 \in \mathcal{R}^r_{M_2}$, we obtain $p_2 = 2.p'_2$ such that $|\langle Q_1, p'_1\rangle|^c = |\langle Q_2, p'_2\rangle|^c$. Hence, $M_1 = (\lambda x.P_1)Q_1 \xrightarrow{p_1}_r (\lambda x.P_1)Q'_1 = M'_1$ and $M_2 = (\lambda x.P_2)Q_2 \xrightarrow{p_2}_r (\lambda x.P_2)Q'_2 = M'_2$ such that $Q_1 \xrightarrow{p'_1}_r Q'_1$ and $Q_2 \xrightarrow{p'_2}_r Q'_2$. By IH, $|\langle Q'_1, \mathcal{R}^r_{Q'_1}\rangle|^c \subseteq |\langle Q'_2, \mathcal{R}^r_{Q'_2}\rangle|^c$. Since $M_1, M_2 \in \mathcal{M}_c$, by lemma 2.10, $M'_1, M'_2 \in \mathcal{M}_c$. By lemma 2.5 and lemma 2.8, $\mathcal{R}^r_{M'_1} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{\lambda x.P_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q'_1}\}$ and $\mathcal{R}^r_{M'_2} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{\lambda x.P_2}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q'_2}\}$, so $|\langle M'_1, \mathcal{R}^r_{M'_1}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q'_1, \mathcal{R}^r_{Q'_1}\rangle|^c\}$ and $|\langle M'_2, \mathcal{R}^r_{M'_2}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle \lambda x.P_2, \mathcal{R}^r_{\lambda x.P_2}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q'_2, \mathcal{R}^r_{Q'_2}\rangle|^c\}$. Let $p \in |\langle M'_1, \mathcal{R}^r_{M'_1}\rangle|^c$. Either $p = 0 \in |\langle M'_2, \mathcal{R}^r_{M'_2}\rangle|^c$. Or $p = 1.p'$ such that $p' \in |\langle \lambda x.P_1, \mathcal{R}^r_{\lambda x.P_1}\rangle|^c \subseteq |\langle \lambda x.P_2, \mathcal{R}^r_{\lambda x.P_2}\rangle|^c$. So $p \in |\langle M'_2, \mathcal{R}^r_{M'_2}\rangle|^c$. Or $p = 2.p'$ such that $p' \in |\langle Q'_1, \mathcal{R}^r_{Q'_1}\rangle|^c \subseteq |\langle Q'_2, \mathcal{R}^r_{Q'_2}\rangle|^c$. So $p \in |\langle M'_2, \mathcal{R}^r_{M'_2}\rangle|^c$.

- Let $M_2 = cN_2 \in \mathcal{M}_c = \Lambda\eta_c$ such that $N_2 \in \Lambda\eta_c$. So $|N_2|^c = |M_2|^c = |M_1|^c$. By lemma 2.7.5, $\mathcal{R}^{\beta\eta}_{M_2} = \{2.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2}\}$ and $|\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c =^{2.13} |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c$. Because $p_2 \in \mathcal{R}^{\beta\eta}_{M_2}$, we obtain $p_2 = 2.p'_2$ such that $p'_2 \in \mathcal{R}^{\beta\eta}_{N_2}$. So, $M_2 = cN_2 \xrightarrow{p_2}_{\beta\eta} cN'_2 = M'_2$ such that $N_2 \xrightarrow{p'_2}_{\beta\eta} N'_2$. Since $|\langle N_2, p'_2\rangle|^c =^{2.14} |\langle M_2, p_2\rangle|^c = |\langle M_1, p_1\rangle|^c$, by IH, $|\langle M'_1, \mathcal{R}^{\beta\eta}_{M'_1}\rangle|^c \subseteq |\langle N'_2, \mathcal{R}^{\beta\eta}_{N'_2}\rangle|^c =^{2.13} |\langle M'_2, \mathcal{R}^{\beta\eta}_{M'_2}\rangle|^c$.

7. Let $M_1 = cN_1 \in \mathcal{M}_c = \Lambda\eta_c$ such that $N_1 \in \Lambda\eta_c$. So $|N_1|^c = |M_1|^c = |M_2|^c$. By lemma 2.7.5, $\mathcal{R}^{\beta\eta}_{M_1} = \{2.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1}\}$ and $|\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c =^{2.13} |\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c$. Because $p_1 \in \mathcal{R}^{\beta\eta}_{M_1}$, we obtain $p_1 = 2.p'_1$ such that $p'_1 \in \mathcal{R}^{\beta\eta}_{N_1}$. So, $M_1 = cN_1 \xrightarrow{p_1}_{\beta\eta} cN'_1 = M'_1$ such that $N_1 \xrightarrow{p'_1}_{\beta\eta} N'_1$. Because $|\langle N_1, p'_1\rangle|^c =^{2.14} |\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c$, by IH, $|\langle M'_1, \mathcal{R}^{\beta\eta}_{M'_1}\rangle|^c =^{2.13} |\langle N'_1, \mathcal{R}^{\beta\eta}_{N'_1}\rangle|^c \subseteq |\langle M'_2, \mathcal{R}^{\beta\eta}_{M'_2}\rangle|^c$. □

B Proofs of section 5

Lemma 5.2. 1. (a) By induction on the structure of $M \in \Lambda I$.

- Let $M = x \neq c$. Then $\Phi^c(x, \mathcal{F}) = x$, $\mathcal{F} = \varnothing$ and $\mathrm{fv}(x) = \mathrm{fv}(x) \setminus \{c\}$.
- Let $M = \lambda x.N$ such that $x \neq c$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_N$. Then, $\mathrm{fv}(M) = \mathrm{fv}(N) \setminus \{x\} =^{IH} \mathrm{fv}(\Phi^c(N, \mathcal{F}')) \setminus \{c, x\} = \mathrm{fv}(\lambda x.\Phi^c(N, \mathcal{F}')) \setminus \{c\} = \mathrm{fv}(\Phi^c(M, \mathcal{F})) \setminus \{c\}$.
- Let $M = M_1 M_2$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_{M_1}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_{M_2}$.
  - If $0 \in \mathcal{F}$ then, $\Phi^c(M, \mathcal{F}) = \Phi^c(M_1, \mathcal{F}_1)\Phi^c(M_2, \mathcal{F}_2)$.

– Else, $\Phi^c(M,\mathcal{F}) = c\Phi^c(M_1,\mathcal{F}_1)\Phi^c(M_2,\mathcal{F}_2)$.

In both cases, $\mathrm{fv}(M) = \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2) =^{IH} (\mathrm{fv}(\Phi^c(M_1,\mathcal{F}_1)) \setminus \{c\}) \cup (\mathrm{fv}(\Phi^c(M_2,\mathcal{F}_2)) \setminus \{c\}) = \mathrm{fv}(\Phi^c(M,\mathcal{F})) \setminus \{c\}$.

(b) By induction on the structure of $M \in \Lambda\mathrm{I}$.

- Let $M \in \mathcal{V}$, then $M \neq c$. So $\mathcal{F} = \varnothing$ and $\Phi^c(M,\mathcal{F}) = M \in \Lambda\mathrm{I}_c$.
- Let $M = \lambda x.N$ such that $x \neq c$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_N$. By IH, $\Phi^c(N,\mathcal{F}') \in \Lambda\mathrm{I}_c$. By lemma 5.2.1a, $x \in \mathrm{fv}(\Phi^c(N,\mathcal{F}'))$. Hence, $\Phi^c(M,\mathcal{F}) = \lambda x.\Phi^c(N,\mathcal{F}') \in \Lambda\mathrm{I}_c$.
- Let $M = M_1M_2$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_{M_1}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_{M_2}$.
  - If $0 \in \mathcal{F}$ then $\Phi^c(M,\mathcal{F}) = \Phi^c(M_1,\mathcal{F}_1)\Phi^c(M_2,\mathcal{F}_2)$. By IH, $\Phi^c(M_1,\mathcal{F}_1), \Phi^c(M_2,\mathcal{F}_2) \in \Lambda\mathrm{I}_c$ and as M_1 is a λ-abstraction, $\Phi^c(M_1,\mathcal{F}_1)$ is a λ-abstraction. Hence $\Phi^c(M,\mathcal{F}) \in \Lambda\mathrm{I}_c$.
  - Else, $\Phi^c(M,\mathcal{F}) = c\Phi^c(M_1,\mathcal{F}_1)\Phi^c(M_2,\mathcal{F}_2)$. By IH, $\Phi^c(M_1,\mathcal{F}_1), \Phi^c(M_2,\mathcal{F}_2) \in \Lambda\mathrm{I}_c$, hence, $\Phi^c(M,\mathcal{F}) \in \Lambda\mathrm{I}_c$.

(c) By induction on the structure of $M \in \Lambda\mathrm{I}$.

- Let $M = x \neq c$. Then, $\mathcal{F} = \varnothing$ and $\Phi^c(x,\mathcal{F}) = x = |x|^c$.
- Let $M = \lambda x.N$ such that $x \neq c$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_N$. Then, $|\Phi^c(M,\mathcal{F})|^c = |\lambda x.\Phi^c(N,\mathcal{F}')|^c = \lambda x.|\Phi^c(N,\mathcal{F}')|^c =^{IH} \lambda x.N$.
- Let $M = M_1M_2$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_{M_1}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_{M_2}$.
  - If $0 \in \mathcal{F}$ then M_1 is a λ-abstraction, hence, $\Phi^c(M_1,\mathcal{F}_1)$ is a λ-abstraction. So, $|\Phi^c(M,\mathcal{F})|^c = |\Phi^c(M_1,\mathcal{F}_1)\Phi^c(M_2,\mathcal{F}_2)|^c = |\Phi^c(M_1,\mathcal{F}_1)|^c|\Phi^c(M_2,\mathcal{F}_2)|^c =^{IH} M_1M_2 = M$.
  - Else, $|\Phi^c(M,\mathcal{F})|^c = |c\Phi^c(M_1,\mathcal{F}_1)\Phi^c(M_2,\mathcal{F}_2)|^c = |\Phi^c(M_1,\mathcal{F}_1)|^c|\Phi^c(M_2,\mathcal{F}_2)|^c =^{IH} M_1M_2 = M$.

(d) By induction on the structure of $M \in \Lambda\mathrm{I}$.

- If $M = x \neq c$ then $\Phi^c(M,\mathcal{F}) = M$ and $\mathcal{F} = \varnothing =^{2.5} |\langle M, \mathcal{R}^{\beta I}_M\rangle|^c$.
- Let $M = \lambda x.N$ such that $x \neq c$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_N$. Then $\mathcal{F} =^{2.6} \{1.p \mid p \in \mathcal{F}'\} =^{IH} \{1.p \mid p \in |\langle\Phi^c(N,\mathcal{F}'), \mathcal{R}^{\beta I}_{\Phi^c(N,\mathcal{F}')}\rangle|^c\} = \{1.|\langle\Phi^c(N,\mathcal{F}'),p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(N,\mathcal{F}')}\} = \{|\langle\Phi^c(M,\mathcal{F}),1.p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(N,\mathcal{F}')}\} =^{2.5} |\langle\Phi^c(M,\mathcal{F}), \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F})}\rangle|^c$.
- Let $M = M_1M_2$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_{M_1}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_{M_2}$.
  - If $0 \in \mathcal{F}$ then $\Phi^c(M,\mathcal{F}) = \Phi^c(M_1,\mathcal{F}_1)\Phi^c(M_2,\mathcal{F}_2)$. Since M_1 is a λ-abstraction then $\Phi^c(M_1,\mathcal{F}_1)$ too. By lemma 5.2.1b, $\Phi^c(M,\mathcal{F}) \in \Lambda\mathrm{I}_c$ then $\Phi^c(M,\mathcal{F}) \in \mathcal{R}^{\beta I}$. Hence, $\mathcal{F} =^{2.6} \{0\} \cup \{1.p \mid p \in \mathcal{F}_1\} \cup \{2.p \mid p \in \mathcal{F}_2\} =^{IH} \{0\} \cup \{1.p \mid p \in |\langle\Phi^c(M_1,\mathcal{F}_1), \mathcal{R}^{\beta I}_{\Phi^c(M_1,\mathcal{F}_1)}\rangle|^c\} \cup \{2.p \mid p \in |\langle\Phi^c(M_2,\mathcal{F}_2), \mathcal{R}^{\beta I}_{\Phi^c(M_2,\mathcal{F}_2)}\rangle|^c\} =$

$\{0\} \cup \{1.|\langle\Phi^c(M_1,\mathcal{F}_1),p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(M_1,\mathcal{F}_1)}\} \cup \{2.|\langle\Phi^c(M_2,\mathcal{F}_2),p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(M_2,\mathcal{F}_2)}\} = \{0\} \cup \{|\langle\Phi^c(M,\mathcal{F}),1.p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(M_1,\mathcal{F}_1)}\} \cup \{|\langle\Phi^c(M,\mathcal{F}),2.p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(M_2,\mathcal{F}_2)}\} =^{2.5} |\langle\Phi^c(M,\mathcal{F}),\mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F})}\rangle|^c$.

– Else, $\Phi^c(M,\mathcal{F}) = c\Phi^c(M_1,\mathcal{F}_1)\Phi^c(M_2,\mathcal{F}_2)$. Then, $\mathcal{F} =^{2.6} \{1.p \mid p \in \mathcal{F}_1\} \cup \{2.p \mid p \in \mathcal{F}_2\} =^{IH} \{1.p \mid p \in |\langle\Phi^c(M_1,\mathcal{F}_1),\mathcal{R}^{\beta I}_{\Phi^c(M_1,\mathcal{F}_1)}\rangle|^c\} \cup \{2.p \mid p \in |\langle\Phi^c(M_2,\mathcal{F}_2),\mathcal{R}^{\beta I}_{\Phi^c(M_2,\mathcal{F}_2)}\rangle|^c\} = \{1.|\langle\Phi^c(M_1,\mathcal{F}_1),p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(M_1,\mathcal{F}_1)}\} \cup \{2.|\langle\Phi^c(M_2,\mathcal{F}_2),p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(M_2,\mathcal{F}_2)}\} = \{|\langle\Phi^c(M,\mathcal{F}),1.2.p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(M_1,\mathcal{F}_1)}\} \cup \{|\langle\Phi^c(M,\mathcal{F}),2.p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(M_2,\mathcal{F}_2)}\} =^{2.5} |\langle\Phi^c(M,\mathcal{F}),\mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F})}\rangle|^c$.

2. (a) By induction on the construction of $M \in \Lambda\mathrm{I}_c$. By lemma 2.21, $|M|^c \in \Lambda\mathrm{I}$

   - Let $M \in \mathcal{V} \setminus \{c\}$. Hence $|M|^c = M$, by lemma 2.5, $|\langle M,\mathcal{R}^{\beta I}_M\rangle|^c = \varnothing = \mathcal{R}^{\beta I}_{|M|^c}$ and $M = \Phi^c(|M|^c, |\langle M,\mathcal{R}^{\beta I}_M\rangle|^c)$.
   - Let $M = \lambda x.P$ such that $x \neq c$, $P \in \Lambda\mathrm{I}_c$ and $x \in \mathrm{fv}(P)$. Then, $|M|^c = \lambda x.|P|^c$. By IH, $|\langle P,\mathcal{R}^{\beta I}_P\rangle|^c \subseteq \mathcal{R}^{\beta I}_{|P|^c}$ and $P = \Phi^c(|P|^c, |\langle P,\mathcal{R}^{\beta I}_P\rangle|^c)$. Hence, $|\langle M,\mathcal{R}^{\beta I}_M\rangle|^c =^{2.5} \{|\langle M,1.p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_P\} = \{1.p \mid p \in |\langle P,\mathcal{R}^{\beta I}_P\rangle|^c\} \subseteq \{1.p \mid p \in \mathcal{R}^{\beta I}_{|P|^c}\} =^{2.5} \mathcal{R}^{\beta I}_{|M|^c}$. Moreover, $M = \Phi^c(|M|^c, |\langle M,\mathcal{R}^{\beta I}_M\rangle|^c)$.
   - Let $M = cPQ$ where $P,Q \in \Lambda\mathrm{I}_c$ then $|M|^c = |P|^c|Q|^c$. By IH, $|\langle P,\mathcal{R}^{\beta I}_P\rangle|^c \subseteq \mathcal{R}^{\beta I}_{|P|^c}$, $|\langle Q,\mathcal{R}^{\beta I}_Q\rangle|^c \subseteq \mathcal{R}^{\beta I}_{|Q|^c}$, $P = \Phi^c(|P|^c, |\langle P,\mathcal{R}^{\beta I}_P\rangle|^c)$ and $Q = \Phi^c(|Q|^c, |\langle Q,\mathcal{R}^{\beta I}_Q\rangle|^c)$. Hence, $|\langle M,\mathcal{R}^{\beta I}_M\rangle|^c =^{2.5} \{|\langle M,1.2.p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_P\} \cup \{|\langle M,2.p\rangle|^c \mid p\mathcal{R}^{\beta I}_Q\} = \{1.p \mid p \in |\langle P,\mathcal{R}^{\beta I}_P\rangle|^c\} \cup \{2.p \mid p \in |\langle Q,\mathcal{R}^{\beta I}_Q\rangle|^c\} \subseteq \{1.p \mid p \in \mathcal{R}^{\beta I}_{|P|^c}\} \cup \{2.p \mid p \in \mathcal{R}^{\beta I}_{|Q|^c}\} \subseteq^{2.5} \mathcal{R}^{\beta I}_{|M|^c}$. Moreover $M = \Phi^c(|M|^{\beta I}, |\langle M,\mathcal{R}^{\beta I}_M\rangle|^c)$.
   - Let $M = PQ$ where $P,Q \in \Lambda\mathrm{I}_c$ and P is a λ-abstraction. Then, $|M|^c = |P|^c|Q|^c$, where $|P|^c$ is a λ-abstraction. By IH, $|\langle P,\mathcal{R}^{\beta I}_P\rangle|^c \subseteq \mathcal{R}^{\beta I}_{|P|^c}$, $|\langle Q,\mathcal{R}^{\beta I}_Q\rangle|^c \subseteq \mathcal{R}^{\beta I}_{|Q|^c}$, $P = \Phi^c(|P|^c, |\langle P,\mathcal{R}^{\beta I}_P\rangle|^c)$ and $Q = \Phi^c(|Q|^c, |\langle Q,\mathcal{R}^{\beta I}_Q\rangle|^c)$. Hence, $|\langle M,\mathcal{R}^{\beta I}_M\rangle|^c =^{2.5} \{0\} \cup \{|\langle M,1.p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_P\} \cup \{|\langle M,2.p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_Q\} = \{0\} \cup \{1.p \mid p \in |\langle P,\mathcal{R}^{\beta I}_P\rangle|^c\} \cup \{2.p \mid p \in |\langle Q,\mathcal{R}^{\beta I}_Q\rangle|^c\} \subseteq \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta I}_{|P|^c}\} \cup \{2.p \mid p \in \mathcal{R}^{\beta I}_{|Q|^c}\} =^{2.5} \mathcal{R}^{\beta I}_{|M|^c}$. Moreover $M = \Phi^c(|M|^{\beta I}, |\langle M,\mathcal{R}^{\beta I}_M\rangle|^c)$.

   (b) By lemma 2.21, $|M|^c \in \Lambda\mathrm{I}$. By lemma 2.19 $c \notin \mathrm{fv}(|M|^c)$. By lemma 5.2.2a, $|\langle M,\mathcal{R}^{\beta I}_M\rangle|^c \subseteq \mathcal{R}^{\beta I}_{|M|^c}$ and $M = \Phi^c(|M|^c, |\langle M,\mathcal{R}^{\beta I}_M\rangle|^c)$. To prove unicity, assume that $\langle N',\mathcal{F}'\rangle$ is another such pair. So $\mathcal{F}' \subseteq \mathcal{R}^{\beta I}_{N'}$ and $M = \Phi^c(N',\mathcal{F}')$. Then, $|M|^c = |\Phi^c(N',\mathcal{F}')|^c =^{5.2.1c} N'$ and $\mathcal{F}' =^{5.2.1d} |\langle\Phi^c(N',\mathcal{F}'),\mathcal{R}^{\beta I}_{\Phi^c(N',\mathcal{F}')}\rangle|^c = |\langle M,\mathcal{R}^{\beta I}_M\rangle|^c$. $\square$

Lemma 5.3. By lemma 5.2.1c and lemma 2.16, there exists a unique $p' \in \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F})}$, such that $|\langle \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F})}, p'\rangle|^c = p$. By lemma 2.2.8, there exists P such that $\Phi^c(M,\mathcal{F}) \xrightarrow{p'}_{\beta I} P$. By lemma 2.22, $M =^{5.2.1c} |\Phi^c(M,\mathcal{F})|^c \xrightarrow{p_0}_{\beta I} |P|^c$, such that $|\langle \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F})}, p'\rangle|^c = p_0$. So $p = p_0$ and by lemma 2.2.9, $M' = |P|^c$. Let $\mathcal{F}' = |\langle P, \mathcal{R}^{\beta I}_P\rangle|^c$. Because, $\Phi^c(M,\mathcal{F}) \xrightarrow{p'}_{\beta I} P$, by lemma 2.10 and lemma 5.2.1b, $P \in \Lambda\mathrm{I}_c$. By lemma 5.2.2a, $P = \Phi^c(M',\mathcal{F}')$ and $\mathcal{F}' \subseteq \mathcal{R}^{\beta I}_{M'}$. By lemma 5.2.2b, $\mathcal{F}'$ is unique. □

Lemma 5.6. It sufficient to prove:

$$\langle M,\mathcal{F}\rangle \to_{\beta Id} \langle M',\mathcal{F}'\rangle \iff \Phi^c(M,\mathcal{F}) \to_{\beta I} \Phi^c(M',\mathcal{F}')$$

- $\Rightarrow$) let $\langle M,\mathcal{F}\rangle \to_{\beta Id} \langle M',\mathcal{F}'\rangle$. Then by definition 5.5, there exists $p \in \mathcal{F}$ such that $M \xrightarrow{p}_{\beta I} M'$ and $\mathcal{F}'$ is the set of βI-residuals in M' of the set of redexes $\mathcal{F}$ in M relative to p. By definition 5.4 we obtain $\Phi^c(M,\mathcal{F}) \to_{\beta I} \Phi^c(M',\mathcal{F}')$.
- $\Leftarrow$) Let $\Phi^c(M,\mathcal{F}) \to_{\beta I} \Phi^c(M',\mathcal{F}')$ then by lemma 2.2.8, there exists $p\mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F})}$ such that $\Phi^c(M,\mathcal{F}) \xrightarrow{p}_{\beta I} \Phi^c(M',\mathcal{F}')$. Because, by lemma 5.2.1b, $\Phi^c(M,\mathcal{F}) \in \Lambda\mathrm{I}_c$, by lemma 2.22 and lemma 5.2.1c, $M = |\Phi^c(M,\mathcal{F})|^c \xrightarrow{p_0}_{\beta I} |\Phi^c(M',\mathcal{F}')|^c = M'$ such that $|\langle \Phi^c(M,\mathcal{F}), p_0\rangle|^c = p$. By definition 5.4, $\mathcal{F}'$ is the set of βI-residuals in M' of the set of redexes $\mathcal{F}$ in M relative to p_0. By definition 5.5 we obtain $\langle M,\mathcal{F}\rangle \to_{\beta d} \langle M',\mathcal{F}'\rangle$. □

Lemma 5.7. By lemma 5.2.1b, $\Phi^c(M,\mathcal{F}_1), \Phi^c(M,\mathcal{F}_2) \in \Lambda\mathrm{I}_c$. By lemma 5.2.1c, $|\Phi^c(M,\mathcal{F}_1)|^c = |\Phi^c(M,\mathcal{F}_2)|^c$. By lemma 5.2.1d, $|\langle \Phi^c(M,\mathcal{F}_1), \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F}_1)}\rangle|^c = \mathcal{F}_1 \subseteq \mathcal{F}_2 = |\langle \Phi^c(M,\mathcal{F}_2), \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F}_2)}\rangle|^c$.

If $\langle M,\mathcal{F}_1\rangle \to_{\beta Id} \langle M',\mathcal{F}'_1\rangle$ then by lemma 5.6, $\Phi^c(M,\mathcal{F}_1) \to_{\beta I} \Phi^c(M',\mathcal{F}'_1)$. By lemma 2.2.8, there exists $p_1 \in \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F}_1)}$ such that $\Phi^c(M,\mathcal{F}_1) \xrightarrow{p_1}_{\beta I} \Phi^c(M',\mathcal{F}'_1)$. Let $p_0 = |\langle \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F}_1)}, p_1\rangle|^c$, so by lemma 5.2.1d, $p_0 \in \mathcal{F}_1$. By lemma 2.22 and lemma 5.2.1c, $M \xrightarrow{p_0}_{\beta I} M'$.

By lemma 5.3 there exists a unique set $\mathcal{F}' \subseteq \mathcal{R}^{\beta I}_{M'}$, such that $\Phi^c(M,\mathcal{F}_1) \xrightarrow{p'}_{\beta I} \Phi^c(M',\mathcal{F}')$ and $|\langle \Phi^c(M,\mathcal{F}_1), p'\rangle|^c = p_0$. By lemma 2.2.8, $p' \in \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F}_1)}$. Since $p', p_1 \in \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F}_1)}$, by lemma 2.16, $p' = p_1$. So, by lemma 2.2.9, $\Phi^c(M',\mathcal{F}') = \Phi^c(M',\mathcal{F}'_1)$. By lemma 5.2.1d, $\mathcal{F}' = \mathcal{F}'_1$ and $\mathcal{F}'_1 = |\langle \Phi^c(M',\mathcal{F}'_1), \mathcal{R}^{\beta I}_{\Phi^c(M',\mathcal{F}'_1)}\rangle|^c$.

By lemma 5.3 there exists a unique set $\mathcal{F}'_2 \subseteq \mathcal{R}^{\beta I}_{M'}$, such that $\Phi^c(M,\mathcal{F}_2) \xrightarrow{p_2}_{\beta I} \Phi^c(M',\mathcal{F}'_2)$ and $|\langle \Phi^c(M,\mathcal{F}_2), p_2\rangle|^c = p_0$. By lemma 2.2.8, $p_2 \in \Phi^c(M,\mathcal{F}_2)$. By lemma 5.2.1d, $\mathcal{F}'_2 = |\langle \Phi^c(M',\mathcal{F}'_2), \mathcal{R}^{\beta I}_{\Phi^c(M',\mathcal{F}'_2)}\rangle|^c$.

Hence, by lemma 2.24, $\mathcal{F}'_1 \subseteq \mathcal{F}'_2$ and by lemma 5.6, $\langle M,\mathcal{F}_2\rangle \to_{\beta Id} \langle M',\mathcal{F}'_2\rangle$. □

Lemma 5.9. 1. By induction on $\Gamma \vdash^{\beta I} M : \sigma$. 2. By induction on $\Gamma \vdash^{\beta\eta} M : \sigma$. 3. First prove (*): if $\Gamma \vdash^r M : \sigma$, and $\sigma \sqsubseteq \sigma'$ then $\Gamma \vdash^r M : \sigma'$ by induction on

$\sigma \sqsubseteq \sigma'$. Then, do the proof of 3. by induction on $\Gamma \vdash^r M : \sigma$. For the latter we do:

- Case (ax): If $\Gamma, x : \sigma \vdash^{\beta\eta} x : \sigma$, $\Gamma', x : \sigma' \sqsubseteq \Gamma, x : \sigma$ and $\sigma \sqsubseteq \sigma''$ then $\sigma' \sqsubseteq \sigma$ and so $\sigma' \sqsubseteq \sigma''$. By (ax) $\Gamma', x : \sigma' \vdash^{\beta\eta} x : \sigma'$. By (*), $\Gamma', x : \sigma' \vdash^{\beta\eta} x : \sigma''$.

- Case $(\to_{E^I})$: If $\frac{\Gamma \vdash^{\beta I} M:\sigma\to\tau \quad \Delta \vdash^{\beta I} N:\sigma}{\Gamma \sqcap \Delta \vdash^{\beta I} MN:\tau}$, $\Gamma = \Gamma_1, \Gamma_2$, $\Delta = \Delta_1, \Delta_2$, $\Gamma \sqcap \Delta = \Gamma_3, \Gamma_2, \Delta_2$, $\Gamma' = \Gamma'_3, \Gamma'_2, \Delta'_2 \sqsubseteq \Gamma$ where, $\Gamma_1 = (x_i : \sigma_i)_n$, $\Gamma_2 = (y_j, \tau_j)_m$, $\Gamma_3 = (x_i : \sigma_i \cap \sigma'_i)_n$, $\Delta_1 = (x_i : \sigma'_i)_n$, $\Delta_2 = (z_l, \rho_l)_k$, $\mathrm{dom}(\Gamma_2) \cap \mathrm{dom}(\Delta_2) = \varnothing$, $\Gamma'_3 = (x_i : \overline{\sigma}_i)_n$, $\Gamma'_2 = (y_j, \overline{\tau}_j)_m$, $\Delta'_2 = (z_l, \overline{\rho}_l)_k$, $\overline{\sigma_i} \sqsubseteq \sigma_i \cap \sigma'_i$, $\overline{\tau_j} \sqsubseteq \tau_j$ and $\overline{\rho_l} \sqsubseteq \rho_l$ then $\Gamma'_3, \Gamma'_2 \sqsubseteq \Gamma$ and $\Gamma'_3, \Delta'_2 \sqsubseteq \Delta$. By IH, $\Gamma'_3, \Gamma'_2 \vdash^{\beta I} M : \sigma \to \tau$ and $\Gamma'_3, \Delta'_2 \vdash^{\beta I} N : \sigma$, so by $(\to_{E^I})$, $\Gamma'_3 \sqcap \Gamma'_3, \Gamma'_2, \Delta'_2 \vdash^{\beta I} MN : \tau$. By (*), and since $\Gamma'_3 \sqcap \Gamma'_3 = \Gamma'_3$, we have: $\Gamma'_3, \Gamma'_2, \Delta'_2 \vdash^{\beta I} MN : \tau$. □

Lemma 5.10. When $M \to_r^* N$ and $M \to_r^* P$, we write $M \to_r^* \{N, P\}$.

1. By induction on $\sigma \in \mathsf{Type}^1$.
   - If $\sigma \in \mathcal{A}$ then $\mathsf{CR}_0^r \subseteq \mathsf{CR}^r = [\![\sigma]\!]^r$.
   - If $\sigma = \tau \cap \rho$ then by IH, $\mathsf{CR}_0^r \subseteq [\![\tau]\!]^r, [\![\rho]\!]^r \subseteq \mathsf{CR}^r$, so $\mathsf{CR}_0^r \subseteq [\![\tau \cap \rho]\!]^r \subseteq \mathsf{CR}^r$.
   - If $\sigma = \tau \to \rho$ then by IH, $\mathsf{CR}_0^r \subseteq [\![\tau]\!]^r, [\![\rho]\!]^r \subseteq \mathsf{CR}^r$ and $[\![\sigma]\!]^r \subseteq \mathsf{CR}^r$ by definition. Let $M \in \mathsf{CR}_0^r$, so $M = xN_1 \dots N_n$ such that $n \geq 0$ and $N_1, \dots, N_n \in \mathsf{CR}^r$. Let $P \in [\![\tau]\!]^r$ so $P \in \mathsf{CR}^r$, hence, $MP \in \mathsf{CR}_0^r \subseteq [\![\rho]\!]^r$ and $M \in [\![\sigma]\!]^r$.
2. Let $M[x := N]N_1 \dots N_n \in \mathsf{CR}^{\beta I}$ where $n \geq 0$, $x \in \mathrm{fv}(M)$ and $(\lambda x.M)NN_1 \dots N_n \to_{\beta I}^* \{M_1, M_2\}$. By lemma 2.2.7, there exist M'_1 and M'_2 such that $M_1 \to_{\beta I}^* M'_1$, $M[x := N]N_1 \dots N_n \to_{\beta I}^* M'_1$, $M_2 \to_{\beta I}^* M'_2$ and $M[x := N]N_1 \dots N_n \to_{\beta I}^* M'_2$. Then we conclude using $M[x := N]N_1 \dots N_n \in \mathsf{CR}^{\beta I}$.
3. Let $M[x := N]N_1 \dots N_n \in \mathsf{CR}^{\beta\eta}$ where $n \geq 0$ and $(\lambda x.M)NN_1 \dots N_n \to_{\beta\eta}^* \{M_1, M_2\}$. By lemma 2.2.7, there exist M'_1 and M'_2 such that $M_1 \to_{\beta\eta}^* M'_1$, $M[x := N]N_1 \dots N_n \to_{\beta\eta}^* M'_1$, $M_2 \to_{\beta\eta}^* M'_2$ and $M[x := N]N_1 \dots N_n \to_{\beta\eta}^* M'_2$. Then we conclude using $M[x := N]N_1 \dots N_n \in \mathsf{CR}^{\beta\eta}$.
4. By induction on σ.
   - If $\sigma \in \mathcal{A}$, then the statement is true by 2.
   - If $\sigma = \tau \cap \rho$, then by IH, $[\![\tau]\!]^{\beta I}$ and $[\![\rho]\!]^{\beta I}$ are I-saturated. Let M, N, N_1,…, $N_n \in \Lambda$, $x \in \mathrm{fv}(M)$, $n \geq 0$, and $M[x := N]N_1 \dots N_n \in [\![\sigma]\!]^{\beta I} = [\![\tau]\!]^{\beta I} \cap [\![\rho]\!]^{\beta I}$. Then by I-saturation, $(\lambda x.M)NN_1 \dots N_n \in [\![\tau]\!]^{\beta I}$ and $(\lambda x.M)NN_1 \dots N_n \in [\![\rho]\!]^{\beta I}$. Done.
   - If $\sigma = \tau \to \rho$, then by IH, $[\![\tau]\!]^{\beta I}$ and $[\![\rho]\!]^{\beta I}$ are I-saturated. Let $n \geq 0$, $M, N, N_1, \dots, N_n \in \Lambda$, $x \in \mathrm{fv}(M)$, and $M[x := N]N_1 \dots N_n \in [\![\sigma]\!]^{\beta I}$. Let $P \in [\![\tau]\!]^{\beta I} \neq \varnothing$, then $M[x := N]N_1 \dots N_n P \in [\![\rho]\!]^{\beta I}$. By I-saturation, $(\lambda x.M)NN_1 \dots N_n P \in [\![\rho]\!]^{\beta I}$ so $(\lambda x.M)NN_1 \dots N_n \in [\![\tau]\!]^{\beta I} \Rightarrow [\![\rho]\!]^{\beta I}$. Since, $M[x := N]N_1 \dots N_n \in [\![\sigma]\!]^{\beta I} \subseteq CR^{\beta I}$ and $CR^{\beta I}$ is saturated by 2, then $(\lambda x.M)NN_1 \dots N_n \in CR^{\beta I}$.

5. By induction on σ.

   - If $\sigma \in \mathcal{A}$, then the statement is true by 3.
   - If $\sigma = \tau \cap \rho$, then by IH, $[\![\tau]\!]^{\beta\eta}$ and $[\![\rho]\!]^{\beta\eta}$ are saturated.
     Let $M[x := N]N_1 \dots N_n \in [\![\sigma]\!]^{\beta\eta} = [\![\tau]\!]^{\beta\eta} \cap [\![\rho]\!]^{\beta\eta}$. Then by saturation, $(\lambda x.M)NN_1 \dots N_n \in [\![\tau]\!]^{\beta\eta}$ and $(\lambda x.M)NN_1 \dots N_n \in [\![\rho]\!]^{\beta\eta}$. Done.
   - If $\sigma = \tau \to \rho$, then by IH, $[\![\tau]\!]^{\beta\eta}$ and $[\![\rho]\!]^{\beta\eta}$ are saturated. Let $n \geq 0$, $M, N, N_1, \dots, N_n \in \Lambda$, $x \in \mathcal{V}$, and $M[x := N]N_1 \dots N_n \in [\![\sigma]\!]^{\beta\eta}$. Let $P \in [\![\tau]\!]^{\beta\eta} \neq \varnothing$, then $M[x := N]N_1 \dots N_n P \in [\![\rho]\!]^{\beta\eta}$. By saturation, $(\lambda x.M)NN_1 \dots N_n P \in [\![\rho]\!]^{\beta\eta}$ so $(\lambda x.M)NN_1 \dots N_n \in [\![\tau]\!]^{\beta\eta} \Rightarrow [\![\rho]\!]^r$. Since, $M[x := N]N_1 \dots N_n \in [\![\sigma]\!]^{\beta\eta} \subseteq CR^{\beta\eta}$ and $CR^{\beta\eta}$ is saturated by 3, then $(\lambda x.M)NN_1 \dots N_n \in CR^{\beta\eta}$.

$\square$

Lemma 5.11. By induction on $x_1 : \sigma_1, \dots, x_n : \sigma_n \vdash^r M : \sigma$.

- If the last rule is (ax) or (ax^I), use the hypothesis.

- If the last rule is $(\to_{E^I})$. Let $\Gamma_1 \sqcap \Gamma_2 = (x_i : \sigma_i \cap \sigma'_i)_n, (y_i : \tau_i)_p, (z_i : \rho_i)_q$ such that $\Gamma_1 = (x_i : \sigma_i)_n, (y_i : \tau_i)_p$ and $\Gamma_2 = (x_i : \sigma'_i)_n, (z_i : \rho_i)_q$. Let $\forall i \in \{1, \dots, n\}, N_i \in [\![\sigma_i \cap \sigma'_i]\!]^{\beta I}$ so $N_i \in [\![\sigma_i]\!]^{\beta I}$ and $N_i \in [\![\sigma'_i]\!]^{\beta I}$, $\forall i \in \{1, \dots, p\}, P_i \in [\![\tau_i]\!]^{\beta I}$ and $\forall i \in \{1, \dots, q\}, P'_i \in [\![\rho_i]\!]^{\beta I}$. So by IH, $M[(x_i := N_i)_n, (y_i := P_i)_p] \in [\![\sigma \to \tau]\!]^{\beta I}$ and $N[(x_i := N_i)_n, (z_i := P'_i)_q] \in [\![\sigma]\!]^{\beta I}$. Hence, $(MN)[(x_i := N_i)_n, (y_i := P_i)_p, (z_i := P'_i)_q] \in [\![\tau]\!]^{\beta I}$.

- If the last rule is $(\to_E)$. Let $\Gamma = (x_i : \sigma_i)_n$ and $\forall i \in \{1, \dots, n\}, N_i \in [\![\sigma_i]\!]^{\beta\eta}$. So by IH, $M[(x_i := N_i)_n] \in [\![\sigma \to \tau]\!]^{\beta\eta}$ and $N[(x_i := N_i)_n] \in [\![\sigma]\!]^{\beta\eta}$. Hence, $(MN)[(x_i := N_i)_n] \in [\![\tau]\!]^{\beta\eta}$.

- If the last rule is $(\to_I)$. Let $\Gamma = (x_i : \sigma_i)_n$ and $\forall i \in \{1, \dots, n\}, N_i \in [\![\sigma_i]\!]^r$. Let $P \in [\![\sigma]\!]^r \neq \varnothing$. So by IH, $M[(x_i := N_i)_n, x := P] \in [\![\tau]\!]^r$. Moreover $((\lambda x.M)[(x_i := N_i)_n])P = (\lambda x.M[(x_i := N_i)_n])P$.

  - For $\vdash^{\beta I}$, since $x \in \mathrm{fv}(M)$ by lemma 2.2.4, $(\lambda x.M[(x_i := N_i)_n]) \to_{\beta I} M[(x_i := N_i)_n, x := P]$ and since by lemma 5.10, $[\![\tau]\!]^{\beta I}$ is I-saturated, $((\lambda x.M)[(x_i := N_i)_n])P \in [\![\tau]\!]^{\beta I}$.
  - For $\vdash^{\beta\eta}$, $(\lambda x.M[(x_i := N_i)_n]) \to_\beta M[(x_i := N_i)_n, x := P]$ and since by lemma 5.10, $[\![\tau]\!]^{\beta\eta}$ is saturated, $((\lambda x.M)[(x_i := N_i)_n])P \in [\![\tau]\!]^{\beta\eta}$.

  So $(\lambda x.M)[(x_i := N_i)_n] \in [\![\sigma]\!]^r \Rightarrow [\![\tau]\!]^r$. Since $x \in [\![\sigma]\!]^r$, $M[(x_i := N_i)_n] \in [\![\tau]\!]^r \subseteq CR^r$, so $\lambda x.M[(x_i := N_i)_n] = (\lambda x.M)[(x_i := N_i)_n] \in CR^r$.

- If the last rule is $(\cap_I)$. Let $\Gamma = (x_i : \sigma_i)_n$ and $\forall i \in \{1, \dots, n\}, N_i \in [\![\sigma_i]\!]^r$. So by IH, $M[(x_i := N_i)_n] \in [\![\tau]\!]^r$ and $M[(x_i := N_i)_n] \in [\![\rho]\!]^r$. So $M[(x_i := N_i)_n] \in [\![\sigma]\!]^r$.

- If the last rule is $(\cap_{E1})$. Let $\Gamma = (x_i : \sigma_i)_n$ and $\forall i \in \{1, \dots, n\}, N_i \in [\![\sigma_i]\!]^r$. So by IH, $M[(x_i := N_i)_n] \in [\![\sigma \cap \tau]\!]^r$, so $M[(x_i := N_i)_n] \in [\![\sigma]\!]^r$.

- If the last rule is $(\cap_{E2})$. Let $\Gamma = (x_i : \sigma_i)_n$ and $\forall i \in \{1, \dots, n\}, N_i \in [\![\sigma_i]\!]^r$. So by IH, $M[(x_i := N_i)_n] \in [\![\sigma \cap \tau]\!]^r$, so $M[(x_i := N_i)_n] \in [\![\tau]\!]^r$. $\square$

Lemma 5.13. By induction on M. Note that by Lemma 2.4, $M \neq c$.

- Let $M = x \neq c$. Then $\Gamma = \Gamma_1, x : \tau$, $\Gamma' = x : \tau$, $\Gamma' \vdash^{\beta I} x : \tau$ and $\forall \sigma$, $\Gamma_1, x : \tau, c : \sigma \vdash^{\beta\eta} x : \tau$.

- Let $M = \lambda x.N \in \Lambda\mathrm{I}_c$ then by lemma 2.4, $N \in \Lambda\mathrm{I}_c$ and $x \in \mathrm{fv}(N)$. $\forall \rho$:
  - If $c \in \mathrm{fv}(M)$ then $c \in \mathrm{fv}(N)$ and by IH, $\exists \sigma, \tau$ where $\Gamma', x : \rho, c : \sigma \vdash^{\beta I} N : \tau$, hence $\Gamma', c : \sigma \vdash^{\beta I} \lambda x.N : \rho \to \tau$.
  - If $c \notin \mathrm{fv}(M)$ then by IH, $\exists \tau$ where $\Gamma', x : \rho \vdash^{\beta I} N : \tau$, hence $\Gamma' \vdash^{\beta I} \lambda x.N : \tau$.

- Let $M = \lambda x.N \in \Lambda\eta_c$ then by lemma 2.4.11.11a, $N \in \Lambda\eta_c$. By IH, $\forall \rho$, $\exists \sigma, \tau$ such that $\Gamma, x : \rho, c : \sigma \vdash^{\beta\eta} N : \tau$. Hence, $\Gamma, c : \sigma \vdash^{\beta\eta} \lambda x.N : \tau$.

- Let $M = cNP$ where $N, P \in \Lambda\mathrm{I}_c$. Let $\Gamma_1' = \Gamma \upharpoonright \mathrm{fv}(N)$ and $\Gamma_2' = \Gamma \upharpoonright \mathrm{fv}(P)$. Note that $\Gamma' = \Gamma \upharpoonright \mathrm{fv}(cNP) = \Gamma_1' \sqcap \Gamma_2'$.
  - If $c \notin \mathrm{fv}(N) \cup \mathrm{fv}(P)$ then by IH, $\exists \tau_1, \tau_2$ such that $\Gamma_1' \vdash^{\beta I} N : \tau_1$ and $\Gamma_2' \vdash^{\beta I} P : \tau_2$. Let $\rho \in \mathsf{Type}^1$ and $\sigma = \tau_1 \to \tau_2 \to \rho$. By $(\to_{E_I})$ twice, $\Gamma_1' \sqcap \Gamma_2', c : \sigma \vdash^{\beta I} cNP : \rho$.
  - If $c \in \mathrm{fv}(N)$ and $c \notin \mathrm{fv}(P)$ then by IH, $\exists \sigma_1, \tau_1, \tau_2$ such that $\Gamma_1', c : \sigma_1 \vdash^{\beta I} N : \tau_1$ and $\Gamma_2' \vdash^{\beta I} P : \tau_2$. Let $\rho \in \mathsf{Type}^1$ and let $\sigma = \sigma_1 \cap (\tau_1 \to \tau_2 \to \rho)$. By (ax^I) and $(\cap_E)$, $c : \sigma \vdash^{\beta I} c : \tau_1 \to \tau_2 \to \rho$. By lemma 5.9.3, $\Gamma_1', c : \sigma \vdash^{\beta I} N : \tau_1$. By $(\to_{E_I})$ twice, $\Gamma_1' \sqcap \Gamma_2', c : \sigma \vdash^{\beta I} cNP : \rho$.
  - If $c \in \mathrm{fv}(N) \cap \mathrm{fv}(P)$ then by IH, $\exists \sigma_1, \sigma_2, \tau_1, \tau_2$ such that $\Gamma_1', c : \sigma_1 \vdash^{\beta I} N : \tau_1$ and $\Gamma_2', c : \sigma_2 \vdash^{\beta I} N : \tau_2$. Let $\rho \in \mathsf{Type}^1$ and let $\sigma = \sigma_1 \cap (\sigma_2 \cap (\tau_1 \to \tau_2 \to \rho))$. By (ax^I) and $(\cap_E)$, $c : \sigma \vdash^{\beta I} c : \tau_1 \to \tau_2 \to \rho$. By lemma 5.9.3, $\Gamma_1', c : \sigma \vdash^{\beta I} N : \tau_1$, and $\Gamma_2', c : \sigma \vdash^{\beta I} P : \tau_2$. By $(\to_{E_I})$ twice, $\Gamma_1' \sqcap \Gamma_2', c : \sigma \vdash^{\beta I} cNP : \rho$.

- Let $M = cNP$ where $N, P \in \Lambda\eta_c$. by IH, $\exists \sigma_1, \sigma_2, \tau_1, \tau_2$ such that $\Gamma, c : \sigma_1 \vdash^{\beta\eta} N : \tau_1$ and $\Gamma, c : \sigma_2 \vdash^{\beta\eta} N : \tau_2$. Let $\rho \in \mathsf{Type}^1$ and let $\sigma = \sigma_1 \cap (\sigma_2 \cap (\tau_1 \to \tau_2 \to \rho))$. By (ax^I) and $(\cap_E)$, $c : \sigma \vdash^{\beta\eta} c : \tau_1 \to \tau_2 \to \rho$. By lemma 5.9.3, $\Gamma, c : \sigma \vdash^{\beta\eta} N : \tau_1$, and $\Gamma, c : \sigma \vdash^{\beta\eta} P : \tau_2$. By $(\to_{E_I})$ twice, $\Gamma, c : \sigma \vdash^{\beta\eta} cNP : \rho$.

- Let $M = NP$ where $N, P \in \Lambda\mathrm{I}_c$ and $N = \lambda x.N_0$. So $N_0 \in \Lambda\mathrm{I}_c$ and $x \in \mathrm{fv}(N_0)$. Let $\Gamma_1' = \Gamma \upharpoonright \mathrm{fv}(N)$ and $\Gamma_2' = \Gamma \upharpoonright \mathrm{fv}(P)$. Note that $\Gamma' = \Gamma \upharpoonright \mathrm{fv}(NP) = \Gamma_1' \sqcap \Gamma_2'$. By BC, $x \neq c$ and $x \notin \mathrm{fv}(P)$.

– If $c \notin \mathrm{fv}(\lambda x.N_0) \cup \mathrm{fv}(P)$ then by IH, $\exists \tau_2$ such that $\Gamma'_2 \vdash^{\beta I} P : \tau_2$ and again by IH, $\exists \tau_1$ such that $\Gamma'_1, x : \tau_2 \vdash^{\beta I} N_0 : \tau_1$. By $(\to_I)$ and $(\to_{E_I})$, $\Gamma'_1 \sqcap \Gamma'_2 \vdash^{\beta I} (\lambda x.N_0)P : \tau_1$.

– If $c \in \mathrm{fv}(\lambda x.N_0)$ and $c \notin \mathrm{fv}(P)$ then by IH, $\exists \tau_2$ such that $\Gamma'_2 \vdash^{\beta I} P : \tau_2$. Again by IH, $\exists \sigma, \tau_1$ such that $\Gamma'_1, c : \sigma, x : \tau_2 \vdash^{\beta I} N_0 : \tau_1$. By $(\to_I)$ and $(\to_{E_I})$, $\Gamma'_1 \sqcap \Gamma'_2, c : \sigma \vdash^{\beta I} (\lambda x.N_0)P : \tau_1$.

– If $c \in \mathrm{fv}(\lambda x.N_0) \cap \mathrm{fv}(P)$, then by IH, $\exists \sigma_2, \tau_2$ such that $\Gamma'_2, c : \sigma_2 \vdash^{\beta I} P : \tau_2$ and again by IH, $\exists \sigma_1, \tau_1$ such that $\Gamma'_1, c : \sigma_1, x : \tau_2 \vdash^{\beta I} N_0 : \tau_1$. By $(\to_I)$, $\Gamma'_1, c : \sigma_1 \vdash^{\beta I} \lambda x N_0 : \tau_2 \to \tau_1$. By $(\to_{E_I})$, $\Gamma'_1 \sqcap \Gamma'_2, c : \sigma_1 \cap \sigma_2 \vdash^{\beta I} (\lambda x.N_0)P : \tau_1$.

- Let $M = NP$ where $N, P \in \Lambda\eta_c$ and $N = \lambda x.N_0$ then by lemma 2.4.11.11a, $N_0 \in \Lambda\eta_c$. By IH, $\exists \sigma_2, \tau_2$ such that $\Gamma, c : \sigma_2 \vdash^{\beta\eta} P : \tau_2$ and again by IH, $\exists \sigma_1, \tau_1$ such that $\Gamma, c : \sigma_1, x : \tau_2 \vdash^{\beta\eta} N_0 : \tau_1$. By $(\to_I)$, $\Gamma, c : \sigma_1 \vdash^{\beta\eta} \lambda x.N_0 : \tau_2 \to \tau_1$. Let $\sigma = \sigma_1 \cap \sigma_2$. By Lemma 5.9.3, $\Gamma, c : \sigma \vdash^{\beta\eta} \lambda x.N_0 : \tau_2 \to \tau_1$ and $\Gamma, c : \sigma \vdash^{\beta\eta} P : \tau_2$. Hence, by $(\to_E)$, $\Gamma, c : \sigma \vdash^{\beta\eta} (\lambda x.N_0)P : \tau_1$.

- Let $M = cN$ where $N \in \Lambda\eta_c$. By IH, $\exists \sigma, \tau$ such that $\Gamma, c : \sigma \vdash^{\beta\eta} N : \tau$. Let $\rho \in \mathsf{Type}^1$ and $\sigma' = \sigma \cap (\tau \to \rho)$. By Lemma 5.9.3, $\Gamma, c : \sigma' \vdash^{\beta\eta} N : \tau$ and $\Gamma, c : \sigma' \vdash^{\beta\eta} c : \tau \to \rho$. Hence, by $(\to_E)$, $\Gamma, c : \sigma' \vdash^{\beta\eta} cN : \rho$. □

Lemma 5.14. If $M \overset{\mathcal{F}_1}{\to}_{\beta Id} M_1$ and $M \overset{\mathcal{F}_2}{\to}_{\beta Id} M_2$, then there exists $\mathcal{F}''_1, \mathcal{F}''_2$ such that $\langle M, \mathcal{F}_1 \rangle \to^*_{\beta Id} \langle M_1, \mathcal{F}''_1 \rangle$ and $\langle M, \mathcal{F}_2 \rangle \to^*_{\beta Id} \langle M_2, \mathcal{F}''_2 \rangle$. By definitions 5.4 and 5.5, $\mathcal{F}''_1 \subseteq \mathcal{R}^{\beta I}_{M_1}$ and $\mathcal{F}''_2 \subseteq \mathcal{R}^{\beta I}_{M_2}$. Note that by definition 5.5 and lemma 2.2.4, $M_1, M_2 \in \Lambda\mathrm{I}$. By lemma 5.7, there exist $\mathcal{F}'''_1 \subseteq \mathcal{R}^{\beta I}_{M_1}$ and $\mathcal{F}'''_2 \subseteq \mathcal{R}^{\beta I}_{M_2}$ such that $\langle M, \mathcal{F}_1 \cup \mathcal{F}_2 \rangle \to^*_{\beta Id} \langle M_1, \mathcal{F}''_1 \cup \mathcal{F}'''_1 \rangle$ and $\langle M, \mathcal{F}_1 \cup \mathcal{F}_2 \rangle \to^*_{\beta Id} \langle M_2, \mathcal{F}''_2 \cup \mathcal{F}'''_2 \rangle$. By lemma 5.6, $T \to^*_{\beta I} T_1$ and $T \to^*_{\beta I} T_2$ where $T = \Phi^c(M, \mathcal{F}_1 \cup \mathcal{F}_2)$, $T_1 = \Phi^c(M_1, \mathcal{F}''_1 \cup \mathcal{F}'''_1)$ and $T_2 = \Phi^c(M_2, \mathcal{F}''_2 \cup \mathcal{F}'''_2)$. Since by lemma 5.2.1b, $T \in \Lambda\mathrm{I}_c$ and by lemma 5.13.1, T is typable in the type system DI, so $T \in \mathsf{CR}^{\beta I}$ by corollary 5.12. So, by lemma 2.10.2, there exists $T_3 \in \Lambda\mathrm{I}_c$, such that $T_1 \to^*_{\beta I} T_3$ and $T_2 \to^*_{\beta I} T_3$. Let $\mathcal{F}_3 = |\langle T_3, \mathcal{R}^{\beta I}_{T_3} \rangle|^c$ and $M_3 = |T_3|^{\beta I}$, then by lemma 5.2.2b, $T_3 = \Phi^c(M_3, \mathcal{F}_3)$. Hence, by lemma 5.6, $\langle M_1, \mathcal{F}''_1 \cup \mathcal{F}'''_1 \rangle \to^*_{\beta Id} \langle M_3, \mathcal{F}_3 \rangle$ and $\langle M_2, \mathcal{F}''_2 \cup \mathcal{F}'''_2 \rangle \to^*_{\beta Id} \langle M_3, \mathcal{F}_3 \rangle$, i.e. $M_1 \overset{\mathcal{F}''_1 \cup \mathcal{F}'''_1}{\to}_{\beta Id} M_3$ and $M_2 \overset{\mathcal{F}''_2 \cup \mathcal{F}'''_2}{\to}_{\beta Id} M_3$. □

Lemma 5.16. Note that $\varnothing \subseteq \mathcal{R}^{\beta I}_M$. We prove this statement by induction on the structure of M.

- Let $M \in \mathcal{V}$ then $\Phi^c(M, \varnothing) = M$ and $\mathcal{R}^{\beta I}_M = \varnothing$ by lemma 2.5.

- Let $M = \lambda x.N$ such that $x \neq c$ then $\Phi^c(M, \varnothing) = \lambda x.\Phi^c(N, \varnothing)$. By IH, $\mathcal{R}^{\beta I}_{\Phi^c(N,\varnothing)} = \varnothing$ and by lemma 2.5, $\mathcal{R}^{\beta I}_{\Phi^c(M,\varnothing)} = \varnothing$.

- Let $M = M_1 M_2$ then $\Phi^c(M, \varnothing) = c\Phi^c(M_1, \varnothing)\Phi^c(M_2, \varnothing)$. By IH, $\mathcal{R}^{\beta I}_{\Phi^c(M_1,\varnothing)} = \varnothing$ and $\mathcal{R}^{\beta I}_{\Phi^c(M_2,\varnothing)} = \varnothing$ and by lemma 2.5, $\mathcal{R}^{\beta I}_{\Phi^c(M,\varnothing)} = \varnothing$. □

Lemma 5.17. We prove the statement by induction on the structure of M.

- let $M \in \mathcal{V}$, then $\Phi^c(M, \varnothing) = M$.
  - Either $M = x$, then $\Phi^c(M, \varnothing)[x := \Phi^c(N, \varnothing)] = \Phi^c(N, \varnothing)$ and by lemma 5.16, $\mathcal{R}^{\beta I}_{\Phi^c(N,\varnothing)} = \varnothing$.
  - Or $M \neq x$, then $\Phi^c(M, \varnothing)[x := \Phi^c(N, \varnothing)] = M$ and by lemma 2.5, $\mathcal{R}^{\beta I}_M = \varnothing$.
- Let $M = \lambda y.M'$ such that $y \neq c$ then $\Phi^c(M, \varnothing) = \lambda y.\Phi^c(M', \varnothing)$. So, $\mathcal{R}^{\beta I}_{\Phi^c(M,\varnothing)[x:=\Phi^c(N,\varnothing)]} = \mathcal{R}^{\beta I}_{\lambda y.\Phi^c(M',\varnothing)[x:=\Phi^c(N,\varnothing)]}$ such that $y \notin \mathrm{fv}(\Phi^c(N, \varnothing)) \cup \{x\}$. By IH, $\mathcal{R}^{\beta I}_{\Phi^c(M',\varnothing)[x:=\Phi^c(N,\varnothing)]} = \varnothing$. By lemma 2.5, $\mathcal{R}^{\beta I}_{\Phi^c(M,\varnothing)[x:=\Phi^c(N,\varnothing)]} = \varnothing$.
- Let $M = M_1M_2$ then $\Phi^c(M, \varnothing) = c\Phi^c(M_1, \varnothing)\Phi^c(M_2, \varnothing)$. So, $\mathcal{R}^{\beta I}_{\Phi^c(M,\varnothing)[x:=\Phi^c(N,\varnothing)]} = \mathcal{R}^{\beta I}_{c\Phi^c(M_1,\varnothing)[x:=\Phi^c(N,\varnothing)]\Phi^c(M_2,\varnothing)[x:=\Phi^c(N,\varnothing)]}$. By IH, $\mathcal{R}^{\beta I}_{\Phi^c(M_1,\varnothing)[x:=\Phi^c(N,\varnothing)]} = \mathcal{R}^{\beta I}_{\Phi^c(M_2,\varnothing)[x:=\Phi^c(N,\varnothing)]} = \varnothing$ and by lemma 2.5, $\mathcal{R}^{\beta I}_{\Phi^c(M,\varnothing)[x:=\Phi^c(N,\varnothing)]} = \varnothing$. □

Lemma 5.18. We prove the statement by induction on the structure of M.

- Let $M \in \mathcal{V}$ then by lemma 2.5, $\mathcal{R}^{\beta I}_M = \varnothing$.
- Let $M = \lambda x.N$ such that $x \neq c$ then by lemma 2.5, $\mathcal{R}^{\beta I}_M = \{1.p \mid p \in \mathcal{R}^{\beta I}_N\}$. Let $p \in \mathcal{R}^{\beta I}_M$, then $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta I}_N$. Then, $\Phi^c(M, \{p\}) = \lambda x.\Phi^c(N, \{p'\})$ By lemma 2.5, $\mathcal{R}^{\beta I}_{\Phi^c(M,\{p\})} = \{1.p \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(N,\{p'\})}\}$. So, By lemma 2.2.8, if $\Phi^c(M, \{p\}) \xrightarrow{p_0}_{\beta I} P$ then $p_0 = 1.p_1$, $P = \lambda x.P'$ and $\Phi^c(N, \{p'\}) \xrightarrow{p_1}_{\beta I} P'$. By IH, $\mathcal{R}^{\beta I}_{P'} = \varnothing$, so by lemma 2.5, $\mathcal{R}^{\beta I}_P = \varnothing$.
- Let $M = M_1M_2$.
  - Let $M \in \mathcal{R}^{\beta I}$, then $M_1 = \lambda x.M_0$ and by lemma 2.5, $\mathcal{R}^{\beta I}_M = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta I}_{M_1}\} \cup \{2.p \mid p \in \mathcal{R}^{\beta I}_{M_2}\}$.
    * Either $p = 0$ then $\Phi^c(M, \{0\}) = \Phi^c(M_1, \varnothing)\Phi^c(M_2, \varnothing)$. By lemma 5.16, $\mathcal{R}^{\beta I}_{\Phi^c(M_1,\varnothing)} = \mathcal{R}^{\beta I}_{\Phi^c(M_2,\varnothing)} = \varnothing$. Because $\Phi^c(M, \{0\}) \to_{\beta I} M'$ then by definition there exists p_0 such that $\Phi^c(M, \{0\}) \xrightarrow{p_0}_{\beta I} M'$. By lemma 2.2.8, $p_0 \in \mathcal{R}^{\beta I}_{\Phi^c(M,\{0\})}$. Because $\Phi^c(M_1, \varnothing) = \lambda x.\Phi^c(M_0, \varnothing)$ such that $x \neq c$, by lemma 2.5, we obtain $\mathcal{R}^{\beta I}_{\Phi^c(M,\{0\})} = \{0\}$ if $\Phi^c(M, \{0\}) \in \mathcal{R}^{\beta I}$, $\mathcal{R}^{\beta I}_{\Phi^c(M,\{0\})} = \varnothing$ otherwise. So p_0 and $\Phi^c(M, \{0\}) \in \mathcal{R}^{\beta I}$. Hence, $M' = \Phi^c(M_0, \varnothing)[x := \Phi^c(M_2, \varnothing)]$ and by lemma 5.17, $\mathcal{R}^{\beta I}_{\Phi^c(M_0,\varnothing)[x:=\Phi^c(M_2,\varnothing)]} = \varnothing$.
    * Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta I}_{M_1}$. So, $\Phi^c(M, \{p\}) = c\Phi^c(M_1, \{p'\})\Phi^c(M_2, \varnothing)$. By lemma 5.16, $\mathcal{R}^{\beta I}_{\Phi^c(M_2,\varnothing)} = \varnothing$. By lemma 2.5, $\mathcal{R}^{\beta I}_{\Phi^c(M,\{p\})} =$

$\{1.2.p \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(M_1,\{p'\})}\}$. So, By lemma 2.2.8, if $\Phi^c(M,\{p\}) \xrightarrow{p_0}_{\beta I} M'$ then $p_0 = 1.2.p'_0$, $p'_0 \in \mathcal{R}^{\beta I}_{\Phi^c(M_1,\{p'\})}$, $M' = cM'_1\Phi^c(M_2,\varnothing)$ and $\Phi^c(M_1,\{p'\}) \xrightarrow{p'_0}_{\beta I} M'_1$. By IH, $\mathcal{R}^{\beta I}_{M'_1} = \varnothing$ and by lemma 2.5, $\mathcal{R}^{\beta I}_{M'} = \varnothing$.

  * Or $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta I}_{M_2}$. So, $\Phi^c(M,\{p\}) = c\Phi^c(M_1,\varnothing)\Phi^c(M_2,\{p'\})$. By lemma 5.16, $\mathcal{R}^{\beta I}_{\Phi^c(M_1,\varnothing)} = \varnothing$. By lemma 2.5, $\mathcal{R}^{\beta I}_{\Phi^c(M,\{p\})} = \{2.p \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(M_2,\{p'\})}\}$. So, By lemma 2.2.8, if $\Phi^c(M,\{p\}) \xrightarrow{p_0}_{\beta I} M'$ then $p_0 = 2.p'_0$, $p'_0 \in \mathcal{R}^{\beta I}_{\Phi^c(M_2,\{p'\})}$, $M' = c\Phi^c(M_1,\varnothing)M'_2$ and $\Phi^c(M_2,\{p'\}) \xrightarrow{p'_0}_{\beta I} M'_2$. By IH, $\mathcal{R}^{\beta I}_{M'_2} = \varnothing$ and by lemma 2.5, $\mathcal{R}^{\beta I}_{M'} = \varnothing$.

- Let $M \notin \mathcal{R}^{\beta I}$, then by lemma 2.5, $\mathcal{R}^{\beta I}_M = \{1.p \mid p \in \mathcal{R}^{\beta I}_{M_1}\} \cup \{2.p \mid p \in \mathcal{R}^{\beta I}_{M_2}\}$.
  * Either $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta I}_{M_1}$. So, $\Phi^c(M,\{p\}) = c\Phi^c(M_1,\{p'\})\Phi^c(M_2,\varnothing)$. By lemma 5.16, $\mathcal{R}^{\beta I}_{\Phi^c(M_2,\varnothing)} = \varnothing$. By lemma 2.5, $\mathcal{R}^{\beta I}_{\Phi^c(M,\{p\})} = \{1.2.p \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(M_1,\{p'\})}\}$. So, By lemma 2.2.8, if $\Phi^c(M,\{p\}) \xrightarrow{p_0}_{\beta I} M'$ then $p_0 = 1.2.p'_0$, $p'_0 \in \mathcal{R}^{\beta I}_{\Phi^c(M_1,\{p'\})}$, $M' = cM'_1\Phi^c(M_2,\varnothing)$ and $\Phi^c(M_1,\{p'\}) \xrightarrow{p'_0}_{\beta I} M'_1$. By IH, $\mathcal{R}^{\beta I}_{M'_1} = \varnothing$ and by lemma 2.5, $\mathcal{R}^{\beta I}_{M'} = \varnothing$.
  * Or $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta I}_{M_2}$. So, $\Phi^c(M,\{p\}) = c\Phi^c(M_1,\varnothing)\Phi^c(M_2,\{p'\})$. By lemma 5.16, $\mathcal{R}^{\beta I}_{\Phi^c(M_1,\varnothing)} = \varnothing$. By lemma 2.5, $\mathcal{R}^{\beta I}_{\Phi^c(M,\{p\})} = \{2.p \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(M_2,\{p'\})}\}$. So, By lemma 2.2.8, if $\Phi^c(M,\{p\}) \xrightarrow{p_0}_{\beta I} M'$ then $p_0 = 2.p'_0$, $p'_0 \in \mathcal{R}^{\beta I}_{\Phi^c(M_2,\{p'\})}$, $M' = c\Phi^c(M_1,\varnothing)M'_2$ and $\Phi^c(M_2,\{p'\}) \xrightarrow{p'_0}_{\beta I} M'_2$. By IH, $\mathcal{R}^{\beta I}_{M'_2} = \varnothing$ and by lemma 2.5, $\mathcal{R}^{\beta I}_{M'} = \varnothing$.

□

Lemma 5.19. By lemma 2.2.8, $p \in \mathcal{R}^{\beta I}_M$. By lemma 5.3, there exists a unique set $\mathcal{F}' \subseteq \mathcal{R}^{\beta I}_{M'}$, such that $\Phi^c(M,\{p\}) \to_{\beta I} \Phi^c(M',\mathcal{F}')$. By lemma 5.18, $\mathcal{R}^{\beta I}_{\Phi^c(M',\mathcal{F}')} = \varnothing$, so $|\langle \Phi^c(M',\mathcal{F}'), \mathcal{R}^{\beta I}_{\Phi^c(M',\mathcal{F}')}\rangle|^c = \varnothing$ and by lemma 5.2.1d, $\mathcal{F}' = \varnothing$. Finally, by lemma 5.6, $\langle M,\{p\}\rangle \to_{\beta Id} \langle M',\varnothing\rangle$. □

Lemma 5.20. It is obvious that $\to^*_{1I} \subseteq \to^*_{\beta I}$. We only prove that $\to^*_{\beta I} \subseteq \to^*_{1I}$. Let $M, M' \in \Lambda\mathrm{I}$ such that $M \to^*_{\beta I} M'$. We prove this claim by induction on the length of $M \to^*_{\beta I} M'$.

- Let $M = M'$ then it is done since $\langle M,\mathcal{F}\rangle \to^*_{\beta Id} \langle M,\mathcal{F}\rangle$ for some $\mathcal{F}$.

- Let $M \rightarrow^*_{\beta I} M'' \rightarrow_{\beta I} M'$. By IH, $M \rightarrow^*_{1I} M''$. By definition there exists p such that $M'' \xrightarrow{p}_{\beta I} M'$ then by lemma 5.19 $\langle M'', \{p\}\rangle \rightarrow_{\beta Id} \langle M', \varnothing\rangle$, so $M'' \rightarrow_{1I} M'$. Hence $M \rightarrow^*_{1I} M'' \rightarrow_{1I} M'$. □

Lemma 5.21. Let $M \in \Lambda\mathrm{I}$ and c be a variable such that $c \notin \mathrm{fv}(M)$. Assume $M \rightarrow^*_{\beta I} M_1$ and $M \rightarrow^*_{\beta I} M_2$. Then by lemma 5.20, $M \rightarrow^*_{1I} M_1$ and $M \rightarrow^*_{1I} M_2$. We prove the statement by induction on the length of $M \rightarrow^*_{1I} M_1$.

- Let $M = M_1$. Hence $M_1 \rightarrow^*_{1I} M_2$ and $M_2 \rightarrow^*_{1I} M_2$.
- Let $M \rightarrow^*_{1I} M'_1 \rightarrow_{1I} M_1$. By IH, $\exists M'_3, M'_1 \rightarrow^*_{1I} M'_3$ and $M_2 \rightarrow^*_{1I} M'_3$. We prove that $\exists M_3, M_1 \rightarrow^*_{1I} M_3$ and $M'_3 \rightarrow_{1I} M_3$, by induction on $M'_1 \rightarrow^*_{1I} M'_3$.
  - let $M'_1 = M'_3$, hence $M'_3 \rightarrow_{1I} M_1$ and $M_1 \rightarrow^*_{1I} M_1$.
  - Let $M'_1 \rightarrow^*_{1I} M''_3 \rightarrow_{1I} M'_3$. By IH, $\exists M'''_3, M_1 \rightarrow^*_{1I} M'''_3$ and $M''_3 \rightarrow_{1I} M'''_3$. By lemma 2.2.4, $c \notin \mathrm{fv}(M''_3)$. Since $M''_3 \rightarrow_{1I} M'_3$ and $M''_3 \rightarrow_{1I} M'''_3$, by lemma 5.14, $\exists M_3, M'_3 \rightarrow_{1I} M_3$ and $M'''_3 \rightarrow_{1I} M_3$.

□

C Proofs of section 6

Lemma 6.3. 1. (a) By induction on the structure of M.

- Let $M \in \mathcal{V} \setminus \{c\}$, then $\mathcal{F} =^{2.6} \varnothing$ and $\Psi^c_0(M, \varnothing) = \{M\} = \{c^0(M)\} \subseteq \Psi^c(M, \varnothing)$.
- Let $M = \lambda x.N$ such that $x \neq c$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq^{2.6} \mathcal{R}^{\beta\eta}_N$.
  - If $0 \in \mathcal{F}$ then $\Psi^c_0(M, \mathcal{F}) = \{\lambda x.N' \mid N' \in \Psi^c_0(N, \mathcal{F}')\} = \{c^0(\lambda x.N') \mid N' \in \Psi^c_0(N, \mathcal{F}')\} \subseteq \Psi^c(M, \mathcal{F})$.
  - Else $\Psi^c_0(M, \mathcal{F}) = \{\lambda x.N'[x := c(cx)] \mid N' \in \Psi^c(N, \mathcal{F}')\} = \{c^0(\lambda x.N'[x := c(cx)]) \mid N' \in \Psi^c(N, \mathcal{F}')\} \subseteq \Psi^c(M, \mathcal{F})$.
- Let $M = NP$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq^{2.6} \mathcal{R}^{\beta\eta}_N$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq^{2.6} \mathcal{R}^{\beta\eta}_P$.
  - If $0 \in \mathcal{F}$ then $\Psi^c_0(M, \mathcal{F}) = \{N'P' \mid N' \in \Psi^c_0(N, \mathcal{F}_1) \wedge P' \in \Psi^c_0(P, \mathcal{F}_2)\} = \{c^0(N'P') \mid N' \in \Psi^c_0(N, \mathcal{F}_1) \wedge P' \in \Psi^c_0(P, \mathcal{F}_2)\}$. By IH, $\Psi^c_0(P, \mathcal{F}_2) \subseteq \Psi^c(P, \mathcal{F}_2)$, so by definition, $\Psi^c_0(M, \mathcal{F}) \subseteq \Psi^c(M, \mathcal{F})$.
  - Else $\Psi^c_0(M, \mathcal{F}) = \{cN'P' \mid N' \in \Psi^c(N, \mathcal{F}_1) \wedge P' \in \Psi^c_0(P, \mathcal{F}_2)\} = \{c^0(cN'P') \mid N' \in \Psi^c(N, \mathcal{F}_1) \wedge P' \in \Psi^c_0(P, \mathcal{F}_2)\}$. By IH, $\Psi^c_0(P, \mathcal{F}_2) \in \Psi^c(P, \mathcal{F}_2)$, so by definition, $\Psi^c_0(M, \mathcal{F}) \subseteq \Psi^c(M, \mathcal{F})$.

(b) By induction on the structure of M.

- Let $M \in \mathcal{V} \setminus \{c\}$, then $\mathcal{F} = \varnothing$, $\Psi^c(M, \mathcal{F}) = \{c^n(M) \mid n \geq 0\}$ and $\forall N \in \Psi^c(M, \mathcal{F}).\ \mathrm{fv}(M) = \{M\} = \mathrm{fv}(N) \setminus \{c\}$.

- Let $M = \lambda x.N$ such that $x \neq x$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_N$.
  - If $0 \in \mathcal{F}$ then $\Psi^c(M, \mathcal{F}) = \{c^n(\lambda x.N') \mid n \geq 0 \wedge N' \in \Psi^c_0(N, \mathcal{F}')\}$. Let $P \in \Psi^c(M, \mathcal{F})$, so $\exists n \geq 0$ and $N' \in \Psi^c_0(N, \mathcal{F}')$ such that $P = c^n(\lambda x.N')$. Hence, $\mathrm{fv}(M) = \mathrm{fv}(N) \setminus \{x\} =^{IH,1a} \mathrm{fv}(N') \setminus \{c, x\} = \mathrm{fv}(P) \setminus \{c\}$.
  - Else $\Psi^c(M, \mathcal{F}) = \{c^n(\lambda x.N'[x := c(cx)]) \mid n \geq 0 \wedge N' \in \Psi^c(N, \mathcal{F}')\}$. Let $P \in \Psi^c(M, \mathcal{F})$, so $\exists n \geq 0$ and $\exists N' \in \Psi^c(N, \mathcal{F}')$ such that, $P = c^n(\lambda x.N'[x := c(cx)])$. Hence, $\mathrm{fv}(M) = \mathrm{fv}(N) \setminus \{x\} =^{IH} \mathrm{fv}(N') \setminus \{c, x\} = \mathrm{fv}(P) \setminus \{c\}$.
- Let $M = M_1 M_2$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_{M_1}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_{M_2}$.
  - If $0 \in \mathcal{F}$ then, $\Psi^c(M, \mathcal{F}) =$
    $\{c^n(N'P') \mid n \geq 0 \wedge N' \in \Psi^c_0(M_1, \mathcal{F}_1) \wedge P' \in \Psi^c(M_2, \mathcal{F}_2)\}$.
    Let $P \in \Psi^c(M, \mathcal{F})$, so $\exists n \geq 0$, $N' \in \Psi^c_0(M_1, \mathcal{F}_1)$ and $P' \in \Psi^c(M_2, \mathcal{F}_2)$ such that $P = c^n(N'P')$.
    Hence, $\mathrm{fv}(M) = \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2) =^{IH,1a} (\mathrm{fv}(N') \setminus \{c\}) \cup (\mathrm{fv}(P') \setminus \{c\}) = (\mathrm{fv}(N') \cup \mathrm{fv}(P')) \setminus \{c\} = \mathrm{fv}(P) \setminus \{c\}$.
  - Else $\Psi^c(M, \mathcal{F}) = \{c^n(cN'P') \mid n \geq 0 \wedge N' \in \Psi^c(M_1, \mathcal{F}_1) \wedge P' \in \Psi^c(M_2, \mathcal{F}_2)\}$. Let $P \in \Psi^c(M, \mathcal{F})$, so $\exists n \geq 0$, $N' \in \Psi^c(M_1, \mathcal{F}_1)$ and $P' \in \Psi^c(M_2, \mathcal{F}_2)$ such that $P = c^n(cN'P')$. Hence, $\mathrm{fv}(M) = \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2) =^{IH} (\mathrm{fv}(N') \cup \mathrm{fv}(P')) \setminus \{c\} = \mathrm{fv}(P) \setminus \{c\}$.

(c) By induction on the structure of M.

- If $M \in \mathcal{V} \setminus \{c\}$ then $\mathcal{F} = \varnothing$ and $\Psi^c(M, \mathcal{F}) = \{c^n(M) | n \geq 0\}$. Use lemma 6.2.
- Let $M = \lambda x.N$ such that $x \neq c$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_N$.
  - If $0 \in \mathcal{F}$, then $N = Px$ such that $x \notin \mathrm{fv}(P)$ and $\Psi^c(M, \mathcal{F}) = \{c^n(\lambda x.N') \mid n \geq 0 \wedge N' \in \Psi^c_0(N, \mathcal{F}')\}$. Let $\mathcal{F}'' = \{p \mid 1.p \in \mathcal{F}'\} \subseteq^{2.6} \mathcal{R}^{\beta\eta}_P$.
    * If $0 \in \mathcal{F}'$ then, $\Psi^c_0(N, \mathcal{F}') = \{P'x \mid P' \in \Psi^c_0(P, \mathcal{F}'')\}$. Let $M' \in \Psi^c(M, \mathcal{F})$, so $M' = c^n(\lambda x.P'x)$ where $n \geq 0$ and $P' \in \Psi^c_0(P, \mathcal{F}'')$. Since $x \notin \mathrm{fv}(P)$, by lemmas 6.3.1b and 6.3.1a, $x \notin \mathrm{fv}(P')$. By IH and lemma 6.3.1a, $P', P'x \in \Lambda\eta_c$. By lemma 2.4, $P' \neq c$. Hence, by $(R1).4$, $\lambda x.P'x \in \Lambda\eta_c$. We conclude using lemma 6.2.
    * Else $\Psi^c_0(N, \mathcal{F}') = \{cP'x \mid P' \in \Psi^c(P, \mathcal{F}'')\}$. Let $M' \in \Psi^c(M, \mathcal{F})$, so $M' = c^n(\lambda x.cP'x)$ where $n \geq 0$ and $P' \in \Psi^c(P, \mathcal{F}'')$. Since $x \notin \mathrm{fv}(P)$, by lemmas 6.3.1b, $x \notin \mathrm{fv}(P')$, so $x \notin \mathrm{fv}(cP')$. By IH and lemma 6.3.1a, $cP'x \in \Lambda\eta_c$. Since $cP' \neq c$, by $(R1).4$, $\lambda x.cP'x \in \Lambda\eta_c$. We conclude using lemma 6.2.
  - Else $\Psi^c(M, \mathcal{F}) = \{c^n(\lambda x.N'[x := c(cx)]) \mid n \geq 0 \wedge N' \in \Psi^c(N, \mathcal{F}')\}$. Let $N' \in \Psi^c(N, \mathcal{F}')$ and $n \geq 0$. Since by IH

$N' \in \Lambda\eta_c$, by lemma 6.2 and $(R1).3$, $c^n(\lambda x.N'[x := c(cx)]) \in \Lambda\eta_c$.

- Let $M = NP$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_N$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_P$.
  - If $0 \in \mathcal{F}$ then $\Psi^c(M,\mathcal{F}) = \{c^n(N'P') \mid n \geq 0 \wedge N' \in \Psi^c_0(N,\mathcal{F}_1) \wedge P' \in \Psi^c(P,\mathcal{F}_2)\}$. Let $P = c^n(N'P') \in \Psi^c(M,\mathcal{F})$ such that $n \geq 0$, $N' \in \Psi^c_0(N,\mathcal{F}_1)$ and $P' \in \Psi^c(P,\mathcal{F}_2)$. By IH and lemma 6.3.1a, $N', P' \in \Lambda\eta_c$. Since N is a λ-abstraction then by definition N' too. Hence, by $(R3)$, $N'P' \in \Lambda\eta_c$. By lemma 6.2, $c^n(N'P') \in \Lambda\eta_c$.
  - Else $\Psi^c(M,\mathcal{F}) = \{c^n(cN'P') \mid n \geq 0 \wedge N' \in \Psi^c(N,\mathcal{F}_1) \wedge P' \in \Psi^c(P,\mathcal{F}_2)\}$. Let $c^n(cN'P') \in \Psi^c(M,\mathcal{F})$ such that $n \geq 0$, $N' \in \Psi^c(N,\mathcal{F}_1)$ and $P' \in \Psi^c(P,\mathcal{F}_2)$. By IH, $N', P' \in \Lambda\eta_c$. Hence by $(R2)$, $cN'P' \in \Lambda\eta_c$ and by lemma 6.2, $c^n(cN'P') \in \Lambda\eta_c$.

(d) We prove this lemma by case on the belonging of 0 in $\mathcal{F}$. Let $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_N$.

- If $0 \in \mathcal{F}$ then $\Psi^c_0(Nx,\mathcal{F}) = \{N'x \mid N' \in \Psi^c_0(N,\mathcal{F}')\}$. Hence, $P = N'x$ such that $N' \in \Psi^c_0(N,\mathcal{F}')$. Since $x \notin \mathrm{fv}(N)$, by lemmas 6.3.1b and 6.3.1a, $x \notin \mathrm{fv}(N')$. So $\lambda x.P = \lambda x.N'x \in \mathcal{R}^{\beta\eta}$ and by lemma 2.5, $\mathcal{R}^{\beta\eta}_{\lambda x.P} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_P\}$.
- Else $\Psi^c_0(Nx,\mathcal{F}) = \{cN'x \mid N' \in \Psi^c(N,\mathcal{F}')\}$ and $P = cN'x$ such that $N' \in \Psi^c(N,\mathcal{F}')$. Since $x \notin \mathrm{fv}(N)$, by lemmas 6.3.1b, $x \notin \mathrm{fv}(N')$ and so $x \notin \mathrm{fv}(cN')$. Since $\lambda x.cN'x \in \mathcal{R}^{\beta\eta}$, by lemma 2.5, $\mathcal{R}^{\beta\eta}_{\lambda x.P} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_P\}$.

(e) Let $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_N$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_x =^{2.5} \varnothing$. We prove this lemma by case on the belonging of 0 in $\mathcal{F}$.

- If $0 \in \mathcal{F}$ then $\Psi^c(Nx,\mathcal{F}) = \{c^n(N'Q) \mid n \geq 0 \wedge N' \in \Psi^c_0(N,\mathcal{F}_1) \wedge Q \in \Psi^c(x,\mathcal{F}_2)\}$. So $Px = c^n(N'Q)$ such that $n \geq 0$, $N' \in \Psi^c_0(N,\mathcal{F}_1)$ and $Q \in \Psi^c(x,\mathcal{F}_2)$. So $n = 0$, $N' = P$ and $Q = x$. Since $x \in \Psi^c_0(x,\varnothing)$, $Px \in \Psi^c_0(Nx,\mathcal{F})$.
- Else $\Psi^c(Nx,\mathcal{F}) = \{c^n(cN'Q) \mid n \geq 0 \wedge N' \in \Psi^c_0(N,\mathcal{F}_1) \wedge Q \in \Psi^c(x,\mathcal{F}_2)\}$. So $Px = c^n(cN'Q)$ such that $n \geq 0$, $N' \in \Psi^c_0(N,\mathcal{F}_1)$ and $Q \in \Psi^c(x,\mathcal{F}_2)$. So $n = 0$, $cN' = P$ and $Q = x$. Since $x \in \Psi^c_0(x,\varnothing)$, $Px \in \Psi^c_0(Nx,\mathcal{F})$.

(f) Easy by case on the structure of M and induction on n.

(g) By induction on the structure of M.

- Let $M \in \mathcal{V}\setminus\{c\}$. Then $\Psi^c(M,\mathcal{F}) = \{c^n(M) \mid n \geq 0\}$ and $\mathcal{F} = \varnothing$. Now, use lemma 2.12.
- Let $M = \lambda x.N$ such that $x \neq c$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_N$.
  - If $0 \in \mathcal{F}$ then $\Psi^c(M,\mathcal{F}) = \{c^n(\lambda x.N') \mid n \geq 0 \wedge N' \in \Psi^c_0(N,\mathcal{F}')\}$. Let $c^n(\lambda x.N') \in \Psi^c(M,\mathcal{F})$ where $n \geq 0$ and

$N' \in \Psi^c_0(N, \mathcal{F}')$. Then, $|c^n(\lambda x.N')|^c =^{2.12} |\lambda x.N'|^c = \lambda x.|N'|^c =^{IH,1a} \lambda x.N$.

  - Else $\Psi^c(M, \mathcal{F}) = \{c^n(\lambda x.N'[x := c(cx)]) \mid n \geq 0 \wedge N' \in \Psi^c(N, \mathcal{F}')\}$. Let $c^n(\lambda x.N'[x := c(cx)]) \in \Psi^c(M, \mathcal{F})$ where $n \geq 0$ and $N' \in \Psi^c(N, \mathcal{F}')$. Then, $|c^n(\lambda x.N'[x := c(cx)])|^c =^{2.12} |\lambda x.N'[x := c(cx)]|^c = \lambda x.|N'[x := c(cx)]|^c =^{2.17} \lambda x.|N'|^c =^{IH} \lambda x.N$.

- Let $M = M_1M_2$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_{M_1}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_{M_2}$.
  - If 0 then $\Psi^c(M, \mathcal{F}) = \{c^n(N'P') \mid n \geq 0 \wedge N' \in \Psi^c_0(M_1, \mathcal{F}_1) \wedge P' \in \Psi^c(M_2, \mathcal{F}_2)\}$. Let $c^n(N'P') \in \Psi^c(M, \mathcal{F})$ where $n \geq 0$, $N' \in \Psi^c_0(M_1, \mathcal{F}_1)$ and $P' \in \Psi^c(M_2, \mathcal{F}_2)$. Since M_1 is a λ-abstraction, by definition N' too. Then, $|c^n(N'P')|^c =^{2.12} |N'P'|^c = |N'|^c|P'|^c =^{IH,1a} M_1M_2$.
  - Else $\Psi^c(M, \mathcal{F}) = \{c^n(cP_1P_2) \mid n \geq 0 \wedge P_1 \in \Psi^c(M_1, \mathcal{F}_1) \wedge P_2 \in \Psi^c(M_2, \mathcal{F}_2)\}$. Let $c^n(cP_1P_2) \in \Psi^c(M, \mathcal{F})$ where $n \geq 0$, $P_1 \in \Psi^c(M_1, \mathcal{F}_1)$ and $P_2 \in \Psi^c(M_2, \mathcal{F}_2)$. Then $|c^n(cP_1P_2)|^c =^{2.12} |cP_1P_2|^c = |cP_1|^c|P_2|^c = |P_1|^c|P_2|^c =^{IH} M_1M_2$.

(h) We prove the statement by induction on M.

- Let $M \in \mathcal{V} \setminus \{c\}$. Then $\Psi^c(M, \mathcal{F}) = \{c^n(x) \mid n \geq 0\}$ and $\mathcal{F} = \varnothing$. If $P \in \Psi^c(M, \mathcal{F})$ then $\mathcal{R}^{\beta\eta}_P =^{2.7.5} \varnothing$. Hence, $\mathcal{F} = |\langle P, \mathcal{R}^{\beta\eta}_P\rangle|^c$.
- Let $M = \lambda x.N$ such that $x \neq c$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_N$.
  - If $0 \in \mathcal{F}$ then $N = Px$ where $x \notin \mathrm{fv}(P)$ and $\Psi^c(M, \mathcal{F}) = \{c^n(\lambda x.N') \mid n \geq 0 \wedge N' \in \Psi^c_0(N, \mathcal{F}')\}$. Let $N_0 = c^n(\lambda x.N') \in \Psi^c(M, \mathcal{F})$ where $n \geq 0$ and $N' \in \Psi^c_0(N, \mathcal{F}')$. Then, $|\langle N_0, \mathcal{R}^{\beta\eta}_{N_0}\rangle|^c = \{|\langle N_0, p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{N_0}\} =^{2.7.5} \{|\langle \lambda x.N', p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{\lambda x.N'}\} =^{1d} \{0\} \cup \{|\langle \lambda x.N', 1.p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{N'}\} = \{0\} \cup \{1.|\langle N', p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{N'}\} = \{0\} \cup \{1.p \mid p \in |\langle N', \mathcal{R}^{\beta\eta}_{N'}\rangle|^c\} =^{IH,1a} \{0\} \cup \{1.p \mid p \in \mathcal{F}'\} =^{2.6} \mathcal{F}$.
  - Else $\Psi^c(M, \mathcal{F}) = \{c^n(\lambda x.P[x := c(cx)]) \mid n \geq 0 \wedge P \in \Psi^c(N, \mathcal{F}')\}$. Let $N_0 = c^n(\lambda x.P[x := c(cx)]) \in \Psi^c(M, \mathcal{F})$ where $n \geq 0$ and $P \in \Psi^c(N, \mathcal{F}')$. Then, $|\langle N_0, \mathcal{R}^{\beta\eta}_{N_0}\rangle|^c = \{|\langle N_0, p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{N_0}\} =^{2.7.5} \{|\langle \lambda x.P[x := c(cx)], p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{\lambda x.P[x:=c(cx)]}\} =^{2.7.3} \{|\langle \lambda x.P[x := c(cx)], 1.p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{P[x:=c(cx)]}\} =^{2.7.4} \{|\langle \lambda x.P[x := c(cx)], 1.p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_P\} = \{1.|\langle P[x := c(cx)], p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_P\} =^{2.18} \{1.|\langle P, p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_P\} = \{1.p \mid p \in |\langle P, \mathcal{R}^{\beta\eta}_P\rangle|^c\} =^{IH} \{1.p \mid p \in \mathcal{F}'\} =^{2.6} \mathcal{F}$.
- Let $M = M_1M_2$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_{M_1}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_{M_2}$.
  - If $0 \in \mathcal{F}$ then $\Psi^c(M, \mathcal{F}) = \{c^n(NP) \mid n \geq 0 \wedge N \in \Psi^c_0(M_1, \mathcal{F}_1) \wedge P \in \Psi^c(M_2, \mathcal{F}_2)\}$. Let $N_0 = c^n(NP) \in \Psi^c(M, \mathcal{F})$ where

$n \geq 0$, $N \in \Psi^c_0(M_1, \mathcal{F}_1)$ and $P \in \Psi^c(M_2, \mathcal{F}_2)$. Since M_1 is a λ-abstraction, by definition N too. Then, $|\langle N_0, \mathcal{R}^{\beta\eta}_{N_0}\rangle|^c = \{|\langle N_0, p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{c^n(NP)}\} =^{2.7.5} \{|\langle NP, p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{NP}\} =^{2.5} \{0\} \cup \{|\langle NP, 1.p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{N}\} \cup \{|\langle NP, 2.p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{P}\} = \{0\} \cup \{1.|\langle N, p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{N}\} \cup \{2.|\langle P, p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{P}\} = \{0\} \cup \{1.p \mid p \in |\langle N, \mathcal{R}^{\beta\eta}_{N}\rangle|^c\} \cup \{2.p \mid p \in |\langle P, \mathcal{R}^{\beta\eta}_{P}\rangle|^c\} =^{IH} \{0\} \cup \{1.p \mid p \in \mathcal{F}_1\} \cup \{2.p \mid p \in \mathcal{F}_2\} =^{2.6} \mathcal{F}$.

  - Else $\Psi^c(M, \mathcal{F}) = \{c^n(cP_1P_2) \mid n \geq 0 \wedge P_1 \in \Psi^c(M_1, \mathcal{F}_1) \wedge P_2 \in \Psi^c(M_2, \mathcal{F}_2)\}$. Let $N_0 = c^n(cP_1P_2) \in \Psi^c(M, \mathcal{F})$ where $n \geq 0$, $P_1 \in \Psi^c(M_1, \mathcal{F}_1)$ and $P_2 \in \Psi^c(M_2, \mathcal{F}_2)$. Then, $|\langle N_0, \mathcal{R}^{\beta\eta}_{N_0}\rangle|^c = \{|\langle N_0, p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{N_0}\} =^{2.7.5} \{|\langle cP_1P_2, p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{cP_1P_2}\} =^{2.5} \{|\langle cP_1P_2, 1.2.p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{P_1}\} \cup \{|\langle cP_1P_2, 2.p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{P_2}\} = \{1.|\langle P_1, p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{P_1}\} \cup \{2.|\langle P_2, p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{P_2}\} = \{1.p \mid p \in |\langle P_1, \mathcal{R}^{\beta\eta}_{P_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle P_2, \mathcal{R}^{\beta\eta}_{P_2}\rangle|^c\} =^{IH} \{1.p \mid p \in \mathcal{F}_1\} \cup \{2.p \mid p \in \mathcal{F}_2\} =^{2.6} \mathcal{F}$.

2. (a) By induction on the construction of M.
   - Let $M \in \mathcal{V} \setminus \{c\}$. So $|M|^c = M$, by lemma 2.5, $\mathcal{R}^{\beta\eta}_M = \varnothing = \mathcal{R}^{\beta\eta}_{|M|^c}$ and $M \in \Psi^c(|M|^c, |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c) = \Psi^c(M, \varnothing) = \{c^n(M) \mid n \geq 0\}$.
   - Let $M = \lambda x.N[x := c(cx)]$ such that $x \neq c$ and $N \in \Lambda\eta_c$. Then, $|M|^c = \lambda x.|N|^c$ and $|\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c = \{|\langle M, p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_M\} =^{2.7.3} \{|\langle M, 1.p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{N[x:=c(cx)]}\} =^{2.7.4} \{|\langle M, 1.p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_N\} =^{2.18} \{1.|\langle N, p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_N\} = \{1.p \mid p \in |\langle N, \mathcal{R}^{\beta\eta}_N\rangle|^c\} \subseteq^{IH} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{|N|^c}\} =^{2.17} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{|N[x:=c(cx)]|^c}\} \subseteq^{2.5} \mathcal{R}^{\beta\eta}_{\lambda x.|N[x:=c(cx)]|^c} = \mathcal{R}^{\beta\eta}_{|\lambda x.N[x:=c(cx)]|^c}$.
     We just proved that $|\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c = \{1.p \mid p \in |\langle N, \mathcal{R}^{\beta\eta}_N\rangle|^c\}$, so $0 \notin |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c$ and $|\langle N, \mathcal{R}^{\beta\eta}_N\rangle|^c = \{p \mid 1.p \in |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c\}$. By definition, $\Psi^c(|M|^c, |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c) = \{c^n(\lambda x.N'[x := c(cx)]) \mid n \geq 0 \wedge N' \in \Psi^c(|N|^c, |\langle N, \mathcal{R}^{\beta\eta}_N\rangle|^c)\}$. By IH, $N \in \Psi^c(|N|^c, |\langle N, \mathcal{R}^{\beta\eta}_N\rangle|^c)$, so $M \in \Psi^c(|M|^c, |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c)$.
   - Let $M = \lambda x.Nx$ such that $Nx \in \Lambda\eta_c$, $N \neq c$ and $x \notin \mathrm{fv}(N) \cup \{c\}$. By lemma 2.4.7, $N \in \Lambda\eta_c$ and by lemma 2.19, $x \notin \mathrm{fv}(|N|^c)$. $|M|^c = \lambda x.|Nx|^c = \lambda x.|N|^c x$. Since $M, |M|^c \in \mathcal{R}^{\beta\eta}$, by lemma 2.5, $\mathcal{R}^{\beta\eta}_M = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{Nx}\}$, so $|\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle Nx, \mathcal{R}^{\beta\eta}_{Nx}\rangle|^c\} \subseteq^{IH} \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{|Nx|^c}\} = \mathcal{R}^{\beta\eta}_{|M|^c}$.
     We proved $|\langle Nx, \mathcal{R}^{\beta\eta}_{Nx}\rangle|^c = \{p \mid 1.p \in |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c\}$ and $0 \in |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c$. By definition, $\Psi^c(|M|^c, |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c) = \{c^n(\lambda x.N') \mid n \geq 0 \wedge N' \in \Psi^c_0(|Nx|^c, |\langle Nx, \mathcal{R}^{\beta\eta}_{Nx}\rangle|^c)\}$. By IH, $Nx \in \Psi^c(|Nx|^{\beta\eta}, |\langle Nx, \mathcal{R}^{\beta\eta}_{Nx}\rangle|^c)$, so by lemma 6.3.1e, $Nx \in \Psi^c_0(|Nx|^{\beta\eta}, |\langle Nx, \mathcal{R}^{\beta\eta}_{Nx}\rangle|^c)$. Hence $M \in \Psi^c(|M|^c, |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c)$.

- Let $M = cNP$ where $N, P \in \Lambda\eta_c$, so $cN \in \Lambda\eta_c$. $|M|^c = |cN|^c|P|^c = |N|^c|P|^c$. Because $M, cN \notin \mathcal{R}^{\beta\eta}$, By lemma 2.5, $\mathcal{R}_M^{\beta\eta} = \{1.2.p \mid p \in \mathcal{R}_N^{\beta\eta}\} \cup \{2.p \mid \in \mathcal{R}_P^{\beta\eta}\}$. So $|\langle M, \mathcal{R}_M^{\beta\eta}\rangle|^c = \{1.p \mid p \in |\langle N, \mathcal{R}_N^{\beta\eta}\rangle|^c\} \cup \{2.p \mid p \in |\langle P, \mathcal{R}_P^{\beta\eta}\rangle|^c\} \subseteq^{IH} \{1.p \mid p \in \mathcal{R}_{|N|^c}^{\beta\eta}\} \cup \{2.p \mid p \in \mathcal{R}_{|P|^c}^{\beta\eta}\} \subseteq^{2.5} \mathcal{R}_{|M|^c}^{\beta\eta}$.

  We just proved that $0 \notin |\langle M, \mathcal{R}_M^{\beta\eta}\rangle|^c$ and $|\langle N, \mathcal{R}_N^{\beta\eta}\rangle|^c = \{p \mid 1.p \in |\langle M, \mathcal{R}_M^{\beta\eta}\rangle|^c\}$ and $|\langle P, \mathcal{R}_P^{\beta\eta}\rangle|^c = \{p \mid 2.p \in |\langle M, \mathcal{R}_M^{\beta\eta}\rangle|^c\}$. By definition, $\Psi^c(|M|^c, |\langle M, \mathcal{R}_M^{\beta\eta}\rangle|^c) = \{c^n(cN'P') \mid n \geq 0 \wedge N' \in \Psi^c(|N|^c, |\langle N, \mathcal{R}_N^{\beta\eta}\rangle|^c) \wedge P' \in \Psi^c(|P|^c, |\langle P, \mathcal{R}_P^{\beta\eta}\rangle|^c)\}$. By IH, $N \in \Psi^c(|N|^{\beta\eta}, |\langle N, \mathcal{R}_N^{\beta\eta}\rangle|^c)$ and $P \in \Psi^c(|P|^{\beta\eta}, |\langle P, \mathcal{R}_P^{\beta\eta}\rangle|^c)$, so $M \in \Psi^c(|M|^c, |\langle M, \mathcal{R}_M^{\beta\eta}\rangle|^c)$.
- Let $M = NP$ where $N, P \in \Lambda\eta_c$ and N is a λ-abstraction. So by definition $|N|^c$ is a λ-abstraction too and $|M|^c = |N|^c|P|^c$. Since $M \in \mathcal{R}^{\beta\eta}$, By lemma 2.5, $\mathcal{R}_M^{\beta\eta} = \{0\} \cup \{1.p \mid p \in \mathcal{R}_N^{\beta\eta}\} \cup \{2.p \mid p \in \mathcal{R}_P^{\beta\eta}\}$. So $|\langle M, \mathcal{R}_M^{\beta\eta}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle N, \mathcal{R}_N^{\beta\eta}\rangle|^c\} \cup \{2.p \mid p \in |\langle P, \mathcal{R}_P^{\beta\eta}\rangle|^c\} \subseteq^{IH} \{0\} \cup \{1.p \mid p \in \mathcal{R}_{|N|^c}^{\beta\eta}\} \cup \{2.p \mid p \in \mathcal{R}_{|P|^c}^{\beta\eta}\} =^{2.5} \mathcal{R}_{|M|^c}^{\beta\eta}$.

  We just proved that $0 \in |\langle M, \mathcal{R}_M^{\beta\eta}\rangle|^c$, $|\langle N, \mathcal{R}_N^{\beta\eta}\rangle|^c = \{p \mid 1.p \in |\langle M, \mathcal{R}_M^{\beta\eta}\rangle|^c\}$ and $|\langle P, \mathcal{R}_P^{\beta\eta}\rangle|^c = \{p \mid 2.p \in |\langle M, \mathcal{R}_M^{\beta\eta}\rangle|^c\}$. By definition, $\Psi^c(|M|^c, |\langle M, \mathcal{R}_M^{\beta\eta}\rangle|^c) = \{c^n(N'P') \mid n \geq 0 \wedge N' \in \Psi_0^c(|N|^c, |\langle N, \mathcal{R}_N^{\beta\eta}\rangle|^c) \wedge P' \in \Psi^c(|P|^c, |\langle P, \mathcal{R}_P^{\beta\eta}\rangle|^c)\}$. By IH, $N \in \Psi^c(|N|^c, |\langle N, \mathcal{R}_N^{\beta\eta}\rangle|^c)$ and $P \in \Psi^c(|P|^c, |\langle P, \mathcal{R}_P^{\beta\eta}\rangle|^c)$, so $N \in \Psi_0^c(|N|^c, |\langle N, \mathcal{R}_N^{\beta\eta}\rangle|^c)$ and $M \in \Psi^c(|M|^c, |\langle M, \mathcal{R}_M^{\beta\eta}\rangle|^c)$.
- Let $M = cN$ where $N \in \Lambda\eta_c$ then $|M|^c = |N|^c$. By lemma 2.5, $\mathcal{R}_M^{\beta\eta} = \{2.p \mid p \in \mathcal{R}_N^{\beta\eta}\}$ so $|\langle M, \mathcal{R}_M^{\beta\eta}\rangle|^c = |\langle N, \mathcal{R}_N^{\beta\eta}\rangle|^c \subseteq^{IH} \mathcal{R}_{|N|^c}^{\beta\eta} = \mathcal{R}_{|M|^c}^{\beta\eta}$.

  By IH, $N \in \Psi^c(|N|^c, |\langle N, \mathcal{R}_N^{\beta\eta}\rangle|^c) = \Psi^c(|M|^c, |\langle M, \mathcal{R}_M^{\beta\eta}\rangle|^c)$, so by lemma 6.3.1f, $M \in \Psi^c(|M|^c, |\langle M, \mathcal{R}_M^{\beta\eta}\rangle|^c)$.

(b) By lemma 2.19, $c \notin \mathrm{fv}(|M|^c)$. By lemma 6.3.2a, $|\langle M, \mathcal{R}_M^{\beta\eta}\rangle|^c \subseteq \mathcal{R}_{|M|^c}^{\beta\eta}$ and $M \in \Psi^c(|M|^c, |\langle M, \mathcal{R}_M^{\beta\eta}\rangle|^c)$. To prove unicity, assume that $\langle N', \mathcal{F}'\rangle$ is another such pair. So $\mathcal{F}' \subseteq \mathcal{R}_{N'}^{\beta\eta}$ and $M \in \Psi^c(N', \mathcal{F}')$. By lemma 6.3.1g, $|M|^c = N'$ and by lemma 6.3.1h, $\mathcal{F}' = |\langle M, \mathcal{R}_M^{\beta\eta}\rangle|^c$. □

Lemma 6.4. Let $N_1 \in \Psi^c(M, \mathcal{F})$. By lemma 6.3.1c, $N_1 \in \Lambda\eta_c$. By lemma 6.3.1h and lemma 2.16, there exists a unique $p_1 \in \mathcal{R}_{N_1}^{\beta\eta}$, such that $|\langle N_1, p_1\rangle|^c = p$. By lemma 2.2.8, there exists N_1' such that $N_1 \xrightarrow{p_1}_{\beta\eta} N_1'$. By lemma 2.10, $N_1' \in \Lambda\eta_c$. By lemma 2.22, $|N_1|^c \xrightarrow{p_1'}_{\beta\eta} |N_1'|^c$ such that $p_1' = |\langle N_1, p_1\rangle|^c = p$. By lemma 6.3.1g, $M = |N_1|^c$. So by lemma 2.2.9, $M' = |N_1'|^c$. Let $\mathcal{F}' = |\langle N_1', \mathcal{R}_{N_1'}^{\beta\eta}\rangle|^c$. By lemma 6.3.2b, $(M', \mathcal{F}')$ is the one and only pair such that

$c \notin \mathrm{fv}(M')$, $\mathcal{F}' \subseteq \mathcal{R}^{\beta\eta}_{M'}$ and $N'_1 \in \Psi^c(M', \mathcal{F}')$.

Let $N_2 \in \Psi^c(M, \mathcal{F})$. By lemma 6.3.1c, $N_2 \in \Lambda\eta_c$. By lemma 6.3.1h and lemma 2.16, there exists a unique $p_2 \in \mathcal{R}^{\beta\eta}_{N_2}$, such that $|\langle N_2, p_2\rangle|^c = p$. By lemma 2.2.8, there exists N'_2 such that $N_2 \xrightarrow{p_2}_{\beta\eta} N'_2$. By lemma 2.10, $N'_2 \in \Lambda\eta_c$. By lemma 2.22, $|N_2|^c \xrightarrow{p'_2}_{\beta\eta} |N'_2|^c$ such that $p'_2 = |\langle N_2, p_2\rangle|^c = p$. By lemma 6.3.1g, $M = |N_2|^c$. So by lemma 2.2.9, $M' = |N'_2|^c$. Let $\mathcal{F}'' = |\langle N'_2, \mathcal{R}^{\beta\eta}_{N'_2}\rangle|^c$. By lemma 6.3.2b, $(M', \mathcal{F}'')$ is the one and only pair such that $c \notin \mathrm{fv}(M')$, $\mathcal{F}'' \subseteq \mathcal{R}^{\beta\eta}_{M'}$ and $N'_2 \in \Psi^c(M', \mathcal{F}'')$.

Because $N_1, N_2 \in \Psi^c(M, \mathcal{F})$, by lemma 6.3.1h, $|\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c = |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c$ and by lemma 6.3.1g, $|N_1|^c = |N_2|^c$. Finally, by lemma 2.24, $\mathcal{F}' = |\langle N'_1, \mathcal{R}^{\beta\eta}_{N'_1}\rangle|^c = |\langle N'_2, \mathcal{R}^{\beta\eta}_{N'_2}\rangle|^c = \mathcal{F}''$. $\square$

Lemma 6.15. Note that $\Psi^c(M, \mathcal{F}) \neq \varnothing$. Then, it is sufficient to prove:

- $\langle M, \mathcal{F}\rangle \to^*_{\beta\eta d} \langle M', \mathcal{F}'\rangle \Rightarrow \forall N \in \Psi^c(M, \mathcal{F}).\ \exists N' \in \Psi^c(M', \mathcal{F}').\ N \to^*_{\beta\eta} N'$ by induction on the reduction $\langle M, \mathcal{F}\rangle \to^*_{\beta\eta d} \langle M', \mathcal{F}'\rangle$.
  - If $\langle M, \mathcal{F}\rangle = \langle M', \mathcal{F}'\rangle$ then it is done.
  - Let $\langle M, \mathcal{F}\rangle \to_{\beta\eta d} \langle M'', \mathcal{F}''\rangle \to^*_{\beta\eta d} \langle M', \mathcal{F}'\rangle$. By IH: $\forall N'' \in \Psi^c(M'', \mathcal{F}'').\ \exists N' \in \Psi^c(M', \mathcal{F}').\ N \to^*_{\beta\eta} N''$. By definition 6.6, there exist $p \in \mathcal{F}$ such that $M \xrightarrow{p}_{\beta\eta} M''$ and $\mathcal{F}''$ is the set of $\beta\eta$-residuals in M'' of the set of redexes $\mathcal{F}$ in M relative to p. By definition 6.5 we obtain: $\forall N \in \Psi^c(M, \mathcal{F}).\ \exists N'' \in \Psi^c(M'', \mathcal{F}'').\ N \to_{\beta\eta} N''$.

- $\exists N \in \Psi^c(M, \mathcal{F}).\ \exists N' \in \Psi^c(M', \mathcal{F}').\ N \to^*_{\beta\eta} N' \Rightarrow \langle M, \mathcal{F}\rangle \to^*_{\beta\eta d} \langle M', \mathcal{F}'\rangle$ by induction on the reduction $N \to^*_{\beta\eta} N'$ such that $N \in \Psi^c(M, \mathcal{F})$ and $N' \in \Psi^c(M', \mathcal{F}')$.
  - If $N = N'$ then by lemma 6.3.2b, $M = M'$ and $\mathcal{F} = \mathcal{F}'$.
  - Let $N \to_{\beta\eta} N'' \to^*_{\beta\eta} N'$. By lemma 6.3.1c, $N \in \Lambda\eta_c$, so by lemma 2.10, $N'' \in \Lambda\eta_c$. By lemma 6.3.2b, $\langle |N''|^c, |\langle N'', \mathcal{R}^{\beta\eta}_{N''}\rangle|^c\rangle$ is the one and only pair such that $c \notin FV(|N''|^c)$, $|\langle N'', \mathcal{R}^{\beta\eta}_{N''}\rangle|^c \subseteq \mathcal{R}^{\beta\eta}_{|N''|^c}$ and $N'' \in \Psi^c(|N''|^c, |\langle N'', \mathcal{R}^{\beta\eta}_{N''}\rangle|^c)$. So by IH, $\langle |N''|^c, |\langle N'', \mathcal{R}^{\beta\eta}_{N''}\rangle|^c\rangle \to^*_{\beta\eta d} \langle M', \mathcal{F}'\rangle$. By definition, there exists p such that $N \xrightarrow{p}_{\beta\eta} N''$ and by lemma 2.2.8, $p \in \mathcal{R}^{\beta\eta}_N$. By lemmas 2.22 and lemma 6.3.1g, $M = |N|^c \xrightarrow{p_0}_{\beta\eta} |N''|^c$ such that $|\langle N, p\rangle|^c = p_0$. So by lemma 2.2.8, $p_0 \in \mathcal{R}^{\beta\eta}_M$. By definition 6.5, there exists a unique $\mathcal{F}' \subseteq \mathcal{R}^{\beta\eta}_{|N''|^c}$, such that for all $P \in \Psi^c(M, \mathcal{F})$, there exist $P' \in \Psi^c(|N''|^c, \mathcal{F}')$ and $p'_0 \in \mathcal{R}^{\beta\eta}_P$ such that $P \xrightarrow{p'_0}_{\beta\eta} P'$ and $|\langle P, p'_0\rangle|^c = p_0 = |\langle N, p\rangle|^c$. Moreover, $\mathcal{F}'$ is called the set of $\beta\eta$-residuals in $|N''|^c$ of the set of redexes $\mathcal{F}$ in M relative to $|\langle N, p\rangle|^c$. Since $N \in \Psi^c(M, \mathcal{F})$, there exist $P' \in \Psi^c(|N''|^c, \mathcal{F}')$ and $p' \in \mathcal{R}^{\beta\eta}_N$ such that $N \xrightarrow{p'}_{\beta\eta} P'$ and $|\langle N, p'\rangle|^c = |\langle N, p\rangle|^c$. By

lemma 2.16, $p = p'$, so by lemma 2.2.9, $P' = N''$. Since $N'' \in \Psi^c(|N''|^c, \mathcal{F}')$, by lemma 6.3.2b, $\mathcal{F}' = |\langle N'', \mathcal{R}^{\beta\eta}_{N''}\rangle|^c$. Finally, by definition 6.6, $\langle M, \mathcal{F}\rangle \to_{\beta\eta d} \langle |N''|^c, |\langle N'', \mathcal{R}^{\beta\eta}_{N''}\rangle|^c\rangle$.

□

Lemma 6.16. By lemma 6.3.1c, $\Psi^c(M, \mathcal{F}_1), \Psi^c(M, \mathcal{F}_2) \subseteq \Lambda\eta_c$. For all $N_1 \in \Psi^c(M, \mathcal{F}_1)$ and $N_2 \in \Psi^c(M, \mathcal{F}_2)$, by lemma 6.3.1g, $|N_1|^c = |N_2|^c$ and by lemma 6.3.1h, $|\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c = \mathcal{F}_1 \subseteq \mathcal{F}_2 = |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c$.

If $\langle M, \mathcal{F}_1\rangle \to_{\beta\eta d} \langle M', \mathcal{F}'_1\rangle$ then by lemma 6.15, there exist $N_1 \in \Psi^c(M, \mathcal{F}_1)$ and $N'_1 \in \Psi^c(M', \mathcal{F}'_1)$ such that $N_1 \to_{\beta\eta} N'_1$. By definition, there exists p_1 such that $N_1 \xrightarrow{p_1}_{\beta\eta} N'_1$, and by lemma 2.2.8, $p_1 \in \mathcal{R}^{\beta\eta}_{N_1}$. Let $p_0 = |\langle N_1, p_1\rangle|^c$, so by lemma 6.3.1h, $p_0 \in \mathcal{F}_1$. By lemma 2.22 and lemma 6.3.1g, $M \xrightarrow{p_0}_{\beta\eta} M'$.

By lemma 6.4 there exists a unique set $\mathcal{F}' \subseteq \mathcal{R}^{\beta\eta}_{M'}$ such that for all $P_1 \in \Psi^c(M, \mathcal{F}_1)$ there exist $P'_1 \in \Psi^c(M', \mathcal{F}')$ and $p' \in \mathcal{R}^{\beta\eta}_{P_1}$ such that $P_1 \xrightarrow{p'}_{\beta\eta} P'_1$ and $|\langle P_1, p'\rangle|^c = p_0$.

Because, $N_1 \in \Psi^c(M, \mathcal{F}_1)$, there exist $P'_1 \in \Psi^c(M', \mathcal{F}')$ and $p' \in \mathcal{R}^{\beta\eta}_{N_1}$ such that $N_1 \xrightarrow{p'}_{\beta\eta} P'_1$ and $|\langle N_1, p'\rangle|^c = p_0$. Since $p', p_1 \in \mathcal{R}^{\beta\eta}_{N_1}$, by lemma 2.16, $p' = p_1$, so by lemma 2.2.9, $P'_1 = N'_1$. By lemma 6.3.1h, $\mathcal{F}' = |\langle N'_1, \mathcal{R}^{\beta\eta}_{N'_1}\rangle|^c = \mathcal{F}'_1$.

By lemma 6.4 there exists a unique set $\mathcal{F}'_2 \subseteq \mathcal{R}^{\beta\eta}_{M'}$, such that for all $P_2 \in \Psi^c(M, \mathcal{F}_2)$ there exist $P'_2 \in \Psi^c(M', \mathcal{F}'_2)$ and $p_2 \in \mathcal{R}^{\beta\eta}_{P_2}$ such that $P_2 \xrightarrow{p_2}_{\beta\eta} P'_2$ and $|\langle P_2, p_2\rangle|^c = p_0$.

Since $\Psi^c(M, \mathcal{F}_2) \neq \varnothing$, let $N_2 \in \Psi^c(M, \mathcal{F}_2)$. So, there exist $N'_2 \in \Psi^c(M', \mathcal{F}'_2)$ and $p_2 \in \mathcal{R}^{\beta\eta}_{N_2}$ such that $N_2 \xrightarrow{p_2}_{\beta\eta} N'_2$ and $|\langle N_2, p_2\rangle|^c = p_0$. By lemma 6.3.1h, $\mathcal{F}'_2 = |\langle N'_2, \mathcal{R}^{\beta\eta}_{N'_2}\rangle|^c$.

Hence, by lemma 2.24, $\mathcal{F}'_1 \subseteq \mathcal{F}'_2$ and by lemma 6.15, $\langle M, \mathcal{F}_2\rangle \to_{\beta\eta d} \langle M', \mathcal{F}'_2\rangle$.

□

Lemma 6.17. If $M \xrightarrow{\mathcal{F}_1}_{\beta\eta d} M_1$ and $M \xrightarrow{\mathcal{F}_2}_{\beta\eta d} M_2$, then there exist $\mathcal{F}''_1, \mathcal{F}''_2$ such that $\langle M, \mathcal{F}_1\rangle \to^*_{\beta\eta d} \langle M_1, \mathcal{F}''_1\rangle$ and $\langle M, \mathcal{F}_2\rangle \to^*_{\beta\eta d} \langle M_2, \mathcal{F}''_2\rangle$. By definitions 6.5 and 6.6, $\mathcal{F}''_1 \subseteq \mathcal{R}^{\beta\eta}_{M_1}$ and $\mathcal{F}''_2 \subseteq \mathcal{R}^{\beta\eta}_{M_2}$. By lemma 6.16, there exist $\mathcal{F}'''_1 \subseteq \mathcal{R}^{\beta\eta}_{M_1}$ and $\mathcal{F}'''_2 \subseteq \mathcal{R}^{\beta\eta}_{M_2}$ such that $\langle M, \mathcal{F}_1 \cup \mathcal{F}_2\rangle \to^*_{\beta\eta d} \langle M_1, \mathcal{F}''_1 \cup \mathcal{F}'''_1\rangle$ and $\langle M, \mathcal{F}_1 \cup \mathcal{F}_2\rangle \to^*_{\beta\eta d} \langle M_2, \mathcal{F}''_2 \cup \mathcal{F}'''_2\rangle$. By lemma 6.15 there exist $T \in \Psi^c(M, \mathcal{F}_1 \cup \mathcal{F}_2)$, $T_1 \in \Psi^c(M_1, \mathcal{F}''_1 \cup \mathcal{F}'''_1)$ and $T_2 \in \Psi^c(M_2, \mathcal{F}''_2 \cup \mathcal{F}'''_2)$ such that $T \to^*_{\beta\eta} T_1$ and $T \to^*_{\beta\eta} T_2$.

Because by lemma 6.3.1c, $T \in \Lambda\eta_c$ and by lemma 5.13.2, T is typable in the type system D, so $T \in \mathsf{CR}^{\beta\eta}$ by corollary 5.12. So, by lemma 2.10.1, there exists $T_3 \in \Lambda\eta_c$, such that $T_1 \to^*_{\beta\eta} T_3$ and $T_2 \to^*_{\beta\eta} T_3$. Let $\mathcal{F}_3 = |\langle T_3, \mathcal{R}^{\beta\eta}_{T_3}\rangle|^c$ and $M_3 = |T_3|^{\beta\eta}$, then by lemma 6.3.2a, $\mathcal{F}_3 \subseteq \mathcal{R}^{\beta\eta}_{M_3}$ and $T_3 \in \Psi^c(M_3, \mathcal{F}_3)$. Hence, by lemma 6.15, $\langle M_1, \mathcal{F}''_1 \cup \mathcal{F}'''_1\rangle \to^*_{\beta\eta d} \langle M_3, \mathcal{F}_3\rangle$ and $\langle M_2, \mathcal{F}''_2 \cup \mathcal{F}'''_2\rangle \to^*_{\beta\eta d} \langle M_3, \mathcal{F}_3\rangle$, i.e. $M_1 \xrightarrow{\mathcal{F}''_1 \cup \mathcal{F}'''_1}_{\beta\eta d} M_3$ and $M_2 \xrightarrow{\mathcal{F}''_2 \cup \mathcal{F}'''_2}_{\beta\eta d} M_3$. □

Lemma 6.19. Note that $\varnothing \subseteq \mathcal{R}^{\beta\eta}_M$. We prove this statement by induction on the structure of M.

- Let $M \in \mathcal{V} \setminus \{c\}$ then $\Psi^c(M, \varnothing) = \{c^n(M) \mid n \geq 0\}$ and $\mathcal{R}^{\beta\eta}_{c^n(M)} = \varnothing$, where $n \geq 0$, by lemma 2.5 and lemma 2.7.5.
- Let $M = \lambda x.N$ such that $x \neq c$ then $\Psi^c(M, \varnothing) = \{c^n(\lambda x.Q[x := c(cx)]) \mid n \geq 0 \wedge Q \in \Psi^c(N, \varnothing)\}$. Let $P \in \Psi^c(M, \varnothing)$, then $P = c^n(\lambda x.Q[x := c(cx)])$ such that $n \geq 0$ and $Q \in \Psi^c(N, \varnothing)$ By IH, $\mathcal{R}^{\beta\eta}_Q = \varnothing$ and by lemma 2.7.4, lemma 2.7.3 and lemma 2.7.5, $\mathcal{R}^{\beta\eta}_P = \varnothing$.
- Let $M = M_1M_2$ then $\Psi^c(M, \varnothing) = \{c^n(cQ_1Q_2) \mid n \geq 0 \wedge Q_1 \in \Psi^c(M_1, \varnothing) \wedge Q_2 \in \Psi^c(M_2, \varnothing)\}$. Let $P \in \Psi^c(M, \varnothing)$, then $P = c^n(cQ_1Q_2)$ such that $n \geq 0$, $Q_1 \in \Psi^c(M_1, \varnothing)$ and $Q_2 \in \Psi^c(M_2, \varnothing)$. By IH, $\mathcal{R}^{\beta\eta}_{Q_1} = \mathcal{R}^{\beta\eta}_{Q_2} = \varnothing$ and by lemma 2.5 and lemma 2.7.5, $\mathcal{R}^{\beta\eta}_P = \varnothing$. □

Lemma 6.20. We prove the statement by induction on the structure of M.

- Let $M \in \mathcal{V} \setminus \{c\}$, then $\Psi^c(M, \varnothing) = \{c^n(M) \mid n \geq 0\}$. Let $P \in \Psi^c(M, \varnothing)$ and $Q \in \Psi^c(N, \varnothing)$, then $P = c^n(M)$ where $n \geq 0$.
  - Either $M = x$, then $P[x := Q] = c^n(Q)$ and by lemma 6.3.1f and lemma 6.19, $\mathcal{R}^{\beta\eta}_{c^n(Q)} = \varnothing$.
  - Or $M \neq x$, then $P[x := Q] = P$ and by lemma 6.19, $\mathcal{R}^{\beta\eta}_P = \varnothing$.
- Let $M = \lambda y.M'$ such that $y \neq c$ then $\Psi^c(M, \varnothing) = \{c^n(\lambda y.P'[y := c(cy)]) \mid n \geq 0 \wedge P' \in \Psi^c(M', \varnothing)\}$. Let $P \in \Psi^c(M, \varnothing)$ and $Q \in \Psi^c(N, \varnothing)$, then $P = c^n(\lambda y.P'[y := c(cy)])$ where $n \geq 0$ and $P' \in \Psi^c(M', \varnothing)$. So, $\mathcal{R}^{\beta\eta}_{P[x:=Q]} = \mathcal{R}^{\beta\eta}_{c^n(\lambda y.P'[x:=Q][y:=c(cy)])}$, such that $y \notin \mathrm{fv}(Q) \cup \{x\}$. By IH, $\mathcal{R}^{\beta\eta}_{P'[x:=Q]} = \varnothing$ and by lemmas 2.7.4, 2.7.3 and 2.7.5, $\mathcal{R}^{\beta\eta}_{P[x:=Q]} = \varnothing$.
- Let $M = M_1M_2$ then $\Psi^c(M, \varnothing) = \{c^n(cP_1P_2) \mid n \geq 0 \wedge P_1 \in \Psi^c(M_1, \varnothing) \wedge P_2 \in \Psi^c(M_2, \varnothing)\}$. Let $P \in \Psi^c(M, \varnothing)$ and $Q \in \Psi^c(N, \varnothing)$ then $P = c^n(cP_1P_2)$ where $n \geq 0$, $P_1 \in \Psi^c(M_1, \varnothing)$ and $P_2 \in \Psi^c(M_2, \varnothing)$. So, $\mathcal{R}^{\beta\eta}_{P[x:=Q]} = \mathcal{R}^{\beta\eta}_{c^n(cP_1[x:=Q]P_2[x:=Q])}$. By IH, $\mathcal{R}^{\beta\eta}_{P_1[x:=Q]} = \mathcal{R}^{\beta\eta}_{P_2[x:=Q]} = \varnothing$ and by lemmas 2.5 and 2.7.5, $\mathcal{R}^{\beta\eta}_{P[x:=Q]} = \varnothing$. □

Lemma 6.21. We prove the statement by induction on the structure of M.

- Let $M \in \mathcal{V} \setminus \{c\}$ then nothing to prove since by lemma 2.5, $\mathcal{R}^{\beta\eta}_M = \varnothing$.
- Let $M = \lambda x.N$ such that $x \neq c$.
  - If $M \in \mathcal{R}^{\beta\eta}$ then $N = N_0x$ such that $x \notin FV(N_0)$ and by lemma 2.5, $\mathcal{R}^{\beta\eta}_M = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_N\}$. Let $p \in \mathcal{R}^{\beta\eta}_M$ then:

  * Either $p = 0$, then $\Psi^c(M, \{p\}) = \{c^n(\lambda x.P') \mid n \geq 0 \wedge P' \in \Psi^c_0(N, \varnothing)\}$. Let $P \in \Psi^c(M, \{p\})$ then $P = c^n(\lambda x.P')$ such that $n \geq 0$ and $P' \in \Psi^c_0(N, \varnothing)$. So $P' = cP'_0x$ such that $P'_0 \in \Psi^c(N_0, \varnothing)$. By lemmas 6.19 and 6.3.1a, $\mathcal{R}^{\beta\eta}_{P'} = \varnothing$. If $P \rightarrow_{\beta\eta} Q$ then by definition, there exists p_0 such that $P \xrightarrow{p_0}_{\beta\eta} Q$. By lemma 2.4.12b and lemma 2.2.8, $Q = c^n(Q')$, $p_0 = 2^n.p'_0$ and $\lambda x.P' \xrightarrow{p'_0}_{\beta\eta} Q'$ such that $p'_0 \in \mathcal{R}^{\beta\eta}_{\lambda x.P'}$. By lemma 6.3.1b, $x \notin \text{fv}(cP'_0)$. By lemmas 2.5, $\mathcal{R}^{\beta\eta}_{\lambda x.P'} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{P'}\} = \{0\}$. So $p'_0 = 0$ and $Q' = cP'_0$. By lemma 6.19, $\mathcal{R}^{\beta\eta}_{P'_0} = \varnothing$ and by lemma 2.7.5, $\mathcal{R}^{\beta\eta}_{Q} = \varnothing$.
  * Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_N$. So $\Psi^c(M, \{p\}) = \{c^n(\lambda x.P'[x := c(cx)]) \mid n \geq 0 \wedge P' \in \Psi^c(N, \{p'\})\}$. Let $P \in \Psi^c(M, \{p\})$ then $P = c^n(\lambda x.P'[x := c(cx)])$ such that $n \geq 0$ and $P' \in \Psi^c(N, \{p'\})$. If $P \rightarrow_{\beta\eta} Q$ then there exists p_0 such that $P \xrightarrow{p_0}_{\beta\eta} Q$. By lemma 2.4.12b, lemma 2.2.8, lemma 2.7.3 and lemma 2.4.12a, $p_0 = 2^n.1.p'_0$ such that $p'_0 \in \mathcal{R}^{\beta\eta}_{P'}$ and $Q = c^n(\lambda x.Q'[x := c(cx)])$ such that $P' \xrightarrow{p'_0}_{\beta\eta} Q'$. By IH, $\mathcal{R}^{\beta\eta}_{Q'} = \varnothing$, so by lemma 2.7.4, lemma 2.7.3 and lemma 2.7.5, $\mathcal{R}^{\beta\eta}_{Q} = \varnothing$.

- Else, by lemma 2.5, $\mathcal{R}^{\beta\eta}_M = \{1.p \mid p \in \mathcal{R}^{\beta\eta}_N\}$. Let $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_N$. So $\Psi^c(M, \{p\}) = \{c^n(\lambda x.P'[x := c(cx)]) \mid n \geq 0 \wedge P' \in \Psi^c(N, \{p'\})\}$. Let $P \in \Psi^c(M, \{p\})$ then $P = c^n(\lambda x.P'[x := c(cx)])$ such that $n \geq 0$ and $P' \in \Psi^c(N, \{p'\})$. If $P \rightarrow_{\beta\eta} Q$ then there exists p_0 such that $P \xrightarrow{p_0}_{\beta\eta} Q$. By lemma 2.4.12b, lemma 2.2.8, lemma 2.7.3 and lemma 2.4.12a, $p_0 = 2^n.1.p'_0$ such that $p'_0 \in \mathcal{R}^{\beta\eta}_{P'}$ and $Q = c^n(\lambda x.Q'[x := c(cx)])$ such that $P' \xrightarrow{p'_0}_{\beta\eta} Q'$. By IH, $\mathcal{R}^{\beta\eta}_{Q'} = \varnothing$, so by lemma 2.7.4, lemma 2.7.3 and lemma 2.7.5, $\mathcal{R}^{\beta\eta}_{Q} = \varnothing$.

• Let $M = M_1M_2$.

- Let $M \in \mathcal{R}^{\beta\eta}$, then $M_1 = \lambda x.M_0$ such that $x \neq c$ and by lemma 2.5, $\mathcal{R}^{\beta\eta}_M = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_1}\} \cup \{2.p \mid p \in \mathcal{R}^{\beta\eta}_{M_2}\}$. Let $p \in \mathcal{R}^{\beta\eta}_M$ then:
  * Either $p = 0$ then $\Psi^c(M, \{p\}) = \{c^n(P_1P_2) \mid n \geq 0 \wedge P_1 \in \Psi^c_0(M_1, \varnothing) \wedge P_2 \in \Psi^c(M_2, \varnothing)\}$. Let $P \in \Psi^c(M, \{p\})$ then $P = c^n(P_1P_2)$ such that $n \geq 0$, $P_1 \in \Psi^c_0(M_1, \varnothing)$ and $P_2 \in \Psi^c(M_2, \varnothing)$. By lemma 6.19 and lemma 6.3.1a, $\mathcal{R}^{\beta\eta}_{P_1} = \mathcal{R}^{\beta\eta}_{P_2} = \varnothing$. Since $P_1 \in \Psi^c_0(M_1, \varnothing)$, $P_1 = \lambda x.P_0[x := c(cx)]$ such that $P_0 \in \Psi^c(M_0, \varnothing)$. If $P \rightarrow_{\beta\eta} Q$ then by definition there exists p_0 such that $P \xrightarrow{p_0}_{\beta\eta} Q$. By lemma 2.4.12b and lemma 2.2.8, $Q = c^n(Q')$, $p_0 = 2^n.p'_0$ and $P_1P_2 \xrightarrow{p'_0}_{\beta\eta} Q'$ such that $p'_0 \in \mathcal{R}^{\beta\eta}_{P_1P_2}$. By lemma 2.5, $\mathcal{R}^{\beta\eta}_{P_1P_2} = \{0\}$. So $p'_0 = 0$ and $Q = c^n(P_0[x := c(cP_2)])$. Because $c(cP_2) \in \Psi^c(M_2, \varnothing)$, by lemma 6.20 and lemma 2.7.5, $\mathcal{R}^{\beta\eta}_{Q} = \varnothing$.

* Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{M_1}$. So, $\Psi^c(M, \{p\}) = \{c^n(cP_1P_2) \mid n \geq 0 \wedge P_1 \in \Psi^c(M_1, \{p'\}) \wedge P_2 \in \Psi^c(M_2, \varnothing)\}$. Let $P \in \Psi^c(M, \{p\})$ then $P = c^n(cP_1P_2)$ such that $n \geq 0$, $P_1 \in \Psi^c(M_1, \{p'\})$ and $P_2 \in \Psi^c(M_2, \varnothing)$. By lemma 6.19, $\mathcal{R}^{\beta\eta}_{P_2} = \varnothing$. If $P \to_{\beta\eta} Q$ then by definition there exists p_0 such that $P \xrightarrow{p_0}_{\beta\eta} Q$. By lemma 2.4.12b and lemma 2.2.8, $p_0 = 2^n.p'_0$ such that $p'_0 \in \mathcal{R}^{\beta\eta}_{cP_1P_2}$ and $Q = c^n(Q')$ such that $cP_1P_2 \xrightarrow{p'_0}_{\beta\eta} Q'$. By lemma 2.5, $\mathcal{R}^{\beta\eta}_{cP_1P_2} = \{1.2.p \mid p \in \mathcal{R}^{\beta\eta}_{P_1}\}$. So $p'_0 = 1.2.p''_0$ such that $p''_0 \in \mathcal{R}^{\beta\eta}_{P_1}$. So $Q' = cQ_1P_2$ and $P_1 \xrightarrow{p''_0}_{\beta\eta} Q_1$. By IH, $\mathcal{R}^{\beta\eta}_{Q_1} = \varnothing$, so by lemma 2.7.5, $\mathcal{R}^{\beta\eta}_{Q} = \varnothing$.
* Or $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{M_2}$. So, $\Psi^c(M, \{p\}) = \{c^n(cP_1P_2) \mid n \geq 0 \wedge P_1 \in \Psi^c(M_1, \{\varnothing\}) \wedge P_2 \in \Psi^c(M_2, p')\}$. Let $P \in \Psi^c(M, \{p\})$ then $P = c^n(cP_1P_2)$ such that $n \geq 0$, $P_1 \in \Psi^c(M_1, \{\varnothing\})$ and $P_2 \in \Psi^c(M_2, p')$. By lemma 6.19, $\mathcal{R}^{\beta\eta}_{P_1} = \varnothing$. If $P \to_{\beta\eta} Q$ then by definition there exists p_0 such that $P \xrightarrow{p_0}_{\beta\eta} Q$. By lemma 2.4.12b and lemma 2.2.8, $p_0 = 2^n.p'_0$ such that $p'_0 \in \mathcal{R}^{\beta\eta}_{cP_1P_2}$ and $Q = c^n(Q')$ such that $cP_1P_2 \xrightarrow{p'_0}_{\beta\eta} Q'$. By lemma 2.5, $\mathcal{R}^{\beta\eta}_{cP_1P_2} = \{2.p \mid p \in \mathcal{R}^{\beta\eta}_{P_2}\}$. So $p'_0 = 2.p''_0$ such that $p''_0 \in \mathcal{R}^{\beta\eta}_{P_2}$. So $Q' = cP_1Q_2$ and $P_2 \xrightarrow{p''_0}_{\beta\eta} Q_2$. By IH, $\mathcal{R}^{\beta\eta}_{Q_2} = \varnothing$, so by lemma 2.7.5, $\mathcal{R}^{\beta\eta}_{Q} = \varnothing$.

– Let $M \notin \mathcal{R}^{\beta\eta}$, then by lemma 2.5, $\mathcal{R}^{\beta\eta}_{M} = \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_1}\} \cup \{2.p \mid p \in \mathcal{R}^{\beta\eta}_{M_2}\}$.

  * Either $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{M_1}$. So, $\Psi^c(M, \{p\}) = \{c^n(cP_1P_2) \mid n \geq 0 \wedge P_1 \in \Psi^c(M_1, \{p'\}) \wedge P_2 \in \Psi^c(M_2, \varnothing)\}$. Let $P \in \Psi^c(M, \{p\})$ then $P = c^n(cP_1P_2)$ such that $n \geq 0$, $P_1 \in \Psi^c(M_1, \{p'\})$ and $P_2 \in \Psi^c(M_2, \varnothing)$. By lemma 6.19, $\mathcal{R}^{\beta\eta}_{P_2} = \varnothing$. If $P \to_{\beta\eta} Q$ then by definition there exists p_0 such that $P \xrightarrow{p_0}_{\beta\eta} Q$. By lemma 2.4.12b and lemma 2.2.8, $p_0 = 2^n.p'_0$ such that $p'_0 \in \mathcal{R}^{\beta\eta}_{cP_1P_2}$ and $Q = c^n(Q')$ such that $cP_1P_2 \xrightarrow{p'_0}_{\beta\eta} Q'$. By lemma 2.5, $\mathcal{R}^{\beta\eta}_{cP_1P_2} = \{1.2.p \mid p \in \mathcal{R}^{\beta\eta}_{P_1}\}$. So $p'_0 = 1.2.p''_0$ such that $p''_0 \in \mathcal{R}^{\beta\eta}_{P_1}$. So $Q' = cQ_1P_2$ and $P_1 \xrightarrow{p''_0}_{\beta\eta} Q_1$. By IH, $\mathcal{R}^{\beta\eta}_{Q_1} = \varnothing$, so by lemma 2.7.5, $\mathcal{R}^{\beta\eta}_{Q} = \varnothing$.
  * Or $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{M_2}$. So, $\Psi^c(M, \{p\}) = \{c^n(cP_1P_2) \mid n \geq 0 \wedge P_1 \in \Psi^c(M_1, \{\varnothing\}) \wedge P_2 \in \Psi^c(M_2, p')\}$. Let $P \in \Psi^c(M, \{p\})$ then $P = c^n(cP_1P_2)$ such that $n \geq 0$, $P_1 \in \Psi^c(M_1, \{\varnothing\})$ and $P_2 \in \Psi^c(M_2, p')$. By lemma 6.19, $\mathcal{R}^{\beta\eta}_{P_1} = \varnothing$. If $P \to_{\beta\eta} Q$ then by definition there exists p_0 such that $P \xrightarrow{p_0}_{\beta\eta} Q$. By lemma 2.4.12b and lemma 2.2.8, $p_0 = 2^n.p'_0$ such that $p'_0 \in$

$\mathcal{R}^{\beta\eta}_{cP_1P_2}$ and $Q = c^n(Q')$ such that $cP_1P_2 \xrightarrow{p'_0}_{\beta\eta} Q'$. By lemma 2.5, $\mathcal{R}^{\beta\eta}_{cP_1P_2} = \{2.p \mid p \in \mathcal{R}^{\beta\eta}_{P_2}\}$. So $p'_0 = 2.p''_0$ such that $p''_0 \in \mathcal{R}^{\beta\eta}_{P_2}$. So $Q' = cP_1Q_2$ and $P_2 \xrightarrow{p''_0}_{\beta\eta} Q_2$. By IH, $\mathcal{R}^{\beta\eta}_{Q_2} = \varnothing$, so by lemma 2.7.5, $\mathcal{R}^{\beta\eta}_{Q} = \varnothing$.

□

Lemma 6.22. By lemma 2.2.8, $p \in \mathcal{R}^{\beta\eta}_{M}$. By lemma 6.4, there exists a unique set $\mathcal{F}' \subseteq \mathcal{R}^{\beta\eta}_{M'}$, such that for all $N \in \Psi^c(M, \{p\})$, there exists $N' \in \Psi^c(M', \mathcal{F}')$ such that $N \rightarrow_{\beta\eta} N'$. Note that $\Psi^c(M, \{p\}) \neq \varnothing$. Let $N \in \Psi^c(M, \{p\})$ then there exists $N' \in \Psi^c(M', \mathcal{F}')$ such that $N \rightarrow_{\beta\eta} N'$. By lemma 6.21, $\mathcal{R}^{\beta\eta}_{N'} = \varnothing$, so $|\langle N', \mathcal{R}^{\beta\eta}_{N'}\rangle|^c = \varnothing$ and by lemma 6.3.1h, $\mathcal{F}' = \varnothing$. Finally, by lemma 6.15, $\langle M, \{p\}\rangle \rightarrow_{\beta\eta d} \langle M', \varnothing\rangle$. □

Lemma 6.23. By definition $\rightarrow^*_1 \subseteq \rightarrow^*_{\beta\eta}$. We prove that $\rightarrow^*_{\beta\eta} \subseteq \rightarrow^*_1$. Let $M, M' \in \Lambda$ such that $c \notin \mathrm{fv}(M)$ and $M \rightarrow^*_{\beta\eta} M'$. We prove this claim by induction on $M \rightarrow^*_{\beta\eta} M'$.

- Let $M = M'$ then it is done since $\langle M, \mathcal{F}\rangle \rightarrow^*_{\beta\eta d} \langle M, \mathcal{F}\rangle$.
- Let $M \rightarrow^*_{\beta\eta} M'' \rightarrow_{\beta\eta} M'$. By IH, $M \rightarrow^*_1 M''$. By definition there exists p such that $M'' \xrightarrow{p}_{\beta\eta} M'$. By lemma 2.2.3, $c \notin \mathrm{fv}(M'')$. By lemma 6.22, $\langle M'', \{p\}\rangle \rightarrow_{\beta\eta d} \langle M', \varnothing\rangle$, so $M'' \rightarrow_1 M'$. Hence $M \rightarrow^*_1 M'' \rightarrow_1 M'$. □

Lemma 6.24. Let $M \in \Lambda$ and let $c \in \mathcal{V}$ such that $c \notin \mathrm{fv}(M)$. Let $M \rightarrow^*_{\beta\eta} M_1$ and $M \rightarrow^*_{\beta\eta} M_2$. Then by lemma 6.23, $M \rightarrow^*_1 M_1$ and $M \rightarrow^*_1 M_2$. We prove the statement by induction on $M \rightarrow^*_1 M_1$.

- Let $M = M_1$. Hence $M_1 \rightarrow^*_1 M_2$ and $M_2 \rightarrow^*_1 M_2$.
- Let $M \rightarrow^*_1 M'_1 \rightarrow_1 M_1$. By IH, $\exists M'_3, M'_1 \rightarrow^*_1 M'_3$ and $M_2 \rightarrow^*_1 M'_3$. We prove that $\exists M_3, M_1 \rightarrow^*_1 M_3$ and $M'_3 \rightarrow_1 M_3$, by induction on $M'_1 \rightarrow^*_1 M'_3$.
  - let $M'_1 = M'_3$, hence $M'_3 \rightarrow_1 M_1$ and $M_1 \rightarrow^*_1 M_1$.
  - Let $M'_1 \rightarrow^*_1 M''_3 \rightarrow_1 M'_3$. By IH, $\exists M'''_3, M_1 \rightarrow^*_1 M'''_3$ and $M''_3 \rightarrow_1 M'''_3$. By lemma 2.2.3, $c \notin \mathrm{fv}(M''_3)$. Since $M''_3 \rightarrow_1 M'_3$ and $M''_3 \rightarrow_1 M'''_3$, By lemma 6.17, $\exists M_3, M'_3 \rightarrow_1 M_3$ and $M'''_3 \rightarrow_1 M_3$.

□