Reducibility proofs in λ-calculi with intersection types

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Abstract

Reducibility has been used to prove a number of properties in the λ-calculus and is well known to offer on one hand very general proofs which can be applied to a number of instantiations, and on the other hand, to be quite mysterious and inflexible. In this paper, we look at two related but different results in λ-calculi with intersection types. We show that one such result (which aims at giving reducibility proofs of Church-Rosser, standardisation and weak normalisation for the untyped λ-calculus) faces serious problems which break the reducibility method and then we provide a proposal to partially repair the method. Then, we consider a second result whose purpose is to use reducibility for typed terms to show Church-Rosser of β-developments for untyped terms (without needing to use strong normalisation), from which Church-Rosser of β-reduction easily follows. We extend the second result to encompass both βI- and $\beta\eta$-reduction rather than simply β-reduction.

1 Introduction

Based on realisability semantics [6], the reducibility method has been developed by Tait [11] in order to prove normalisation of some functional theories. The idea is to interpret types by sets of λ-terms closed under some properties. Krivine [10] uses reducibility to prove the strong normalisation of system D. Koletsos [8] proves that the set of simply typed λ-terms has the Church-Rosser property. Gallier [3, 4] uses some aspects of Koletsos's method to prove a number of results such as the strong normalisation of the λ-terms that are typable in systems like D or $D\Omega$ [10]. In particular, Gallier states some conditions a property needs to satisfy in order to be enjoyed by some typable terms under some restrictions. Similarly, Ghilezan and Likavec [5] state some conditions a property on λ-terms has to satisfy in order to be held by all λ-terms that are typable under some restriction on types in a type system which is close to $D\Omega$. Additionally Ghilezan and Likavec state a condition that a property needs to satisfy in order to step from "a λ-term typable under some restrictions on types holds the property" to "a λ-term of the untyped λ-calculus holds the property". If successful, the method designed by Ghilezan and Likavec would provide an attractive method for establishing properties like Church-Rosser for all the untyped λ-terms, simply by showing easier conditions on typed terms. However, we show in this paper that Ghilezan and Likavec's method fails for the typed terms, and that also the step of passing from typed to untyped terms fails. We show why we also fail to entirely repair the first result and how far we can get when trying to repair it. The second result seems unrepairable. Ghilezan and Likavec also present a weaker method for a type system similar to system D, which allows using reducibility to prove properties of the term typable by this system, namely the strongly normalisable terms. As far as we know, this portion of their result is correct. (They do not actually apply this weaker method to any sets of terms.)

In addition to the method proposed by Ghilezan and Likavec (which does not actually work for the full untyped λ-calculus), other steps of establishing properties like Church-Rosser (also called confluence) for typed λ-terms and concluding the properties for all the untyped λ-terms have been successfully exploited in the literature. Koletsos and Stavrinos [9] use reducibility to state that λ-terms that are typable in system D hold the Church-Rosser property. Using this result together with a method based on β-developments [7, 10], they show that β-developments are Church-Rosser and this in turn will imply the confluence of the untyped λ-calculus. Although Klop proves the confluence of β-developments [1], his proof is based on strong normalisation whereas the Koletsos and Stavrinos's proof only uses an embedding of β-developments in the reduction of typable λ-terms. In this paper, we apply Koletsos and Stavrinos's method to βI-reduction and then generalise it to $\beta\eta$-reduction.

In section 2 we introduce formal machinery. In section 3 we present the reducibility method used by Ghilezan and Likavec and show that it fails at a number of important propositions which makes it inapplicable to the full untyped λ-calculus, although a version of their method works for the strongly normalisable terms. We give counterexamples which show that all the conditions stated in Ghilezan and Likavec's paper are satisfied, yet the claimed property does not hold. In section 4 we give some indications on the limits of the method. We show how these limits affect the salvation of the method (when trying to salvage the method we, in some sense, only go a bit further than the result obtained by Ghilezan and Likavec using a type system similar to D instead of the type system similar to $D\Omega$). We also point out some links between the work done by Ghilezan and Likavec and the work done by Gallier. In section 5 we adapt the Church-Rosser proof of Koletsos and Stavrinos [9] to βI-reduction. In section 6 we non-trivially generalise the Koletsos and Stavrinos's method to handle $\beta\eta$-reduction. We conclude in section 7. For space reasons we omit proofs. However, full proofs can be downloaded from the web page of the authors.

2 The Formal Machinery

In this section we provide some known formal machinery on λ-calculus and type theory. Let n, m be metavariables which range over the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. We assume that if a metavariable v ranges over any set s then the metavariables v_n, v', v'', etc. also range over s. A binary relation is a set of pairs. Let *rel* range over binary relations. Let $\mathrm{dom}(\mathit{rel}) = \{x \mid \langle x, y\rangle \in \mathit{rel}\}$ and $\mathrm{ran}(\mathit{rel}) = \{y \mid \langle x, y\rangle \in \mathit{rel}\}$. A function is a binary relation *fun* such that if $\{\langle x, y\rangle, \langle x, z\rangle\} \subseteq \mathit{fun}$ then $y = z$. Let *fun* range over functions. Let $s \to s' = \{\mathit{fun} \mid \mathrm{dom}(\mathit{fun}) \subseteq s \wedge \mathrm{ran}(\mathit{fun}) \subseteq s'\}$. Given n sets $s_1, \ldots, s_n$, where $n \geq 2$, $s_1 \times \ldots \times s_n$ stands for the set of all the tuples built on the sets $s_1, \ldots, s_n$. If $x \in s_1 \times \ldots \times s_n$, then $x = \langle x_1, \ldots, x_n\rangle$ such that $x_i \in s_i$ for all $i \in \{1, \ldots, n\}$.

Definition 1 (BACKGROUND ON THE λ-CALCULUS)

- We let x, y, z range over $\mathcal{V}$, a countable infinite set of λ-term variables. λ-terms are defined by $M \in \Lambda ::= x \mid (\lambda x.M) \mid (M_1 M_2)$. We let M, N, P, Q, R range over Λ. We assume the usual definition of subterms and write $N \subseteq M$ if N is a subterm of M ($M \subseteq M$). We assume the usual convention for parenthesis and omit these if no confusion arises. Hence, $M\ N_1 ... N_n$, where $n \geq 1$, stands for $(...((M\ N_1)\ N_2)...N_{n-1})\ N_n$.
  We take terms modulo α-conversion and use the Barendregt convention (BC) where the names of bound variables differ from the free ones. When two terms M and N are equal (modulo α), we write $M = N$. We write $\mathrm{fv}(M)$ for the set of the free variables of term M.
  If $\mathcal{F} \subseteq \Lambda$ then let $\mathcal{F} \upharpoonright M = \{N \mid N \in \mathcal{F} \wedge N \subseteq M\}$
  We define $M^n(N)$ by induction on n, as follows: $M^0(N) = N$ and $M^{n+1}(N) = M(M^n(N))$.
- The set of paths is defined as follows: $p \in \mathsf{Path} ::= 0 \mid 1.p \mid 2.p$. We define $M|_p$ as follows: $M|_0 = M$, $(\lambda x.M)|_{1.p} = M|_p$, $(MN)|_{1.p} = M|_p$ and $(MN)|_{2.p} = N|_p$. We define $2^n.p$ by induction on $n \geq 0$: $2^0.p = p$ and $2^{n+1}.p = 2^n.2.p$.
- The set $\Lambda\mathrm{I} \subset \Lambda$, of terms of the λI-calculus is defined by the following rules: – $x \in \Lambda\mathrm{I}$
  – If $x \in \mathrm{fv}(M)$ and $M \in \Lambda\mathrm{I}$ then $\lambda x.M \in \Lambda\mathrm{I}$ – If $M, N \in \Lambda\mathrm{I}$ then $MN \in \Lambda\mathrm{I}$
- We define as usual the substitution $M[x := N]$ of the term N for all free occurrences of x in the term M. We write $M[x_1 := N_1, \ldots, x_n := N_n]$ for the simultaneous substitution of N_i for all free occurrences of x_i in M for $1 \leq i \leq n$.
- Let us define the four following common relations:
  – $\mathsf{Beta} ::= \langle (\lambda x.M)N, M[x := N]\rangle$ – $\mathsf{Eta} ::= \langle \lambda x.Mx, M\rangle$, where $x \notin FV(M)$
  – $\mathsf{BetaI} ::= \langle (\lambda x.M)N, M[x := N]\rangle$, where $x \in FV(M)$ – $\mathsf{BetaEta} = \mathsf{Beta} \cup \mathsf{Eta}$
  Let $\langle r, s\rangle \in \{\langle \mathsf{Beta}, \beta\rangle, \langle \mathsf{BetaI}, \beta I\rangle, \langle \mathsf{Eta}, \eta\rangle, \langle \mathsf{BetaEta}, \beta\eta\rangle\}$. We define $\mathcal{R}^s$ to be $\{L \mid \langle L, R\rangle \in$

$r\}$. If $\langle L, R\rangle \in r$ then we call L a *s-redex* and R the *s-contractum* of L (or the L s-contractum).
We define the ternary relation $\rightarrow_s$ as follows:

– $M \xrightarrow{0}_s M'$ if $\langle M, M'\rangle \in r$ – $\lambda x.M \xrightarrow{1.p}_s \lambda x.M'$ if $M \xrightarrow{p}_s M'$

– $MN \xrightarrow{1.p}_s M'N$ if $M \xrightarrow{p}_s M'$ – $NM \xrightarrow{2.p}_s NM'$ if $M \xrightarrow{p}_s M'$

We define the binary relation $\rightarrow_s$ (we use the same name as for the just defined ternary relation $\rightarrow_s$ to simplify the notations) as follows: $M \rightarrow_s M'$ if there exists p such that $M \xrightarrow{p}_s M'$. We define $\mathcal{R}^s_M$ as $\{p \mid M|_p \in \mathcal{R}^s\}$.
We us define the head and internal reductions:
$\rightarrow_h ::= \langle \lambda x_1. \dots x_n.(\lambda x.M_0)M_1 \dots M_m, \lambda x_1. \dots x_n.M_0[x := M_1]M_2 \dots M_m\rangle$, where $n \geq 0$ and $m \geq 1$.
We define the binary relation $\rightarrow_i$ as $\rightarrow_\beta \setminus \rightarrow_h$.
$s \in \{\beta, \beta I, \eta, \beta\eta, h, i\}$. We define the relation $\rightarrow^*_s$ as the reflexive and transitive closure of $\rightarrow_s$.

- Let $\mathsf{NF}_\beta = \{\lambda x_1. \dots \lambda x_n.x_0 N_1 \dots N_m \mid N_1, \dots, N_m \in \mathsf{NF}_\beta\}$ be the set of β-normal forms, $\mathsf{WN}_\beta = \{M \mid \exists N \in \mathsf{NF}_\beta.\ M \rightarrow^*_\beta N\}$ be the set of weakly β-normalisable terms and $\mathsf{SN}_\beta = \{M \mid \text{all the reductions starting from } M \text{ are finite}\}$ be the set of strongly β-normalisable terms.
Let $r \in \{\beta, \beta I, \beta\eta\}$. We say that M has the Church-Rosser property for r (has r-CR) if whenever $M \rightarrow^*_r M_1$ and $M \rightarrow^*_r M_2$, there exists M_3 such that $M_1 \rightarrow^*_r M_3$ and $M_2 \rightarrow^*_r M_3$. Let $\mathsf{CR}^r = \{M \mid M \text{ has } r\text{-CR}\}$ and $\mathsf{CR}^r_0 = \{xM_1 \dots M_n \mid \forall i \in \{1, \dots, n\}.\ M_i \in \mathsf{CR}^r\}$. We use CR to denote CR^β and CR_0 to denote CR^β_0.
A term is a weak head normal form if it is a λ-abstraction (a term of the form $\lambda x.M$) or if it starts with a variable (a term of the form $xM_1 \dots M_n$). A term is weakly head normalising if it reduces to a weak head normal form. let $\mathsf{W}^r = \{M \mid \exists n \in \mathbb{N}.\ \exists x \in \mathcal{V}.\ \exists P, P_1, \dots, P_n \in \Lambda.\ (M \rightarrow^*_r \lambda x.P \vee M \rightarrow^*_r xP_1 \dots P_n)\}$. We use W to denote W^β.
We say that M has the standardisation property if whenever $M \rightarrow^*_\beta N$ then there is a M' such that $M \rightarrow^*_h M'$ and $M' \rightarrow^*_i N$. Let $\mathsf{S} = \{M \mid M \text{ has the standardisation property}\}$.

Throughout, we take c to be a metavariable ranging over $\mathcal{V}$. The next definition adapts Λ_c and the c-erasure defined by Krivine [10], to deal with βI- and $\beta\eta$-reduction.

Definition 2 (BACKGROUND ON DEVELOPMENTS)

- Let $\mathcal{M}_c$ range over $\{\Lambda\eta_c, \Lambda \mathrm{I}_c\}$ where $\Lambda\eta_c$ and $\Lambda\mathrm{I}_c$ are defined as follows (note that $\Lambda\mathrm{I}_c \subset \Lambda\mathrm{I}$):

(R1) If x is a variable distinct from c then
1. $x \in \mathcal{M}_c$.
2. If $M \in \Lambda\mathrm{I}_c$ and $x \in \mathrm{fv}(M)$ then $\lambda x.M \in \Lambda\mathrm{I}_c$.
3. If $M \in \Lambda\eta_c$ then $\lambda x.(M[x := c(cx)]) \in \Lambda\eta_c$.
4. If $Nx \in \Lambda\eta_c$, $x \notin \mathrm{fv}(N)$ and $N \neq c$ then $\lambda x.Nx \in \Lambda\eta_c$.

(R2) If $M, N \in \mathcal{M}_c$ then $cMN \in \mathcal{M}_c$.
(R3) If $M, N \in \mathcal{M}_c$ and M is a λ-abstraction then $MN \in \mathcal{M}_c$.
(R4) If $M \in \Lambda\eta_c$ then $cM \in \Lambda\eta_c$.

- We define $|M|^c$ and $|\langle M, p\rangle|^c$ inductively as follows:

– $|x|^c = x$ – $|\lambda x.N|^c = \lambda x.|N|^c$, if $x \neq c$

– $|cP|^c = |P|^c$ – $|NP|^c = |N|^c|P|^c$, if $N \neq c$.

– $|\langle M, 0\rangle|^c = 0$ – $|\langle \lambda x.M, 1.p\rangle|^c = 1.|\langle M, p\rangle|^c$, if $x \neq c$

– $|\langle MN, 1.p\rangle|^c = 1.|\langle M, p\rangle|^c$

– $|\langle cM, 2.p\rangle|^c = |\langle M, p\rangle|^c$ – $|\langle NM, 2.p\rangle|^c = 2.|\langle M, p\rangle|^c$, if $N \neq c$

Definition 3 (BACKGROUND ON TYPE SYSTEMS) Let $i \in \{1, 2\}$.

- Let $\mathcal{A}$ be a countable infinite set of type variables, let α range over $\mathcal{A}$ and let $\Omega \notin \mathcal{A}$ be a constant type. The sets of types $\mathsf{Type}^1 \subset \mathsf{Type}^2$ are given by:

(1)	$\tau \le \tau$	(6)	$(\tau_1 \le \tau_2 \wedge \tau_1 \le \tau_3) \Rightarrow \tau_1 \le \tau_2 \cap \tau_3$
(2)	$(\tau_1 \le \tau_2 \wedge \tau_2 \le \tau_3) \Rightarrow \tau_1 \le \tau_3$	(7)	$(\tau_1 \le \tau_1' \wedge \tau_2 \le \tau_2') \Rightarrow \tau_1 \cap \tau_2 \le \tau_1' \cap \tau_2'$
(3)	$\tau_1 \cap \tau_2 \le \tau_1$	(8)	$(\tau_1 \le \tau_1' \wedge \tau_2' \le \tau_2) \Rightarrow \tau_1' \to \tau_2' \le \tau_1 \to \tau_2$
(4)	$\tau_1 \cap \tau_2 \le \tau_2$	(9)	$\tau \le \Omega$
(5)	$(\tau_1 \to \tau_2) \cap (\tau_1 \to \tau_3) \le \tau_1 \to (\tau_2 \cap \tau_3)$	(10)	$\tau \to \Omega \le \Omega \to \Omega$

Figure 1: Ordering axioms on types

$$\frac{}{\Gamma, x : \tau \vdash x : \tau}\ (ax) \qquad \frac{}{x : \tau \vdash x : \tau}\ (ax^I)$$

$$\frac{\Gamma \vdash M : \tau_1 \to \tau_2 \quad \Gamma \vdash N : \tau_1}{\Gamma \vdash MN : \tau_2}\ (\to_E) \qquad \frac{\Gamma_1 \vdash M : \tau_1 \to \tau_2 \quad \Gamma_2 \vdash N : \tau_1}{\Gamma_1 \sqcap \Gamma_2 \vdash MN : \tau_2}\ (\to_{E^I})$$

$$\frac{\Gamma, x : \tau_1 \vdash M : \tau_2}{\Gamma \vdash \lambda x.M : \tau_1 \to \tau_2}\ (\to_I) \qquad \frac{\Gamma \vdash M : \tau_1 \quad \Gamma \vdash M : \tau_2}{\Gamma \vdash M : \tau_1 \cap \tau_2}\ (\cap_I)$$

$$\frac{\Gamma \vdash M : \tau_1 \cap \tau_2}{\Gamma \vdash M : \tau_1}\ (\cap_{E1}) \qquad \frac{\Gamma \vdash M : \tau_1 \cap \tau_2}{\Gamma \vdash M : \tau_2}\ (\cap_{E2})$$

$$\frac{\Gamma \vdash M : \tau_1 \quad \tau_1 \le^\nabla \tau_2}{\Gamma \vdash M : \tau_2}\ (\le^\nabla) \qquad \frac{}{\Gamma \vdash M : \Omega}\ (\Omega)$$

Figure 2: Typing rules

$$\sigma \in \mathsf{Type}^1 ::= \alpha \mid \sigma_1 \to \sigma_2 \mid \sigma_1 \cap \sigma_2$$
$$\tau \in \mathsf{Type}^2 ::= \alpha \mid \tau_1 \to \tau_2 \mid \tau_1 \cap \tau_2 \mid \Omega$$

- Let $\Gamma, \Delta \in \mathcal{B}^1 = \{\{x_1 : \sigma_1, \ldots, x_n : \sigma_n\} \mid \forall j, k \in \{1, \ldots, n\}.\ (k \neq j \Rightarrow x_k \neq x_j)\}$ and $\Gamma, \Delta \in \mathcal{B}^2 = \{\{x_1 : \tau_1, \ldots, x_n : \tau_n\} \mid \forall j, k \in \{1, \ldots, n\}.\ (k \neq j \Rightarrow x_k \neq x_j)\}$. Let $\mathcal{B} \in \{\mathcal{B}^1, \mathcal{B}^2\}$. We define $\mathrm{dom}(\Gamma) = \{x \mid x : \tau \in \Gamma\}$. When $\mathrm{dom}(\Gamma_1) \cap \mathrm{dom}(\Gamma_2) = \varnothing$, we write Γ_1, Γ_2 for $\Gamma_1 \cup \Gamma_2$. Moreover, we write $x : \tau$ for $\{x : \tau\}$. We denote $\Gamma = \{x_m : \tau_m, \ldots, x_n : \tau_n\}$ where $n \ge m \ge 0$, by $(x_i : \tau_i)_n^m$. If $m = 1$, we simply denote Γ by $(x_i : \tau_i)_n$. If $\Gamma_1 = (x_i : \tau_i)_n, (y_i : \tau_i'')_p$ and $\Gamma_2 = (x_i : \tau_i')_n, (z_i : \tau_i''')_q$ where $x_1, \ldots, x_n$ are the only shared variables, then $\Gamma_1 \sqcap \Gamma_2 = (x_i : \tau_i \cap \tau_i')_n, (y_i : \tau_i'')_p, (z_i : \tau_i''')_q$. Let $X \subseteq \mathcal{V}$. We define $\Gamma \upharpoonright X = \Gamma' \subseteq \Gamma$ where $\mathrm{dom}(\Gamma') = \mathrm{dom}(\Gamma) \cap X$.
- We define a type system TS by its set of types $types$, its type derivability relation $deriv$, its set of environments $\mathcal{B}$ and its set X of rules from Figure 2 where $deriv$ is the type derivability relation built on $\mathcal{B}$, Λ and $types$ and generated using the rules X. We write $TS, deriv \sharp (types, \mathcal{B}, X)$ to define the type derivability relation $deriv$ built on $types$, $\mathcal{B}$ and X and to define the type system TS built on $types$, $deriv$, $\mathcal{B}$ and X.
  Referring to Figure 1, let $\nabla_1 = \{(1), (2), (3), (4), (5), (6), (7), (8)\}$, $\nabla_2 = \nabla_1 \cup \{(9), (10)\}$, and $\mathsf{Type}^{\nabla_i} = \mathsf{Type}^i$. Let $\le^i$ be the subtyping relation defined on the set of types Type^i and the set of axioms ∇_i. We write $\tau_1 \sim^i \tau_2$ iff $\tau_1 \le^i \tau_2$ and $\tau_2 \le^i \tau_1$.
  We now define $\lambda\cap^1$, $\lambda\cap^2$, and D, D_I, our four main type systems:
  $\lambda\cap^1, \vdash^1 \sharp(\mathsf{Type}^1, \mathcal{B}^1, \{(ax), (\to_E), (\to_I), (\cap_I), (\le^1)\})$
  $\lambda\cap^2, \vdash^2 \sharp(\mathsf{Type}^2, \mathcal{B}^2, \{(ax), (\to_E), (\to_I), (\cap_I), (\le^2), (\Omega)\})$
  $D, \vdash^{\beta\eta} \sharp(\mathsf{Type}^1, \mathcal{B}^1, \{(ax), (\to_E), (\to_I), (\cap_I), (\cap_{E1}), (\cap_{E2})\})$
  $D\Omega, \vdash^{\Omega} \sharp(\mathsf{Type}^2, \mathcal{B}^2, \{(ax), (\to_E), (\to_I), (\cap_I), (\cap_{E1}), (\cap_{E2}), (\Omega)\})$
  $D_I, \vdash^{\beta I} \sharp(\mathsf{Type}^1, \mathcal{B}^1, \{(ax^I), (\to_E^I), (\to_I), (\cap_I), (\cap_{E1}), (\cap_{E2})\})$. In D_I, we assume $\sigma \cap \sigma = \sigma$.

3 Problems of the Ghilezan and Likavec's reducibility method [5]

We now introduce the Ghilezan and Likavec's method and explain its problems. Throughout, we let $\circledast = \lambda x.xx$.

Definition 4 (TYPES/REDUCIBILITY OF [5]) Let $i \in \{1, 2\}$ and $\mathcal{P}$ ranging over 2^Λ.

- The type interpretation $[\![-]\!]^i_{-}$ is a function in $(\mathsf{Type}^i \times 2^\Lambda) \to 2^\Lambda$, defined by:
  - $[\![\alpha]\!]^i_{\mathcal{P}} = \mathcal{P}$
  - $[\![\tau_1 \to \tau_2]\!]^1_{\mathcal{P}} = \{M \mid \forall N \in [\![\tau_1]\!]^1_{\mathcal{P}}.\ MN \in [\![\tau_2]\!]^1_{\mathcal{P}}\}$
  - $[\![\tau_1 \to \tau_2]\!]^2_{\mathcal{P}} = \{M \in \mathcal{P} \mid \forall N \in [\![\tau_1]\!]^2_{\mathcal{P}}.\ MN \in [\![\tau_2]\!]^2_{\mathcal{P}}\}$
  - $[\![\tau_1 \cap \tau_2]\!]^i_{\mathcal{P}} = [\![\tau_1]\!]^i_{\mathcal{P}} \cap [\![\tau_2]\!]^i_{\mathcal{P}}$
  - $[\![\Omega]\!]^2_{\mathcal{P}} = \Lambda$
- A valuation is a function $\nu : \mathcal{V} \to \Lambda$.
  We let $\nu(x := M)$ be the function ν where $\nu'(x) = M$ and $\nu'(y) = \nu(y)$ if $y \neq x$.
  Let $[\![-]\!]_\nu : \Lambda \to \Lambda$ where $[\![M]\!]_\nu = M[x_1 := \nu(x_1), \ldots, x_n := \nu(x_n)]$ for $\mathrm{fv}(M) = \{x_1, \ldots, x_n\}$.
- – $\nu \models^i M : \tau$ iff $[\![M]\!]_\nu \in [\![\tau]\!]^i_{\mathcal{P}}$ – $\nu \models^i \Gamma$ iff $\forall (x : \tau) \in \Gamma.\ \nu(x) \in [\![\tau]\!]^i_{\mathcal{P}}$
  – $\Gamma \models^i M : \tau$ iff $\forall \nu \in \mathcal{V} \to \Lambda.\ (\nu \models^i \Gamma \Rightarrow \nu \models^i M : \tau)$
- Let $\mathcal{X} \subseteq \Lambda$. Let us recall the variable, saturation, closure and invariance under abstraction predicates defined by Ghilezan and Likavec:
  - $\mathrm{VAR}^i(\mathcal{P}, \mathcal{X}) \iff \mathcal{V} \subseteq \mathcal{X}$.
  - $\mathrm{SAT}^1(\mathcal{P}, \mathcal{X}) \iff (\forall M \in \Lambda.\ \forall x \in \mathcal{V}.\ \forall N \in \mathcal{P}.\ M[x := N] \in \mathcal{X} \Rightarrow (\lambda x.M)N \in \mathcal{X})$.
  - $\mathrm{SAT}^2(\mathcal{P}, \mathcal{X}) \iff (\forall M, N \in \Lambda.\ \forall x \in \mathcal{V}.\ M[x := N] \in \mathcal{X} \Rightarrow (\lambda x.M)N \in \mathcal{X})$.
  - $\mathrm{CLO}^1(\mathcal{P}, \mathcal{X}) \iff (\forall M \in \Lambda.\ \forall x \in \mathcal{V}.\ Mx \in \mathcal{X} \Rightarrow M \in \mathcal{P})$.
  - $\mathrm{CLO}^2(\mathcal{P}, \mathcal{X}) \iff \mathrm{CLO}(\mathcal{P}, \mathcal{X}) \iff (\forall M \in \Lambda.\ \forall x \in \mathcal{V}.\ M \in \mathcal{X} \Rightarrow \lambda x.M \in \mathcal{P})$.
  - $\mathrm{VAR}(\mathcal{P}, \mathcal{X}) \iff (\forall x \in \mathcal{V}.\ \forall n \in \mathbb{N}.\ \forall N_1, \ldots, N_n \in \mathcal{P}.\ xN_1 \ldots N_n \in \mathcal{X})$.
  - $\mathrm{SAT}(\mathcal{P}, \mathcal{X}) \iff (\forall M, N \in \Lambda.\ \forall x \in \mathcal{V}.\ \forall n \in \mathbb{N}.\ \forall N_1, \ldots, N_n \in \mathcal{P}.$
    $M[x := N]N_1 \ldots N_n \in \mathcal{X} \Rightarrow (\lambda x.M)NN_1 \ldots N_n \in \mathcal{X})$.
  - $\mathrm{INV}(\mathcal{P}) \iff (\forall M \in \Lambda.\ \forall x \in \mathcal{V}.\ M \in \mathcal{P} \iff \lambda x.M \in \mathcal{P})$.

  For $\mathcal{R} \in \{\mathrm{VAR}^i, \mathrm{SAT}^i, \mathrm{CLO}^i\}$, let $\mathcal{R}(\mathcal{P}) \iff \forall \tau \in \mathsf{Type}^i.\ \mathcal{R}(\mathcal{P}, [\![\tau]\!]^i_{\mathcal{P}})$.

Lemma 5 (PRINCIPAL BASIC LEMMAS PROVED IN [5])

1. *If* $\mathrm{VAR}^1(\mathcal{P})$ *and* $\mathrm{CLO}^1(\mathcal{P})$ *are satisfied then*
   a. $\forall \sigma \in \mathsf{Type}^1.\ [\![\sigma]\!]^1_{\mathcal{P}} \subseteq \mathcal{P}$.
   b. *If* $\mathrm{SAT}^1(\mathcal{P})$ *and* $\Gamma \vdash^1 M : \sigma$ *then we have* $\Gamma \models^1 M : \sigma$ *and* $M \in \mathcal{P}$
2. $\mathrm{VAR}^2(\mathcal{P}) \wedge \mathrm{SAT}^2(\mathcal{P}) \wedge \mathrm{CLO}^2(\mathcal{P}) \wedge \Gamma \vdash^2 M : \tau \Rightarrow \Gamma \models^2 M : \tau$.
3. $\mathrm{VAR}^2(\mathcal{P}) \wedge \mathrm{SAT}^2(\mathcal{P}) \wedge \mathrm{CLO}^2(\mathcal{P}) \wedge \forall \tau \in \mathsf{Type}^2.\ (\tau \not\sim^2 \Omega \wedge \Gamma \vdash^2 M : \tau) \Rightarrow M \in \mathcal{P}$.
4. $\mathrm{CLO}(\mathcal{P}, \mathcal{P}) \Rightarrow \forall \tau \in \mathsf{Type}^2.\ (\tau \not\sim^2 \Omega \Rightarrow \mathrm{CLO}^2(\mathcal{P}, [\![\tau]\!]^2_{\mathcal{P}}))$.
5. $\mathrm{VAR}(\mathcal{P}, \mathcal{P})$, $\mathrm{SAT}(\mathcal{P}, \mathcal{P})$ *and* $\mathrm{CLO}(\mathcal{P}, \mathcal{P})$ *for* $\mathcal{P} \in \{\mathsf{CR}, \mathsf{S}, \mathsf{W}\}$.

Ghilezan and Likavec claim that if $\mathrm{CLO}^1(\mathcal{P})$, $\mathrm{VAR}^1(\mathcal{P})$ and $\mathrm{SAT}^1(\mathcal{P})$ are true then $\mathsf{SN}_\beta \subseteq \mathcal{P}$ (note that this result does not make any use of the type system $\lambda\cap^1$).

According to Ghilezan and Likavec, VAR^i, SAT^i and CLO^i are sufficient for the reducibility method, and to prove them one needs stronger induction hypotheses which are easier to prove. They sets out to show that when $i = 2$, the stronger conditions are VAR, SAT and CLO. We show that this attempt fails. They do not develop the necessary stronger induction hypotheses for the case when $i = 1$, and $\lambda\cap^1$ can only anyway type strongly normalisable terms, so we will not consider the case $i = 1$ further.

Lemma 6 *For all* $\tau, \tau' \in \mathsf{Type}^2$, $\alpha \to \Omega \to \tau' \not\sim^2 \Omega \to \tau$

Lemma 7 (LEMMA 3.16 OF [5] IS FALSE) *Lemma 3.16 of [5] stated below is false:*
$\mathrm{VAR}(\mathcal{P}, \mathcal{P}) \Rightarrow \forall \tau \in \mathsf{Type}^2.\ (\forall \tau' \in \mathsf{Type}^2.\ (\tau \not\sim^2 \Omega \to \tau') \Rightarrow \mathrm{VAR}(\mathcal{P}, [\![\tau]\!]^2_{\mathcal{P}}))$.

Proof: Note that $\mathrm{VAR}(\mathcal{P}, [\![\tau]\!]^2_{\mathcal{P}}) \Rightarrow \mathcal{V} \subseteq [\![\tau]\!]^2_{\mathcal{P}}$. Let $x \in \mathcal{V}$, τ be $\alpha \to \Omega \to \alpha$ and $\mathcal{P}$ be WN_β. By lemma 6, for all $\tau' \in \mathsf{Type}^2$, $\tau \not\sim^2 \Omega \to \tau'$ and $\mathrm{VAR}(\mathcal{P}, \mathcal{P})$ is true. Assume $\mathrm{VAR}(\mathcal{P}, [\![\tau]\!]^2_{\mathcal{P}})$, then $x \in [\![\tau]\!]^2_{\mathcal{P}}$. Then $x \in [\![\alpha \to \Omega \to \alpha]\!]^2_{\mathcal{P}} = [\![\tau]\!]^2_{\mathcal{P}}$ because $x \in \mathcal{P} = [\![\alpha]\!]^2_{\mathcal{P}}$, and $xx(\circledast\circledast) \in [\![\alpha]\!]^2_{\mathcal{P}} = \mathcal{P}$ because $\circledast\circledast \in \Lambda = [\![\Omega]\!]^2_{\mathcal{P}}$. But $xx(\circledast\circledast) \in \mathcal{P}$ is false, so $\mathrm{VAR}(\mathcal{P}, [\![\tau]\!]^2_{\mathcal{P}})$ is false. $\square$

The proof for Lemma 3.18 of [5] does not work (because of a misused of an induction hypothesis) but we have not yet proved or disproved that lemma:

Remark 8 (IT IS NOT CLEAR THAT LEMMA 3.18 OF [5] HOLDS) It is not clear whether this lemma of [5] holds: $\mathrm{SAT}(\mathcal{P},\mathcal{P}) \Rightarrow \forall\tau \in \mathsf{Type}^2.\ (\forall\tau' \in \mathsf{Type}^2.\ (\tau \not\sim^2 \Omega \to \tau') \Rightarrow \mathrm{SAT}(\mathcal{P}, [\![\tau]\!]^2_{\mathcal{P}}))$.

Then, Ghilezan and Likavec give a proposition (Proposition 3.21) which is the reducibility method for typable terms. However, the proof of that proposition depends on two problematic lemmas (lemma 3.16 which we showed to fail in our lemma 7, and lemma 3.18 which according to remark 8 has not been proved).

Lemma 9 (PROPOSITION 3.21 OF [5] FAILS) *Assume* $\mathrm{VAR}(\mathcal{P},\mathcal{P})$*,* $\mathrm{SAT}(\mathcal{P},\mathcal{P})$ *and* $\mathrm{CLO}(\mathcal{P},\mathcal{P})$*. It is* **not** *the case that:* $\forall\tau \in \mathsf{Type}^2.\ (\tau \not\sim^2 \Omega \wedge \forall\tau' \in \mathsf{Type}^2.\ (\tau \not\sim^2 \Omega \to \tau') \wedge \Gamma \vdash^2 M : \tau \Rightarrow M \in \mathcal{P})$.

Proof: Let $\mathcal{P}$ be WN_β. Note that $\lambda y.\lambda z.\circledast\circledast \notin \mathsf{WN}_\beta$ and $\varnothing \vdash^2 \lambda y.\lambda z.\circledast\circledast : \alpha \to \Omega \to \Omega$ is derivable, where $\alpha \to \Omega \to \Omega \not\sim^2 \Omega$ and by lemma 6, $\alpha \to \Omega \to \Omega \not\sim^2 \Omega \to \tau'$, for all $\tau' \in \mathsf{Type}^2$. Since $\mathrm{VAR}(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$, $\mathrm{CLO}(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$ and $\mathrm{SAT}(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$ hold, we get a counterexample for Proposition 3.21 of [5]. $\square$

Finally, also the Ghilezan and Likavec's proof method for untyped terms fails.

Lemma 10 (PROPOSITION 3.23 OF [5] FAILS) *Proposition 3.23 of [5] which states that "If* $\mathcal{P} \subseteq \Lambda$ *is invariant under abstraction (i.e.,* $\mathrm{INV}(\mathcal{P})$*),* $\mathrm{VAR}(\mathcal{P},\mathcal{P})$ *and* $\mathrm{SAT}(\mathcal{P},\mathcal{P})$ *then* $\mathcal{P} = \Lambda$*" fails.*

Proof: The proof given in [5] depends on Proposition 3.21 which fails. As $\mathrm{VAR}(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$, $\mathrm{SAT}(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$ and $\mathrm{INV}(\mathsf{WN}_\beta)$, we get a counterexample for Proposition 3.23. $\square$

4 How much of the Ghilezan and Likavec's method can we salvage?

Because we proved that the Proposition 3.23 of [5] is false, we know that the given set of properties ($\mathrm{INV}(\mathcal{P})$, $\mathrm{VAR}(\mathcal{P},\mathcal{P})$ and $\mathrm{SAT}(\mathcal{P},\mathcal{P})$) that a set of terms $\mathcal{P}$ has to fulfil to be equal to the set of terms of the untyped λ-calculus is not the right one. So even if one works on the soundness result or on the type interpretation (the set of realisers), to obtain the same result as the one claimed by Ghilezan and Likavec, one should come up with a new set of properties.

Proposition 3.23 of [5] states a set of properties characterising the set of terms of the untyped λ-calculus. The predicate $\mathrm{VAR}(\Lambda,\Lambda)$ states that the variables (and the terms of the form $xNM_1 \cdots M_n$) belong to the untyped λ-calculus. The predicate $\mathrm{INV}(\Lambda)$ states among other things that if a term is a λ-term then the abstraction of a variable over this term is a λ-term too. To get a full characterisation of the set of terms of the untyped λ-calculus, we need a predicate, let us call it $\mathrm{APP}(\mathcal{P})$, stating that $(\lambda x.M)NM_1 \cdots M_n \in \mathcal{P}$ if $M, N, M_1, \ldots, M_n \in \mathcal{P}$, to be true. Is this predicate true if $\mathrm{VAR}(\mathcal{P},\mathcal{P})$, $\mathrm{SAT}(\mathcal{P},\mathcal{P})$ and $\mathrm{INV}(\mathcal{P})$ are true? No, because we saw that we can find a set of terms (WN_β) which satisfies these properties but is not equal to the λ-calculus. For example, we cannot get the non strongly normalisable terms to be in WN_β. So, these properties are not enough to characterise the λ-calculus.

The problem with these properties is that if one tries to salvage the Ghilezan and Likavec's reducibility method, the properties $\mathrm{VAR}(\mathcal{P},\mathcal{P})$ and $\mathrm{CLO}(\mathcal{P},\mathcal{P})$ are going to impose a restriction on the arrow types for which the interpretation is in $\mathcal{P}$ (the realisers of arrow types), as we can see in the arrow type case of the proof of the following lemma 13.4 and in the arrow type case of the proof of the following lemma 14. As shown at the end of this section, even if the obtained result when considering these restrictions is different from (in some sens, is an improvement of) the one given by Ghilezan and Likavec using the type system $\lambda\cap^1$, we do not succeed in salvaging their method.

The use of the non-trivial types (we recall the definition below) introduced by Gallier [4] are not much of a help in this case, because of the precise restriction imposed by $\mathrm{VAR}(\mathcal{P},\mathcal{P})$. One might also want to consider the sets of properties (we do not recall them in this paper for lack of space) stated in his work [4], but which are unfortunately not easy to prove for CR, because a proof of $xM \in \mathsf{CR}$ for all $M \in \Lambda$ is required. Moreover, if one succeeds in proving that the variables are included in the interpretation of a defined set of types containing $\Omega \to \alpha$, where Ω is interpreted as Λ and α as $\mathcal{P}$, then one has proved that $xM \in \mathcal{P}$, so that in the case $\mathcal{P} = \mathsf{CR}$, $M \in \mathsf{CR}$.

It is worth pointing out that a part of the work done by Gallier [4] would still be valid if adapted to the type system $\lambda\cap^2$. Gallier defines the non-trivial types as follows:

$$\psi \in \mathsf{NonTrivial} ::= \alpha \mid \tau \to \psi \mid \tau \cap \psi \mid \psi \cap \tau$$

Types in Type^2 are then interpreted as follows: $[\![\alpha]\!]_{\mathcal{P}} = \mathcal{P}$, $[\![\psi \cap \tau]\!]_{\mathcal{P}} = [\![\tau \cap \psi]\!]_{\mathcal{P}} = [\![\tau]\!]_{\mathcal{P}} \cap [\![\psi]\!]_{\mathcal{P}}$, $[\![\tau]\!]_{\mathcal{P}} = \Lambda$ if $\tau \notin \mathsf{NonTrivial}$ and $[\![\tau \to \psi]\!]_{\mathcal{P}} = \{M \in \mathcal{P} \mid \forall N \in [\![\tau]\!]_{\mathcal{P}}.\ MN \in [\![\psi]\!]_{\mathcal{P}}\}$. We can easily prove that if $\tau_1 \leq^2 \tau_2$ then $[\![\tau_1]\!]_{\mathcal{P}} \subseteq [\![\tau_2]\!]_{\mathcal{P}}$. Hence, considering the type system $\lambda\cap^2$ instead of the type system $D\Omega$, the method of Gallier gets a set of predicates which when satisfied by a set of terms $\mathcal{P}$ implies that the set of terms typable in the system $\lambda\cap^2$ by a non-trivial type is a subset of $\mathcal{P}$. Gallier proved that the set of head-normalising λ-terms satisfies each of the given predicates.

Using a method similar that the Ghilezan and Likavec's method, Gallier proved also that the set of weakly head-normalising terms (W) is equal to the set of terms typable by a weakly non-trivial types in the type system $D\Omega$. The set of weakly non-trivial types is defined as follows:

$$\psi \in \mathsf{WeaklyNonTrivial} ::= \alpha \mid \tau \to \psi \mid \Omega \to \Omega \mid \tau \cap \psi \mid \psi \cap \tau$$

As explain above, when trying to salvage the Ghilezan and Likavec's method we have to restrict the set of realisers when defining the interpretation of the set of types Type^2. The different restrictions lead us to the definition of Type^3.

Definition 11 $\rho \in \mathsf{Type}^3 ::= \alpha \mid \rho_1 \to \rho_2 \mid \rho \cap \tau \mid \tau \cap \rho$.

- $[\![\alpha]\!]^3_{\mathcal{P}} = \mathcal{P}$.
- $[\![\tau_1 \cap \tau_2]\!]^3_{\mathcal{P}} = [\![\tau_1]\!]^3_{\mathcal{P}} \cap [\![\tau_2]\!]^3_{\mathcal{P}}$, if $\tau_1 \cap \tau_2 \in \mathsf{Type}^3$.
- $[\![\tau]\!]^3_{\mathcal{P}} = \Lambda$, if $\tau \notin \mathsf{Type}^3$.
- $[\![\tau_1 \to \tau_2]\!]^3_{\mathcal{P}} = \{M \in \mathcal{P} \mid \forall N \in [\![\tau_1]\!]^3_{\mathcal{P}}.\ MN \in [\![\tau_2]\!]^3_{\mathcal{P}}\}$, if $\tau_1 \to \tau_2 \in \mathsf{Type}^3$.

Because of the defined semantics, we have to consider an additional restriction. As a matter of fact, to prove the soundness lemma 5.2, when considering the case of the rule ($\leq^2$) we need the following result: if $\tau_1 \leq^2 \tau_2$ then $[\![\tau_1]\!]^2_{\mathcal{P}} \subseteq [\![\tau_2]\!]^2_{\mathcal{P}}$. So when proving the soundness result w.r.t. the new semantics, we should have to prove: if $\tau_1 \leq^2 \tau_2$ then $[\![\tau_1]\!]^3_{\mathcal{P}} \subseteq [\![\tau_2]\!]^3_{\mathcal{P}}$. However, by rule (8), $\Omega \to \alpha \leq \alpha \to \alpha$, but $[\![\Omega \to \alpha]\!]^3_{\mathcal{P}} = \Lambda$ and $[\![\alpha \to \alpha]\!]^3_{\mathcal{P}} \subseteq \mathcal{P}$. Hence we define a new type system where we restrict the type system $\lambda\cap^2$ by getting ride of the rule (8).

Definition 12 Let $\leq^3$ be the preorder $\leq^2$ without the rule (8). Let $\vdash^3$ be the relation $\vdash^2$ where $\leq^2$ is replaced by $\leq^3$. Let $\lambda\cap^3$ be the type system $\lambda\cap^2$ where $\vdash^2$ is replaced by $\vdash^3$ and $\leq^2$ is replaced by $\leq^3$. Let $\models^3$ be the relation $\models^2$ where $[\![\tau]\!]^2_{\mathcal{P}}$ is replaced by $[\![\tau]\!]^3_{\mathcal{P}}$.

Lemma 13

1. $[\![\tau_1 \cap \tau_2]\!]^3_{\mathcal{P}} = [\![\tau_1]\!]^3_{\mathcal{P}} \cap [\![\tau_2]\!]^3_{\mathcal{P}}$.
2. $[\![\rho]\!]^3_{\mathcal{P}} \subseteq \mathcal{P}$.
3. *If* $\tau_1 \leq^3 \tau_2$ *then* $[\![\tau_1]\!]^3_{\mathcal{P}} \subseteq [\![\tau_2]\!]^3_{\mathcal{P}}$.
4. *If* $\mathrm{VAR}(\mathcal{P}, \mathcal{P})$ *then for all* $\tau \in \mathsf{Type}^2$, $\mathrm{VAR}(\mathcal{P}, [\![\tau]\!]^3_{\mathcal{P}})$.
5. *If* $\mathrm{SAT}(\mathcal{P}, \mathcal{P})$ *then for all* $\tau \in \mathsf{Type}^2$, $\mathrm{SAT}(\mathcal{P}, [\![\tau]\!]^3_{\mathcal{P}})$.

We now prove the new soundness lemma:

Lemma 14 *If* $\mathrm{VAR}(\mathcal{P}, \mathcal{P})$, $\mathrm{SAT}(\mathcal{P}, \mathcal{P})$, $\mathrm{CLO}(\mathcal{P}, \mathcal{P})$ *and* $\Gamma \vdash^3 M : \tau$ *then* $\Gamma \models^3 M : \tau$

Proposition 15 *If* $\Gamma \vdash^3 M : \rho$ *then* $M \in \mathsf{CR}$, $M \in \mathsf{S}$, *and* $M \in \mathsf{W}$.

The difference with the first result obtained by Ghilezan and Likavec [5] using the type system $\lambda\cap^1$ (beside the fact that the predicates are different. See definition 4.4 and lemma 5.1 above) is that now we are able to prove that even some non strongly normalisable terms belong to the sets CR, S, and W as the following example shows: by (Ω) and (ax) we get $x : \alpha \vdash^3 \circledast\circledast : \Omega$ and $x : \alpha, y : \Omega \vdash^3 x : \alpha$. By $(\to_I)$ we get $x : \alpha \vdash^3 \lambda y.x : \Omega \to \alpha$ and by $\to_E$ we get $x : \alpha \vdash^3 (\lambda y.x)(\circledast\circledast) : \alpha$. Moreover, we conjecture that all the strongly normalisable terms are typable in the type system $\lambda\cap^3$ with a type in Type^3. However, we did not salvage the Ghilezan and Likavec's method because some terms of the untyped λ-calculus are not typable in the type system $\lambda\cap^3$ by a type in Type^3.

5 Adapting the CR proof of Koletsos and Stavrinos [9] to βI-reduction

Koletsos and Stavrinos [9] gave a proof of Church-Rosser for β-reduction for system D given in Definition 3 and showed that this can be used to show confluence of β-developments without using strong normalisation. In this section, we adapt this proof to βI and set the formal ground for generalising the Koletsos and Stavrinos's method for $\beta\eta$ in the next section. After giving the definition of βI-developments, we will introduce the type interpretation which will be used to establish Church-Rosser of both systems D and D_I (for $\beta\eta$- resp. βI-reduction).

The next definition, taken from Krivine [10] (and used by Koletsos and Stavrinos [9]) uses the variable c to destroy the βI-redexes of M which are not in the set $\mathcal{F}$ of βI-redex occurrences in M, and to neutralise applications so that they cannot be transformed into redexes after βI-reduction.

Definition 16 ($\Phi^c(-,-)$) Let $M \in \Lambda\mathrm{I}$, $c \notin \mathrm{fv}(M)$, and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta I}$.

- If $M = x$ then $\mathcal{F} = \varnothing$ and $\Phi^c(x, \mathcal{F}) = x$
- If $M = \lambda x.N$ and $x \neq c$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta I}$ then $\Phi^c(\lambda x.N, \mathcal{F}) = \lambda x.\Phi^c(N, \mathcal{F}')$
- If $M = NP$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta I}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}_P^{\beta I}$ then
  $\Phi^c(NP, \mathcal{F}) = \begin{cases} c\Phi^c(N, \mathcal{F}_1)\Phi^c(P, \mathcal{F}_2) & \text{if } 0 \notin \mathcal{F} \\ \Phi^c(N, \mathcal{F}_1)\Phi^c(P, \mathcal{F}_2) & \text{otherwise} \end{cases}$

Lemma 17 Let $M \in \Lambda I$, such that $c \notin \mathrm{fv}(M)$, $\mathcal{F} \subseteq \mathcal{R}_M^{\beta I}$, $p \in \mathcal{F}$ and $M \xrightarrow{p}_{\beta I} M'$. Then, there exists a unique $\mathcal{F}' \subseteq \mathcal{R}_{M'}^{\beta I}$, such that $P = \Phi^c(M, \mathcal{F}) \xrightarrow{p'}_{\beta I} \Phi^c(M', \mathcal{F}')$ and $|\langle P, p'\rangle|^c = p$.

We define the set of βI-residuals of a set $\mathcal{F}$ of βI-redexes relative to a sequence of βI-redexes.

Definition 18 Let $M \in \Lambda\mathrm{I}$, such that $c \notin \mathrm{fv}(M)$ and let $\mathcal{F} \subseteq \mathcal{R}_M^{\beta I}$.

- Let $p \in \mathcal{F}$ and $M \xrightarrow{p}_{\beta I} M'$. By lemma 17, there exists a unique $\mathcal{F}' \subseteq \mathcal{R}_{M'}^{\beta I}$ such that $P = \Phi^c(M, \mathcal{F}) \xrightarrow{p'}_{\beta I} \Phi^c(M', \mathcal{F}')$ and $|\langle P, p'\rangle|^c = p$. We call $\mathcal{F}'$ the set of **βI-residuals of $\mathcal{F}$ in M' relative to** p.
- A one-step βI-development of $(M, \mathcal{F})$, denoted $(M, \mathcal{F}) \to_{\beta Id} (M', \mathcal{F}')$, is a βI-reduction $M \xrightarrow{p}_{\beta I} M'$ where $p \in \mathcal{F}$ and $\mathcal{F}'$ is the set of βI-residuals of $\mathcal{F}$ in M' relative to p. A **βI-development** is the transitive closure of a one-step βI-development. We write also $M \xrightarrow{\mathcal{F}}_{\beta Id} M_n$ for the βI-development $(M, \mathcal{F}) \to_{\beta Id}^* (M_n, \mathcal{F}_n)$.

Definition 19 1. Let $r \in \{\beta I, \beta\eta\}$. We define $[\![-]\!]^r : \mathsf{Type}^1 \to 2^\Lambda$ by:
- $[\![\alpha]\!]^r = \mathsf{CR}^r$ • $[\![\sigma_1 \cap \sigma_2]\!]^r = [\![\sigma_1]\!]^r \cap [\![\sigma_2]\!]^r$
- $[\![\sigma_1 \to \sigma_2]\!]^r = \{t \in \mathsf{CR}^r \mid \forall u \in [\![\sigma_1]\!]^r.\ tu \in [\![\sigma_2]\!]^r\}$.

2. $\mathcal{X} \subseteq \Lambda$ is saturated iff $\forall n \in \mathbb{N}.\ \forall M, N, M_1, \ldots, M_n \in \Lambda.\ \forall x \in \mathcal{V}$.
   $M[x := N]M_1 \ldots M_n \in \mathcal{X} \Rightarrow (\lambda x.M)NM_1 \ldots M_n \in \mathcal{X}$

$\mathcal{X} \subseteq \Lambda\mathrm{I}$ is I-saturated iff $\forall n \in \mathbb{N}. \forall M, N, M_1, \ldots, M_n \in \Lambda. \forall x \in \mathcal{V}.$
$x \in \mathrm{fv}(M) \Rightarrow M[x := N]M_1 \ldots M_n \in \mathcal{X} \Rightarrow (\lambda x.M)NM_1 \ldots M_n \in \mathcal{X}$

It turns out that if $\sigma \in \mathsf{Type}^1$ then $[\![\sigma]\!]^r$ is saturated and only contains Church-Rosser terms. Krivine [10] gave a proof for β-SN. Koletsos ans Stavrinos [9] adapted Krivine's proof for β-Church-Rosser. In this section we adapt the Koletsos and Stavrinos's method [9] for $\beta\eta$-Church-Rosser. First, we adapt the Krivine's soundness lemma to both $\vdash^{\beta I}$ and $\vdash^{\beta\eta}$.

Lemma 20 *Let* $r \in \{\beta I, \beta\eta\}$. *If* $x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash^r M : \sigma$ *and* $\forall i \in \{1, \ldots, n\}. N_i \in [\![\sigma_i]\!]^r$ *then* $M[(x_i := N_i)_1^n] \in [\![\sigma]\!]^r$.

Next, we adapt a corollary given by Koletsos and Stavrinos to show that if $\Gamma \vdash^r M : \sigma$ then $M \in \mathsf{CR}^r$, for $r \in \{\beta I, \beta\eta\}$. To treat βI- and $\beta\eta$-reduction, we generalise next a lemma given by Krivine [10] (and used by Koletsos and Stavrinos [9]) which states that if $M \in \Lambda\mathrm{I}_c$ (resp. $\Lambda\eta_c$) then M is typable in D (resp. D_I) and hence $M \in \mathsf{CR}^{\beta I}$ (resp. $M \in \mathsf{CR}^{\beta\eta}$).

Lemma 21 *Let* $c \notin \mathrm{dom}(\Gamma) \supseteq \mathrm{fv}(M) \setminus \{c\} = \{x_1, \ldots, x_n\}$.

1. *If* $M \in \Lambda I_c$ *and* $\Gamma' = \Gamma \upharpoonright \mathrm{fv}(M)$, *then there exist* $\sigma_1, \sigma_2 \in \mathsf{Type}^1$ *such that if* $c \in \mathrm{fv}(M)$ *then* $\Gamma', c : \sigma_1 \vdash^{\beta I} M : \sigma_2$, *else* $\Gamma' \vdash^{\beta I} M : \sigma_2$.
2. *If* $M \in \Lambda\eta_c$ *then there exist* $\sigma_1, \sigma_2 \in \mathsf{Type}^1$ *such that* $\Gamma, c : \sigma_1 \vdash^{\beta\eta} M : \sigma_2$.

The next lemma adapts the main Koletsos and Stavrinos's theorem [9] where as far as we know it first appeared.

Lemma 22 (CONFLUENCE OF THE βI-DEVELOPMENTS) *Let* $M \in \Lambda I$, *such that* $c \notin \mathrm{fv}(M)$. *If* $M \overset{\mathcal{F}_1}{\to}_{\beta Id} M_1$ *and* $M \overset{\mathcal{F}_2}{\to}_{\beta Id} M_2$, *then there exist* $\mathcal{F}_1' \subseteq \mathcal{R}_{M_1}^{\beta I}$, $\mathcal{F}_2' \subseteq \mathcal{R}_{M_2}^{\beta I}$ *and* $M_3 \in \Lambda I$ *such that* $M_1 \overset{\mathcal{F}_1'}{\to}_{\beta Id} M_3$ *and* $M_2 \overset{\mathcal{F}_2'}{\to}_{\beta Id} M_3$.

By the notation: $M \to_{1I} M' \iff \exists \mathcal{F}, \mathcal{F}'. (M, \mathcal{F}) \to_{\beta Id}^* (M', \mathcal{F}')$ where $M, M' \in \Lambda\mathrm{I}$, such that $c \notin \mathrm{fv}(M)$, the transitive reflexive closure of $\to_{\beta I}$ is equal to the transitive reflexive closure of $\to_{1I}$. We are now able to prove the inclusion of $\Lambda\mathrm{I}$ in $\mathsf{CR}^{\beta I}$ and so the equality between these two sets.

Lemma 23 *If* $M \in \Lambda I$ *such that* $c \notin \mathrm{fv}(M)$ *then* $M \in \mathsf{CR}^{\beta I}$.

6 Generalisation of the method to $\beta\eta$-reduction

In this section we generalise the method of section 5 to $\beta\eta$-reduction. This generalisation is not trivial since when studying developments involving η-reduction we need closure under η-reduction of a defined set of frozen terms. For example, let $M = \lambda x.cNx \in \Lambda\mathrm{I}_c$ where $x \notin \mathrm{fv}(N)$ and $N \in \Lambda\mathrm{I}_c$, then $M \to_\eta cN \notin \Lambda\mathrm{I}_c$. For such reasons, we extended $\Lambda\mathrm{I}_c$ to $\Lambda\eta_c$. In this section, many of the notions used to prove Church-Rosser of βI-reduction will be extended to deal with $\beta\eta$-reduction.

A full common definition of a $\beta\eta$-residual is given by Curry and Feys [2] (p. 117, 118). Another definition of $\beta\eta$-residual (called λ-residual) is presented by Klop [7] (definition 2.4, p. 254). Klop [7] shows that both definitions enable to prove different properties of developments. Following the definition of a $\beta\eta$-residual given by Curry and Feys [2] (and as pointed out in [2, 7, 1]), if the η-redex $\lambda x.(\lambda y.M)x$, where $x \notin \mathrm{fv}(\lambda y.M)$, is reduced in the term $P = (\lambda x.(\lambda y.M)x)N$ to give the term $Q = (\lambda y.M)N$, then Q is not a $\beta\eta$-residual of P in P (note that following the definition of a λ-residual given by Klop [7], Q is a λ-residual of the redex $(\lambda y.M)x$ in P since the λ of the redex Q is the same than the λ of the redex $(\lambda y.M)x$ in P). Moreover, if the β-redex $(\lambda y.My)x$, where $y \notin \mathrm{fv}(M)$, is reduced in the term $P = \lambda x.(\lambda y.My)x$ to give the term $Q = \lambda x.Mx$, then Q is not a $\beta\eta$-residual of P

in P (note that following the definition of a λ-residual given by Klop [7], Q is a λ-residual of the redex P in P since the λ of the redex Q is the same than the λ of the redex P in P). Our definition 26 differs from the common one stated by Curry and Feys [2] by these cases as we illustrate in the following example: $\Psi^c((\lambda x.(\lambda y.M)x)N, \{1, 1.0, 1.1.0\}) = \{c^n((\lambda x.(\lambda y.P[y := c(cy)])x)Q) \mid n \geq 0 \wedge P \in \Psi^c(M, \varnothing) \wedge Q \in \Psi^c(N, \varnothing)\}$, where $x \notin \mathrm{fv}(\lambda y.M)$. Let $p = 1.0$ then $(\lambda x.(\lambda y.M)x)N \xrightarrow{p}_{\beta\eta} (\lambda y.M)N$. Moreover, $P_0 = c^n((\lambda x.(\lambda y.P[y := c(cy)])x)Q) \xrightarrow{p'}_{\beta\eta} c^n((\lambda y.P[y := c(cy)])Q)$ such that $n \geq 0$, $P \in \Psi^c(M, \varnothing)$, $Q \in \Psi^c(N, \varnothing)$, $|\langle P_0, p'\rangle|^c = |\langle P_0, 2^n.1.0\rangle|^c = p$ (using a lemma stated and proved in the long version of this article) and $c^n((\lambda y.P[y := c(cy)])Q) \in \Psi^c((\lambda y.M)N, \{0\})$.

The next definition adapts definition 16 to deal with $\beta\eta$-reduction.

Definition 24 $(\Psi^c(-,-), \Psi^c_0(-,-))$ Let $c \notin \mathrm{fv}(M)$ and $\mathcal{F} \subseteq \mathcal{R}^{\beta\eta}_M$.

(P1) If $M \in \mathcal{V} \setminus \{c\}$ then $\mathcal{F} = \varnothing$ and
$\Psi^c(M, \mathcal{F}) = \{c^n(M) \mid n > 0\}$ $\qquad \Psi^c_0(M, \mathcal{F}) = \{M\}$

(P2) If $M = \lambda x.N$ and $x \neq c$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_N$ then:
$$\Psi^c(M, \mathcal{F}) = \begin{cases} \{c^n(\lambda x.P[x := c(cx)]) \mid n \geq 0 \wedge P \in \Psi^c(N, \mathcal{F}')\} & \text{if } 0 \notin \mathcal{F} \\ \{c^n(\lambda x.N') \mid n \geq 0 \wedge N' \in \Psi^c_0(N, \mathcal{F}')\} & \text{otherwise} \end{cases}$$
$$\Psi^c_0(M, \mathcal{F}) = \begin{cases} \{\lambda x.N'[x := c(cx)] \mid N' \in \Psi^c(N, \mathcal{F}')\} & \text{if } 0 \notin \mathcal{F} \\ \{\lambda x.N' \mid N' \in \Psi^c_0(N, \mathcal{F}')\} & \text{otherwise} \end{cases}$$

(P3) If $M = NP$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_N$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_P$ then:
$$\Psi^c(M, \mathcal{F}) = \begin{cases} \{c^n(cN'P') \mid n \geq 0 \wedge N' \in \Psi^c(N, \mathcal{F}_1) \wedge P' \in \Psi^c(P, \mathcal{F}_2)\} & \text{if } 0 \notin \mathcal{F} \\ \{c^n(N'P') \mid n \geq 0 \wedge N' \in \Psi^c_0(N, \mathcal{F}_1) \wedge P' \in \Psi^c(P, \mathcal{F}_2)\} & \text{otherwise} \end{cases}$$
$$\Psi^c_0(M, \mathcal{F}) = \begin{cases} \{cN'P' \mid N' \in \Psi^c(N, \mathcal{F}_1) \wedge P' \in \Psi^c_0(P, \mathcal{F}_2)\} & \text{if } 0 \notin \mathcal{F} \\ \{N'P' \mid N' \in \Psi^c_0(N, \mathcal{F}_1) \wedge P' \in \Psi^c_0(P, \mathcal{F}_2) & \text{otherwise} \end{cases}$$

Lemma 25 *Let $M \in \Lambda$, such that $c \notin \mathrm{fv}(M)$, $\mathcal{F} \subseteq \mathcal{R}^{\beta\eta}_M$, $p \in \mathcal{F}$ and $M \xrightarrow{p}_{\beta\eta} M'$. Then, there exists a unique $\mathcal{F}' \subseteq \mathcal{R}^{\beta\eta}_{M'}$ such that for all $N \in \Psi^c(M, \mathcal{F})$, there exists $N' \in \Psi^c(M', \mathcal{F}')$ and $p' \in \mathcal{R}^{\beta\eta}_N$ such that $N \xrightarrow{p'}_{\beta\eta} N'$ and $|\langle N, p'\rangle|^c = p$.*

Definition 26 Let $M \in \Lambda$ and $\mathcal{F} \subseteq \mathcal{R}^{\beta\eta}_M$.

- Let $p \in \mathcal{F}$ and $M \xrightarrow{p}_{\beta\eta} M'$. By lemma 25, there exists a unique $\mathcal{F}' \subseteq \mathcal{R}^{\beta\eta}_{M'}$, such that for all $N \in \Psi^c(M, \mathcal{F})$ there exist $N' \in \Psi^c(M', \mathcal{F}')$ and $p' \in \mathcal{R}^{\beta\eta}_N$ such that $N \xrightarrow{p'}_{\beta\eta} N'$ and $|\langle N, p'\rangle|^c = p$. We call $\mathcal{F}'$ the set of **$\beta\eta$-residuals of $\mathcal{F}$ in M' relative to** p.
- Let $c \notin \mathrm{fv}(M)$. A one-step $\beta\eta$-development of $(M, \mathcal{F})$, denoted $(M, \mathcal{F}) \rightarrow_{\beta\eta d} (M', \mathcal{F}')$, is a $\beta\eta$-reduction $M \xrightarrow{p}_{\beta\eta} M'$ where $p \in \mathcal{F}$ and $\mathcal{F}'$ is the set of $\beta\eta$-residuals of $\mathcal{F}$ in M' relative to p. A **$\beta\eta$-development** is the transitive closure of a one-step $\beta\eta$-development. We write also $M \xrightarrow{\mathcal{F}}_{\beta\eta d} M'$ for the $\beta\eta$-development $(M, \mathcal{F}) \rightarrow^*_{\beta\eta d} (M', \mathcal{F}')$.

Lemma 27 (CONFLUENCE OF THE $\beta\eta$-DEVELOPMENTS) *Let $M, M_1, M_2 \in \Lambda$. If $M \xrightarrow{\mathcal{F}_1}_{\beta\eta d} M_1$ and $M \xrightarrow{\mathcal{F}_2}_{\beta\eta d} M_2$, then there exist sets $\mathcal{F}'_1 \subseteq \mathcal{R}^{\beta\eta}_{M_1}$ and $\mathcal{F}'_2 \subseteq \mathcal{R}^{\beta\eta}_{M_2}$ and a term $M_3 \in \Lambda$ such that $M_1 \xrightarrow{\mathcal{F}'_1}_{\beta\eta d} M_3$ and $M_2 \xrightarrow{\mathcal{F}'_2}_{\beta\eta d} M_3$.*

By the notation: $M \rightarrow_1 M' \iff \exists \mathcal{F}, \mathcal{F}'. (M, \mathcal{F}) \rightarrow^*_{\beta\eta d} (M', \mathcal{F}')$, the transitive reflexive closure of $\rightarrow_{\beta\eta}$ is equal to the transitive reflexive closure of $\rightarrow_1$. We are now able to prove the (non-strict) inclusion of Λ in $\mathsf{CR}^{\beta\eta}$ and the equality between these sets.

Lemma 28 *If $M \in \Lambda$ such that $c \notin \mathrm{fv}(M)$ then $M \in \mathsf{CR}^{\beta\eta}$.*

7 Conclusion/comparison

Reducibility is a powerful method and has been applied to prove using a single method, a number of properties of the λ-calculus (Church-Rosser, strong normalisation, etc.). This paper studied two reducibility methods which exploit the passage from typed (in an intersection type system) to untyped terms. We showed that a first method given by Ghilezan and Likavec [5] fails in its aim and we have only been able to provide a partial solution. We adapted a second method given by Koletsos and Stavrinos [9] from β to βI-reduction and we generalised it to $\beta\eta$-reduction. There are differences in the type systems chosen and the methods of reducibility used by Ghilezan and Likavec on one side and by Koletsos and Stavrinos on the other. Koletsos and Stavrinos use system D [10], which has elimination rules for intersection types whereas Ghilezan and Likavec use $\lambda\cap$ and $\lambda\cap^{\Omega}$ with subtyping. Moreover, the Koletsos and Stavrinos's method depends on the inclusion of typable λ-terms in the set of λ-terms possessing the Church-Rosser property, whereas the Ghilezan and Likavec's method (the working part of their method) is to prove the inclusion of typable terms in an arbitrary subset of the untyped λ-calculus closed by some properties. Moreover, Ghilezan and Likavec consider the $\mathrm{VAR}(\mathcal{P})$, $\mathrm{SAT}(\mathcal{P})$ and $\mathrm{CLO}(\mathcal{P})$ predicates whereas Koletsos and Stavrinos use standard reducibility methods through saturated sets. Koletsos and Stavrinos prove the confluence of developments using the confluence of typable λ-terms in system D (the authors prove that even a simple type system is sufficient). The advantage of the Koletsos and Stavrinos's proof of confluence of developments is that strong normalisation is not needed.

References

[1] H. Barendregt, J. A. Bergstra, J. W. Klop, H. Volken. Degrees, reductions and representability in the lambda calculus. Technical Report Preprint no. 22, University of Utrecht, Department of Mathematics, 1976.

[2] H. B. Curry, R. Feys. Combinatory logic, vol. 1. 1958.

[3] J. Gallier. On the correspondance between proofs and λ-terms. *Cahiers du centre de logique*, 1997. Available at `http://www.cis.upenn.edu/~jean/gbooks/logic.html` (last visited 2008–02–6).

[4] J. Gallier. Typing untyped λ-terms, or reducibility strikes again!. *Annals of Pure and Applied Logic*, 91, 2003.

[5] S. Ghilezan, S. Likavec. Reducibility: A ubiquitous method in lambda calculus with intersection types. *Electr. Notes Theor. Comput. Sci.*, 70(1), 2002.

[6] S. C. Kleene. On the interpretation of intuitionistic number theory. *The Journal of Symbolic Logic*, 10(4), 1945.

[7] J. W. Klop. *Combinatory Reductions Systems.* PhD thesis, Mathematisch Centrum, Amsterdam, 1980.

[8] G. Koletsos. Church-Rosser theorem for typed functional systems. *Journal of Symbolic Logic*, 50(3), 1985.

[9] G. Koletsos, G. Stavrinos. Church-Rosser property and intersection types. *Australian Journal of Logic*, 2008. To appear.

[10] J. L. Krivine. *Lambda-calcul, types et modeles.* Dunod, 1990.

[11] W. W. Tait. Intensional interpretations of functionals of finite type I. *J. Symb. Log.*, 32(2), 1967.