On constructivity of the notion of formal space

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Abstract of our talk

- why formal spaces in MF

- notion of constructive/strong constructive foundation

- strong constructivity of the Minimalist Foundation (MF)

- construction of predicative versions of Hyland’s Effective Topos

- Open problems
**Fundamental issue**

What is a **space**?
What is a space?

in Sambin's talk:

the answer depends

from the underlying conception of mathematics

⇒ it depends from the chosen foundation
Key issue

the notion of Formal Topology/Positive Topology

is a notion of space

in the Minimalist Foundation (for short MF)

with NO known alternatives....
Why we need formal spaces in MF

contrary to most know constructive foundations: Aczel's CZF, Martin-Löf’s type theory,...

the Minimalist Foundation

is a foundation for constructive mathematics

compatible with Weyl’s classical predicative mathematics

⇒ in MF the Continuum must be represented in a pointfree way
Characteristics of *predicative definitions*

in the sense of *Russell-Poincarè*

Whatever involves an apparent variable
must *not be among the possible values* of that variable.
Necessity of a base to describe a point-free topology (=locale) predicatively!

even in strong constructive predicative theories like Aczel’s CZF (+REA)
based on work by Moerdijk-van den Berg-Rathjen and Curi

Theorem:
No complete suplattice is a set
(unless it is the trivial one!)
in Aczel’s CZF (and hence in MF)

reason:
consistency with variations of Troelstra’s Uniformity Principle

∀x ∈ ℙ(1) ∃yεa R(x, y) → ∃yεa ∀x ∈ ℙ(1) R(x, y)
need of two size entities: \textit{collections/sets} to represent a locale as a \textit{collection} closed under suprema \textit{indexed} on a \textit{set}

alternatively:

work with \textit{set of generators} + \textit{relations}

as in Vicker's development
Fundamental issues

What is constructive mathematics?
From Bishop’s “Mathematics as a numerical language”

[Constructive]

“Mathematics describes and predicts the results of certain finitely. computations within the set of integers”
Essence of *Constructive mathematics*

= maths which admits a COMPUTATIONAL intepretation

Constructive Mathematics is a bridge

abstract maths  \(\rightarrow\) bridge  \(\rightarrow\) COMPUTATIONAL maths
Why developing constructive mathematics?

to EXTRACT the computational contents
i.e the meaning of abstract mathematics
in Bishop’s words
**what is constructive mathematics?**

\[
\text{CONSTRUCTIVE mathematics} = \text{IMPLICIT COMPUTATIONAL mathematics}
\]

\[\downarrow\]

constructive mathematician is an *implicit programmer*!!

---

CONSTRUCTIVE proofs

= 

SOME programs
**Fundamental issues**

What is a *constructive* foundation?
Need of a \textit{two level constructive Foundation} (j.w.w. G. Sambin)

A Constructive Foundation should bridge

\begin{itemize}
  \item \textsc{Language} of abstract maths (of commonly used \textit{extensional} set theory) to communicate with people
  \item \textsc{Programming Language} (of \textit{intensional} type theory) to extract programs from constructive proofs to communicate with machines
\end{itemize}
From Bishop’s “Schizophrenia in contemporary mathematics”

informal mathematics must be written in the appropriate language for communicating with people,

formal mathematics must be written in the appropriate language for communicating with machines.
Use an interactive theorem prover...
in order to *speak to machines* as in Bishop’s view....

+ 

in order to *check correctness of mathematical proofs* as strongly advocated by V. Voevodsky

better to **built**

an **interactive theorem prover**
on an **intensional type theory**
like done with proof-assistants:

**Coq, Agda, Matita**
Problem: how to model *extensional concepts* in an *intensional theory*?
**What foundation** for COMPUTER-AIDED formalization of *proofs*?

(j.w.w. G. Sambin)

A constructive foundation should be equipped with

<table>
<thead>
<tr>
<th>Foundation</th>
<th>extensional level</th>
<th>intensional level</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(used by mathematicians to do their proofs)</td>
<td>(language of computer-aided formalized proofs)</td>
</tr>
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</table>

\[ \downarrow \text{interpreted via a QUOTIENT model} \]

\[ \downarrow \text{realizability level (used by computer scientists to extract programs)} \]
our FOUNDATION = ONLY the first TWO LEVELS
linked by a quotient completion
where

our extensional sets = quotients of intensional sets
only implicitly
being formalized in an abstract extensional language
as the usual one of common practice!
may LEVELS in our notion of constructive foundation collapse?

YES, for example in the following two-level foundation

\[
\text{Aczel's CZF (usual math language)} \\
\downarrow \quad \text{(GLOBALLY interpreted in)} \\
\text{Martin-Löf's type theory MLTT}
\]

which serves as the intensional and the realizability levels
may LEVELS in our notion of constructive foundation collapse?

Can all levels be modelled within a single theory?

what about

MLTT + Univalence axiom ??
### our notion of FOUNDATION combines different languages

<table>
<thead>
<tr>
<th>Language of Theory</th>
<th>Level</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>(local)</em> AXIOMATIC SET THEORY</td>
<td>Extensional</td>
</tr>
<tr>
<td>CATEGORY THEORY</td>
<td>Algebraic structure to link intensional/extensional levels via a quotient completion</td>
</tr>
<tr>
<td>TYPE THEORY</td>
<td>Intensional</td>
</tr>
</tbody>
</table>
Our use of **category theory**

to express the abstract link between *extensional/intensional* levels:

- **use**
  - notion of **ELEMENTARY QUOTIENT COMPLETION**
    - \( Q(P) \)
  - (in the language of **CATEGORY THEORY**)

*relative to a suitable Lawvere’s elementary doctrine* \( P \)

in:


### About the plurality of foundations of mathematics

<table>
<thead>
<tr>
<th>Classical mathematics</th>
<th>Constructive mathematics</th>
</tr>
</thead>
<tbody>
<tr>
<td>One <em>standard</em> impredicative foundation</td>
<td>No standard foundation</td>
</tr>
<tr>
<td>ZFC axiomatic set theory</td>
<td>But different incomparable foundations</td>
</tr>
</tbody>
</table>
**Plurality of foundations** ⇒ **need of a minimalist foundation**

<table>
<thead>
<tr>
<th>Classical</th>
<th>Constructive</th>
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</thead>
<tbody>
<tr>
<td>ONE standard</td>
<td>NO standard</td>
</tr>
<tr>
<td>Impredicative</td>
<td>Zermelo-Fraenkel set theory</td>
</tr>
<tr>
<td></td>
<td>internal theory of topoi</td>
</tr>
<tr>
<td></td>
<td>Coquand's Calculus of Constructions</td>
</tr>
<tr>
<td>Predicative</td>
<td>Feferman's explicit maths</td>
</tr>
<tr>
<td></td>
<td>Aczel's CZF</td>
</tr>
<tr>
<td></td>
<td>Martin-Löf's type theory</td>
</tr>
<tr>
<td></td>
<td>HoTT and Voevodsky's Univalent Foundations</td>
</tr>
<tr>
<td></td>
<td>Feferman's constructive expl. maths</td>
</tr>
</tbody>
</table>

what common core ??
Our **TWO-LEVEL Minimalist Foundation**

from [Maietti’09] according to requirements in [M.E.M, G. Sambin05]

- its intensional level
  - = a **PREDICATIVE VERSION** of Coquand’s Calculus of Constructions (Coq).
  - = a **FRAGMENT** of Martin-Löf’s intensional type theory + one UNIVERSE

- its extensional level
  - has a **PREDICATIVE LOCAL** set theory
  - *(NO choice principles)*
ENTITIES in the Minimalist Foundation

- small propositions
- propositions
- sets
- collections
Why we need to have both classes/collections and sets

in MF and in Aczel’s CZF

Constructive predicative notion of Locale

= Formal Topology by P. Martin-Löf and G. Sambin

represented by the fixpoints of a closure operator

on a base of opens \( B \) assumed to be a preorder set:

\[
\mathcal{A}_\triangleleft : \mathcal{P}(B) \longrightarrow \mathcal{P}(B)
\]

\[
U \quad \mapsto \quad \{ x \in B \mid x \triangleleft U \}
\]

satisfying a convergence property:

\[
\mathcal{A}_\triangleleft(U \downarrow V) = \mathcal{A}_\triangleleft(U) \cap \mathcal{A}_\triangleleft(V)
\]

\[
U \downarrow V \equiv \{ a \in B \mid \exists u \in U \quad a \leq u \quad \& \quad \exists v \in V \quad a \leq v \}
\]

NO restriction to inductively generated formal topologies
Why being *predicative*?

for a finer analysis of mathematical concepts and proofs

cfr. *H. Friedman's “Reverse mathematics”*
On the **intensional level** of MF

**Theorem:**

the *intensional level* of MF extended with the following resizing rule

\[
\begin{array}{c}
A \text{ proposition} \\
A \text{ small proposition}
\end{array}
\]

becomes equivalent to the *Coquand’s Calculus of Constructions* with *list types*. 
On the **intensional level** of MF

**Theorem:**

The *extensional level* of MF extended with the following resizing rule

\[
\begin{array}{c}
\text{A proposition} \\
\hline
\text{A small proposition}
\end{array}
\]

becomes equivalent to the generic internal language of *quasi-toposes* with a Natural Numbers Object.
What is the third level of MF?

an extension of Kleene realizability

as required in [M.E.M., G.Sambin05]

This Kleene realizability semantics for MF shows that MF is a strong constructive foundation
What is the role of the third level of a constructive foundation?

It provides a realizability model of the extensional level where to extract programs from constructive proofs of the extensional level i.e. satisfying:

- the choice rule (CR)

\[ \exists x \in A \phi(x) \text{ true under hypothesis } \Gamma \]

\[ \Downarrow \]

there exists a function calculating \( f(x) \) such that

\[ \phi(f(x)) \text{ true under hypothesis } x \in \Gamma \]

- “its functions represents computable functions”
Notion of **strong constructive foundation**

A \textit{two-level foundation} is a \textit{strong constructive foundation} iff

- its intensional level is consistent with
- the \textit{axiom of choice} (AC) + formal Church’s thesis (CT)

i.e. it is a \textit{proofs-as-programs theory}

as in [M. Sambin-2005]

\begin{itemize}
  \item \textbf{paradigmatic example:}
  \item Heyting arithmetics with finite types with Kleene realizability semantics
\end{itemize}
axiom of choice

\[(AC') \quad \forall x \in A \ \exists y \in B \ R(x, y) \quad \rightarrow \quad \exists f \in A \rightarrow B \ \forall x \in A \ R(x, f(x))\]

formal Church’s thesis

\[(CT') \quad \forall f \in \text{Nat} \rightarrow \text{Nat} \quad \exists e \in \text{Nat} \quad \text{(} \forall x \in \text{Nat} \ \exists y \in \text{Nat} \ T(e, x, y) \ & \ U(y) =_{\text{Nat}} f(x) \text{)}\]
NON examples of strongly constructive theories

NO classical theory

NO theory with extensionality of functions can be strongly constructive
NON examples of strongly constructive theories

A theory consistent with AC + CT

CAN NOT BE

- classical

\[
\text{Peano Arithmetics } + \text{ AC } + \text{ CT } \vdash \bot
\]

(because we can define characteristic functions of non-computable predicates)

- extensional even with intuitionistic logic

\[
\text{Intuitionistic arithmetics with finite types } + \text{ AC } + \text{ CT } + \text{ extfun } \vdash \bot
\]

\[
\text{extfun } = \text{ extensionality of functions}
\]
\[
\text{extfun} \quad \frac{f(x) =_B g(x) \quad true \quad [x \in A]}{\lambda x.f(x) =_{A \to B} \lambda x.g(x) \quad true}
\]

extensionality of functions
is Martin-Löf’s intensional type theory strongly constructive? i.e. consistent with formal Church’s thesis?

**key issue**: the presence of the so called $\xi$-rule for lambda terms.
### A realizability semantics for the extensional level

<table>
<thead>
<tr>
<th>$T_{iMF}$</th>
<th>$\rightarrow$</th>
<th>$T_{eff}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>predicative tripos</td>
<td>predicative realizability tripos</td>
<td>model to view iMF proofs-as-programs</td>
</tr>
</tbody>
</table>

\[ \downarrow \]

<table>
<thead>
<tr>
<th>extensional level eMF</th>
<th>effective model of eMF proofs</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \downarrow ) (interpreted)</td>
<td></td>
</tr>
<tr>
<td>( Q(T_{iMF}) )</td>
<td>( Q(T_{eff}) )</td>
</tr>
<tr>
<td>elementary quotient completion of ( T_{iMF} )</td>
<td>elementary quotient completion of ( T_{eff} )</td>
</tr>
</tbody>
</table>

| quotient model of iMF | predicative Hyland’s Eff |

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Crucial categorical tool

the exact completion of a lex category
is represented an elementary completion $Q(P)$
of an elementary Lawvere doctrine $P$

see
A **predicatively generalized elementary topos** is given by

- a finite limit category $C$;
- a **FULL sub-fibration** of the codomain fibration on $C$

$$\pi_S : S \rightarrow C$$

such that

where $i$ is an inclusion functor preserving cartesian morphisms and making the diagram commute satisfying a series of properties:
\[ \pi_S : \mathcal{S} \to \mathcal{C} \]
satisfies the following:

- each fibre in \( \mathcal{S} \) is an LCC pretopos preserved by the inclusion in \( \mathcal{C} \) and by base change functors;
- the **subobject doctrine** associated to \( \mathcal{C} \) is a first order Lawvere hyperdoctrine (represents the logic over collections);
- there is a \( \mathcal{C} \)-object \( \Omega \) classifying \( \mathcal{S} \)-subobjects of \( \mathcal{C} \)-objects:

\[ \text{i.e.} \quad \text{Sub}_\mathcal{S} \cong \mathcal{C}(\_, \Omega) \]

where \( \text{Sub}_\mathcal{S}(A) \) is the full subcategory of \( \text{Sub}_\mathcal{C}(A) \) of those subobjects which are represented by objects in \( \mathcal{S} \);
- there exist **power-objects** of \( \pi_S \) fibre objects
for every $C$-object $A$,
for every object $\alpha: X \to A$ in $S$,
there is an exponential object $(\pi_\Omega)^{\alpha}$ in $C/A$
where $\pi_\Omega: A \times \Omega \to A$ is the first projection, i.e. there is a natural isomorphism

$$C/A(- \times \alpha, \pi_\Omega) \simeq C/A(-, (\pi_\Omega)^{\alpha})$$

as functors on $C/A$. 
Our meta-language: Feferman’s Theory of NON-iterative fixpoints $\overline{ID_1}$

(j.w.w. S. Maschio)

we build a **predicative** version of *Hyland’s Effective Topos* by formalizing it into

the **PREDICATIVE** fragment of 2nd order arithmetics of Feferman’s Theory of NON-iterative fixpoints $\overline{ID_1}$

motivation:

*fixpoints* are needed to interpret iMF-sets

as in

[I. Ishihara, M.E.M., S. Maschio, T.Streicher’18]

“Consistency of the Minimalist Foundation with Church’s thesis and Axiom of Choice”, *AML*. 
A predicative version of Hyland’s Effective Topos

(j.w.w S. Maschio)

it is built as the exact completion $\mathcal{C}_{pEff}$
of the (lex) category $\text{Rec}^{\hat{ID}_1}$ of recursive classes + recursive morphisms
(with extensional function equality)
in Feferman’s Theory of NON-iterative fixpoints $\hat{ID}_1$
and the objects of the subfibration of sets are families of set-theoretic quotients
related to a universe of sets defined by a fix-point in $\hat{ID}_1$
observe that:

\[
\mathcal{C}_{pEff} = \mathcal{Q}(\text{wSub}_{\text{Rec}}) \text{ is the elementary quotient completion of the weak subobjects doctrine of } \text{Rec}^{\hat{ID}_1}
\]

thought of as a predicative tripos \(\mathcal{T}_{eff}\)
the interpretation of the logical connectives and quantifiers
in the hyperdoctrine structure of the subobject functor
is equivalent to Kleene realizability interpretation of intuitionistic logic.

in [M.E. Maietti and S. Maschio’18] "A predicative variant of Hyland’s Effective Topos”
on ArXiv
Embedding in Hyland’s Effective topos

our predicative effective topos

\[ C_{p\text{Eff}} = Q(w\text{Sub}_{\text{Rec}}) \]

can be embedded in Hyland’s Effective Topos \( \text{Eff} \)

\[ Q(w\text{Sub}_{\text{Rec}}) \cong (\text{Rec})_{ex/lex} \leftrightarrow (\text{pAsm})_{ex/lex} \cong \text{Eff} \]

because \( \text{Eff} \) is an exact on lex completion on partitioned assemblies

by embedding the category \( \text{Rec}^{\hat{ID}_1} \) of recursive functions in \( \hat{ID}_1 \)

in the corresponding category of subsets of natural numbers and recursive functions in \( \text{Eff} \).
Key peculiarity of MF: two notions of function

in both levels of MF

- a *primitive notion* of type-theoretic function
  \[ f(x) \in B \ [x \in A] \]

- (syntactically)
  notion of *functional relation*
  \[ \forall x \in A \ \exists! y \in B \ R(x, y) \]
NO axiom of unique choice/ NO choice rules in MF

⇓

MF needs a third level for extraction of programs from proofs from [M.17]
Axiom of unique choice

\[ \forall x \in A \exists! y \in B \ R(x, y) \quad \Rightarrow \quad \exists f \in A \to B \ \forall x \in A \ R(x, f(x)) \]

turns a functional relation into a type-theoretic function.

\[ \Rightarrow \text{identifies the two distinct notions...} \]
Key peculiarities of MF

CONTRARY to Martin-Löf’s type theory and to Aczel’s CZF

for $A, B$ MF-sets:

- Functional relations from $A$ to $B$ do NOT always form a set
- Exponentiation $\text{Fun}(A, B)$ of functional relations is not always a set
- Operations (typed-theoretic terms) from $A$ to $B$ do form a set
- Exponentiation $\text{Op}(A, B)$ is a set
three different kinds of real numbers

in the extensional MF
(even with classical logic!)
in accordance with Weyl's notion of continuum

- reals as Dedekind cuts NOT a set
- reals as Cauchy sequences or Brower's reals NOT a set
- reals as Cauchy sequences as our operations (=Bishop's reals) form a set

recall:
- Aczel's CZF + classical logic = IMPREDICATIVE Zermelo Fraenkel theory
- Martin-Löf's type theory + classical logic = IMPREDICATIVE
How to represent real numbers in the Minimalist Foundation?

Dedekind reals can be represented only in a point-free way via Martin-Löf-Sambin’s FORMAL TOPOLOGY including inductive methods to generate point-free topology by [Coquand, Sambin, Smith, Valentini2003].

⇒ we need to extend MF + inductive/coinductive definitions to represent Sambin’s generated Positive Topologies.
Dedekind reals as ideal points of point-free topology

\[ \text{Dedekind reals} = \text{ideal points (= constructive completely prime filters)} \]

of Joyal's inductively generated formal topology
**pointfree presentation of Dedekind reals**

Joyal's formal topology $\mathcal{R}_d \equiv (\mathbb{Q} \times \mathbb{Q}, \triangleleft_{\mathcal{R}}, \text{Pos}_{\mathcal{R}})$

Basic opens are pairs $\langle p, q \rangle$ of rational numbers

whose cover $\triangleleft_{\mathcal{R}}$ is inductively generated as follows:

\[
\begin{align*}
q \leq p & \quad \implies \quad \langle p, q \rangle \triangleleft_{\mathcal{R}} U \\
\langle p, q \rangle \in U & \quad \implies \quad \langle p, q \rangle \triangleleft_{\mathcal{R}} U \\
p' \leq p \langle q \leq q' & \quad \implies \quad \langle p', q' \rangle \triangleleft_{\mathcal{R}} U \\
\langle p, q \rangle \triangleleft_{\mathcal{R}} U & \quad \implies \quad \langle p, q \rangle \triangleleft_{\mathcal{R}} U \\
p \leq r \langle s \leq q & \quad \implies \quad \langle p, s \rangle \triangleleft_{\mathcal{R}} U \\
\langle r, q \rangle \triangleleft_{\mathcal{R}} U & \quad \implies \quad \langle r, q \rangle \triangleleft_{\mathcal{R}} U \\
\text{wc} \quad \text{wc}(\langle p, q \rangle) \triangleleft_{\mathcal{R}} U & \quad \implies \quad \langle p, q \rangle \triangleleft_{\mathcal{R}} U
\end{align*}
\]

where

\[
\text{wc}(\langle p, q \rangle) \equiv \{ \langle p', q' \rangle \in \mathbb{Q} \times \mathbb{Q} \mid p \langle p' \langle q' \rangle \langle q \rangle \}
\]
on the *Continuum in the Minimalist Foundation* MF

The NON equivalence of the different representations of real numbers in MF provides a paradigmatic example of the peculiar characteristics of MF itself.
Positive Topologies

in \textbf{MF}

\[ (B, \quad A_\triangleleft, \quad J_{\text{Pos}}) \]

defined by

- \( B \) is a preordered set (base of opens)
- \( A_\triangleleft \) is a closure operator on \( B \)

\[ A_\triangleleft : \quad \mathcal{P}(B) \quad \rightarrow \quad \mathcal{P}(B) \]

\[ U \quad \mapsto \quad \{ x \in B \mid x \triangleleft U \} \]

satisfying a convergence property:

\[ A_\triangleleft (U \downarrow V) = A_\triangleleft (U) \cap A_\triangleleft (V) \]

\[ U \downarrow V \equiv \{ a \in B \mid \exists u \in U \ a \leq u \ \& \ \exists v \in V \ a \leq v \} \]

- \( J_{\text{Pos}} \) is an interior operator
\( J_{\text{Pos}} : \mathcal{P}(B) \rightarrow \mathcal{P}(B) \)

\( W \mapsto \{ x \in B \mid \text{Pos}(a, W) \} \)

satisfying

(compatibility)

\[
\begin{align*}
\text{Pos}(a, W) & \quad a \triangleleft U \\
(\exists u \in U) \text{Pos}(u, W)
\end{align*}
\]

related to locale theory to over weakly closed subspaces

in [Vickers’07, Ciraulo-Vickers’16]
The two level extension $\text{MF}_{\text{ind}}$

we extend the extensional level $\text{eMF}$ of $\text{MF}$ as follows:

\[
\text{eMF}_{\text{ind}} = \text{eMF} + \text{inductive covers } a \triangleleft_{I,C} W \text{ as small propositions }
\]

+ coinductive positivity predicates $\text{Pos}_{I,C}(a, W)$ as small propositions

defining a **generated Positive Topology**

for any axiom-set

\[
\begin{align*}
A & \text{ set}_{\text{eMF}} \quad (\text{generators}) \\
I(a) & \text{ set}_{\text{eMF}} \ [a \in A] \\
C(a, j) & \text{ set}_{\text{eMF}} \ [a \in A, j \in I(a)]
\end{align*}
\]

$\Rightarrow$ without iterating the topological generation

small propositions $\text{eMF} \subseteq \text{small propositions } \text{eMF}_{\text{ind}}$

sets $\text{eMF} \subseteq \text{sets } \text{eMF}_{\text{ind}}$
we then extend $\text{iMF}$ to $\text{iMF}_{\text{ind}}$

with fix primitive proofs for rules of

$$(-) \triangleleft I, C (-) \quad \text{Pos}_{I, C}(-, -)$$

by preserving the interpretation in [M.E.M.09]

of the extensional level $\text{eMF}$

into the intensional level $\text{iMF}$
A Kleene realizability semantics for $i\text{MF}_{\text{ind}}$

as suggested by M. Rathjen

we can extend Kleene realizability semantics via suitable inductive definitions to validate

$i\text{MF}_{\text{ind}}$ with Church’s thesis and Axiom of Choice

in CZF+ REA

following [Griffor-Rathjen94]
How to interpret coinductive definitions

the Positivity predicate defined by coinduction on a set $A$ with axiom-set

$$A_{set_{eMF}} \quad \text{(generators)}$$

$$I(a)_{set_{eMF}} [a \in A] \quad C(a, j)_{set_{eMF}} [a \in A, j \in I(a)]$$

is an interior operator defined as the maximum fix point of an operator of the form for any fixed a subset $W$

$$\tau: \mathcal{P}(A) \rightarrow \mathcal{P}(A)$$

$$\tau(X) = \{x \in A \mid x \in W \& (\forall i \in I(x))(\exists y \in C(x, i)) y \in W \cap X\}$$

which can be defined as

$$\text{Pos}(W) = \bigcup \{K \in \mathcal{P}_{eMF}(A) \mid K \subseteq \tau(K) \& K \subseteq W\}$$

if the axiom of choice is valid for $A$

following a suggestion by T. Coquand in [M.E.M., S. Valentini 04]
Key issue to interpret coinduction via inductive definitions

coinductive definitions for generated Positive topologies can be reduced to suitable to inductive definitions if the base $A$ satisfies the axiom of choice
A predicative version of Hyland’s Effective Topos for generated topologies

for $\text{eMF}_{\text{ind}}$

we can extend the construction of the predicative effective topos $C_{\text{peff}}$ for $\text{eMF}$
to one, called $dC_{\text{peff ind}}$ for $\text{eMF}_{\text{ind}}$

thanks to the tool of elementary quotient completions

on the predicative realizability tripos for $\text{iMF}_{\text{ind}}+\text{AC}+\text{CT}$
Story of Joyal’s *arithmetic universes*

introduced in 70’s

\[(\text{Pred}(S))_{ex} = \text{exact completions of a category of predicates } \text{Pred}(S) \text{ for a Skolem theory } S\]

to prove *Gödel’s incompleteness theorems*
from Wraith’s notes

conjecture:  abstract arithmetic universe  =  arithmetic pretopos  ?
             =  pretopos + free monoid actions?
in [M.E.M 2010]
a generic arithmetic universe

= 

a list-arithmetic pretopos

shown by using

extensional dependent type theory à la Martin-Löf

in [M.E.M05]

initial Joyal’s arithmetic universes \(\simeq\) initial list-arithmetic pretopos
Joyal’s *initial arithmetic universe* $\mathcal{A}_0$ embeds in the predicative $\text{Eff}$ via universal properties:

given a *Skolem theory* $\mathcal{S}$

($= \text{cartesian category whose objects are finite products of a Natural numbers object}$)

we can define a *Lawvere doctrine*

with *decidable predicates* $P : \text{Nat}^n \to \text{Nat}$

as fibre objects

defined as *primitive recursive morphisms of* $\mathcal{S}$ such that $P \cdot P = P$:

\[
\widehat{\text{Pred}}_\mathcal{S} : \quad \mathcal{S}^{OP} \quad \longrightarrow \quad \text{InfSL} \\
\text{Nat}^n \quad \mapsto \quad \text{predicates } P \text{ over } \text{Nat}^n \\
\quad \mapsto \quad \text{with pointwise order}
\]
Joyal’s category of predicates $\text{Pred}(S)$

$$= \quad \text{base of the extensional completion}$$

of the free comprehension completion

$$\text{Pred}_S$$

as in


$$\downarrow$$

by universal properties

$$\text{Pred}_S \quad \hookrightarrow \quad \text{Sub}_{C_{peff}}$$

and hence

$$\mathcal{A}_0 = \text{Pred}(S)_{ex/lex} \quad \hookrightarrow \quad C_{peff}$$
Open problems

• the exact proof-theoretic strength of $\text{MF}_\text{ind}$

• provide a type theoretic formulation of Aczel’s Presentation Axiom

  *without the existence of universes of sets*