Arithmetic universes as generalized point-free spaces

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Grothendieck toposes as generalized spaces?
Classifying topos S[T] = "space of models of T"
  - but depends on choice of elementary topos S.
For some T, can use any S with nno.
Use arithmetic universes (AUs) to get base-independence.

Vickers:
"Sketches for arithmetic universes" (arXiv:1608.01559)
"Arithmetic universes and classifying toposes" (arXiv:1701.04611)

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Point-free topology

Point-set topology says:
1 - define collection of points as set
2 - define topology, using open subsets

In constructive mathematics:
Separating the points from the topology damages the space

Evidence?
From topos theory -
- important theorems (Heine-Borel, Tychonoff) fail for point-set spaces
From predicative mathematics -
- points may even fail to form a set.

Point-free topology describes points and opens in one single structure
- a logical theory
- points are models
- opens are propositions

Locales

Frames with morphisms reversed

Frame = complete lattice, binary meet distributes over all joins
Frame homomorphism preserves finite meets, all joins

Locale X = frame OX
Locale map f: X -> Y = frame homomorphism Of: OY -> OX

Categories: Loc = Fr^op

Presentations = propositional geometric theories

Algebra		Logic
generators	G	signature - propositional symbols
relations $\bigwedge_i a_i \leq \bigvee_j \bigwedge_k b_{jk}$	R	axioms $\bigwedge_i a_i \vdash \bigvee_j \bigwedge_k b_{jk}$
presentation	T = (G, R)	theory (signature, axioms)
frame presented	O[T] = Fr<G|R>	Lindenbaum algebra (formulae modulo equivalence)

connectives: finite conjunction, arbitrary disjunction

Universal property of O[T] = Fr<G|R>

For any frame A, and for any -

Algebra

Function f: G -> A
respecting the relations R

Logic

Model of T in A

there is a unique frame homomorphism f': Fr<G|R> -> A
that agrees with f on generators G

Easier to see when X = 1,
A = OX = P(X)
= {truth values}

Locales: write [T] for locale with O[T] = Fr<G|R>
For any locale X,
maps f: X -> [T] in bijection with
models of T in OX - models of T "at X"
Points of [T] = models of T

Another approach: formal topology

Assume G a poset, and a base of opens

$$g_1 \wedge g_2 = \bigvee \{ g \mid g \leq g_1, g \leq g_2 \}$$

Relations take simpler form

$$g \leq \bigvee_{i \in I} g_i \qquad g \lhd \{ g_i \mid i \in I \}$$

Same principles apply

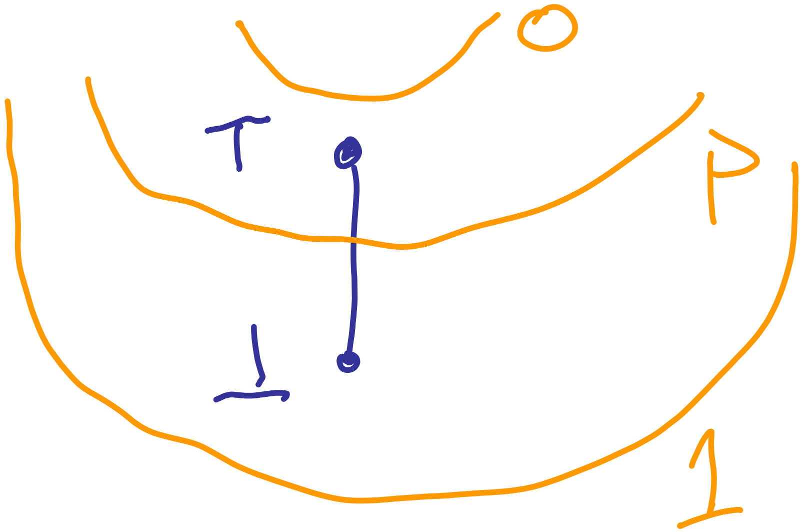
$(G, \leq, \lhd)$ a theory

It has
- formal points (models)
- formal opens (formulae modulo equivalence)

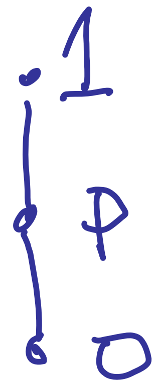
e.g. Sierpinski $

one generator P, no relations

Point = model = truth value



Open = formula



Map X -> $
= model of theory in OX
= open of X

model of theory at X

Grothendieck topos = generalized point-free space

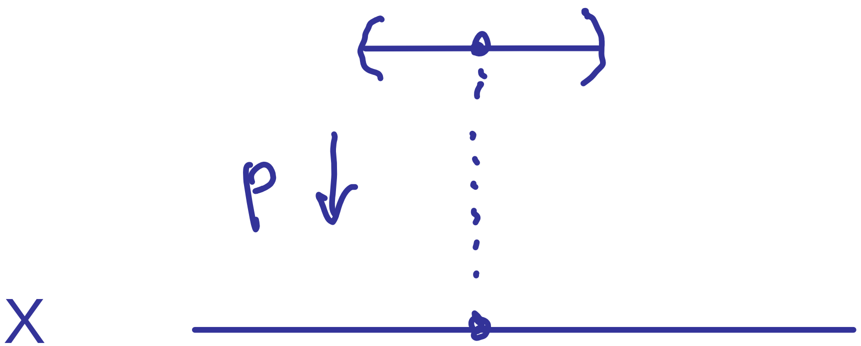
Ungeneralized: locale X
Frame = algebraic theory of opens
  X -> Sierpinski $
Lattice, finite /\, arbitrary \/
Map = function (backwards) preserving those

Generalized: topos X
Grothendieck topos = algebraic theory of sheaves (local homeomorphisms)
  X -> {sets}
Category, finite limits, arbitrary colimits
Map = functor (backwards) preserving those
~ geometric morphism

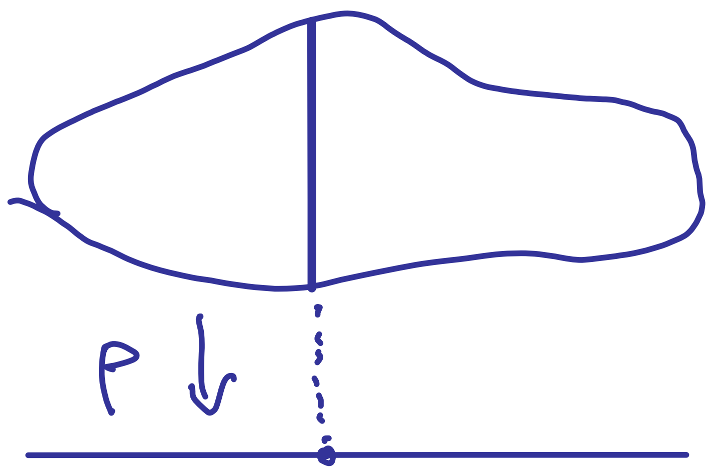
X -> $

$\top$ {*}

$\bot$ $\emptyset$



X -> {sets}
x |-> fibre



Presentations: Geometric theories $\mathbb{T}$

generators = signature: sorts, functions, predicates
relations = axioms

$$\phi(x_1, \dots x_n) \vdash \psi(x_1, \dots x_n)$$

formulae built with $\wedge \bigvee = \exists$

Ungeneralized: propositional
no sorts,
signature just propositional symbols

Present frame by generators and relations:
Lindenbaum algebra
= formulae modulo equivalence

Generalized: predicate

Grothendieck topos generated using finite limits, arbitrary colimits "making axioms hold"
= classifying topos $\mathbf{Set}[\mathbb{T}]$

Injection of generators gives generic model of theory.

Example: "space of sets" (object classifier)

Theory $\emptyset$ one sort, nothing else.

Classifying topos $\mathrm{Set}[\emptyset] = [\mathrm{Fin}, \mathrm{Set}]$

Conceptually object = continuous map {sets} -> {sets}
Continuity is (at least) functorial + preserves filtered colimits
Hence functor {finite sets} -> {sets}

Generic model is the subcategory inclusion Inc: Fin -> Set

Example: "space of pointed sets"

Theory $\emptyset, pt$ one sort X, one constant x: 1 -> X.

Classifying topos $\mathbf{Set}[\emptyset, pt] \simeq [\mathrm{Fin}, \mathbf{Set}]/\mathrm{Inc}$

In slice category: 1 becomes Inc, Inc becomes Inc x Inc

Generic model is Inc with

$$
\begin{tikzcd}
\mathrm{Inc} \arrow[rr, "\Delta"] \arrow[dr, "\mathrm{Id}"'] & & \mathrm{Inc} \times \mathrm{Inc} \arrow[dl, "\pi_2"] \\
& \mathrm{Inc} &
\end{tikzcd}
$$

1 in slice

Inc in slice

Universal property of Set[T]

1. Set[T] has a distinguished "generic" model M of T.

2. For any Grothendieck topos E,
and for any model N of T in E,
there is a unique (up to isomorphism) functor f*: Set[T] -> E
that preserves finite limits and arbitrary colimits
and takes M to N.

Same idea as for frames

f* preserves arbitrary colimits
- can deduce it has right adjoint

These give a geometric morphism f: E -> Set[T]
- topos analogue of continuous map

Reasoning in point-free logic

Box is S[T]
Its internal mathematics is
- geometric mathematics freely generated by a (generic) model of T

Let M be a model of T ...

Reasoning here must be geometric
- finite limits, arbitrary colimits
- includes wide range of free algebras
- e.g. finite powerset
- *not* full powerset or exponentials
- it's predicative

To get f* to another topos E:
Once you know what M maps to (a model in E)
- the rest follows
- by preservation of colimits and finite limits

Reasoning in point-free logic

Let M be a model of T_1 ...

Geometric reasoning
- inside box

Then f(M) = ... is a model of T_2

Outside box

Get map (geometric morphism) f: S[T_1] -> S[T_2]

Reasoning in point-free topology: examples

$+ : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$

Dedekind sections, e.g. (L_x, R_x)

Let $x, y \in \mathbb{R}$

Then $x+y \in \mathbb{R}$ where

$$L_{x+y} = \{q+r \mid q \in L_x, r \in L_y\}$$

$$R_{x+y} = \{q+r \mid q \in R_x, r \in R_y\}$$

Reasoning in point-free topology: examples

Let (x,y) be on the unit circle

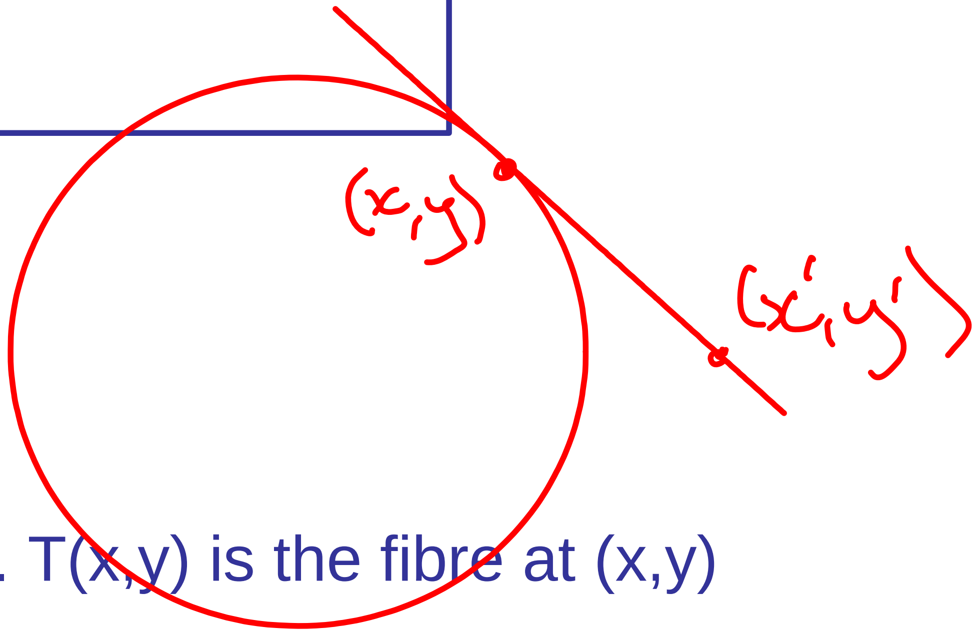
$x^2 + y^2 = 1$

Then can define presentation for a subspace of RxR, the points (x', y') satisfying

xx' + yy' = 1

This construction is geometric

It's the tangent of the circle at (x,y)



Inside the box:
For each point (x,y), a space T(x,y)

Outside the box:
Defines the tangent bundle of the circle. T(x,y) is the fibre at (x,y)

Joyal and Tierney:
Internal point-free space = external bundle

fibrewise topology of bundles

Reasoning in point-free topology: examples

Spec: [BA] -> Spaces

Let B be a Boolean algebra

Then Spec (B) is point-free space of prime filters of B, presented by -

· generators (b) $(b \in B)$

· relations $(b_1 \wedge b_2) = (b_1) \wedge (b_2)$

$(1) = \top$

$(b_1 \vee b_2) = (b_1) \vee (b_2)$

$(0) = \bot$

meet, join in B

logical conjunction, disjunction

Reasoning in point-free topology: examples

Spec : [BA] $\to$ Spaces

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logical conjunction

disjunction

meet, join in B

- B a pt of space of Boolean algebras
- internal point-free space
  = external bundle

$$\begin{array}{c} [BA + \text{prime filter}] \\ \downarrow \\ [BA] \end{array}$$

forget prime filter

Spec(B) is fibre over B

Geometricity => construction is uniform:
- single construction on generic B
- also applies to specific B's
- get those by pullback

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[BA + \text{prime filter}]} \arrow[d] \\ \mathcal{E} \arrow[r, "B"'] & {[BA]} \end{tikzcd}$$

pullback
= generalized fibre of generalized point

Suppose you don't like Set?

the base topos

$\mathcal{S}$

Replace with your favourite elementary topos S.
Needs nno N.

Fin becomes internal category in S.

Finite functions
f: m -> n

$$\mathrm{Fin}_1 \underset{\mathrm{cod}}{\overset{\mathrm{dom}}{\rightrightarrows}} \mathrm{Fin}_0 = N$$

n = {0, ..., n-1}

Classifying topos becomes
- category of internal diagrams on Fin

$$\mathcal{S}[\mathbb{O}] = [\mathrm{Fin}, \mathcal{S}]$$

(f: m -> n, x in X(m))

$$\begin{tikzcd}
\mathrm{dom}^* X \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & X \arrow[d] \\
\mathrm{Fin}_1 \arrow[r, shift left, "\mathrm{dom}"] \arrow[r, shift right, "\mathrm{cod}"'] & N
\end{tikzcd}$$

X(n) = fibre over n

X(f)(x) in X(n)

Suppose you don't like impredicative toposes?

Be patient!

Other classifier is slice, as before.

Roles of S

Infinities are extrinsic to logic
- supplied by S

(1) Supply infinities for infinite disjunctions:
get theories T geometric over S.

(2) Classifying topos built over S: geometric morphism $S[\mathbb{T}] \rightarrow S$

Suppose T has disjunctions all countable

It's geometric over any S with nno.

But different choices of S give different classifying toposes.

Idea: use finitary logic with type theory that provides nno
- replace countable disjunctions by existential quantification over countable types
- they become intrinsic to logic

Arithmetic universes instead of Grothendieck toposes

Pretopos -

finite limits
coequalizers of equivalence relations
finite coproducts

+ all well behaved

&

+ set-indexed coproducts
+ smallness conditions

Giraud's theorem

Grothendieck toposes
bounded S-toposes

extrinsic infinities from S

+ parametrized list objects

$$1 \xrightarrow{\varepsilon} \mathrm{List}(A) \xleftarrow{\mathrm{cons}} A \times \mathrm{List}(A)$$

Arithmetic universes (AUs)

intrinsic infinities
e.g. N = List(1)

Aims

- Finitary formalism for geometric theories

- Dependent type theory of (generalized) spaces

- Use methods of classifying toposes in base-independent way

- Computer support for that

- Foundationally very robust - topos-valid, predicative

- Logic intemalizable in itself
  (cf. Joyal applying AUs to Goedel's theorem)

Classifying AUs

Universal algebra => AUs can be presented by
- generators (objects and morphisms)
- and relations

theory of AUs is cartesian (essentially algebraic)

(G, R) can be used as a logical theory

AU<G|R> has property like that of classifying toposes

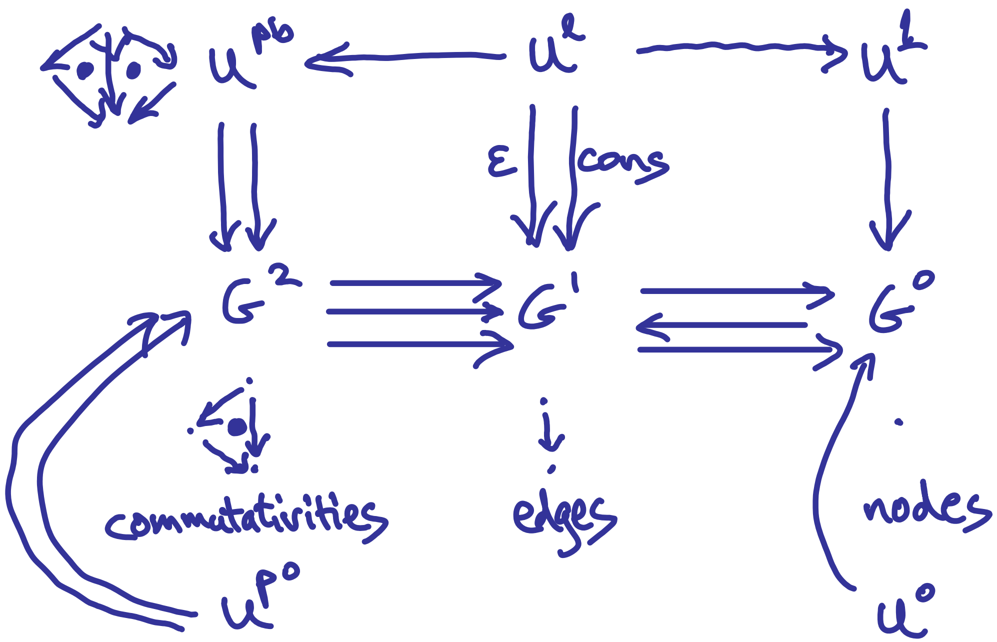
Treat AU<G|R> as "space of models of (G,R)"
- But no dependence on a base topos!

Issues: How to present theories?

"Arithmetic" instead of geometric

Not pure logic - needs ability to construct new sorts

Use sketches - hybrid of logic and category theory
- sorts, unary functions, commutativities
- universals: ability to declare sorts as finite limits, finite colimits or list objects



Issues: strictness

Strict model - interprets pullbacks etc. as the canonical ones
- needed for universal algebra of AUs

But non-strict models are also needed for semantics

Contexts are sketches built in a constrained way
- better behaved than general sketches
- every non-strict model has a canonical strict isomorph

Con is 2-category of contexts
- made by finitary means

A base-independent category of generalized point-free spaces

The assignment T |-> AU<T>
is full and faithful 2-functor
- from contexts
- to AUs and strict AU-functors (reversed)

"Sketches for arithmetic universes" (arXiv:1608.01559)

Bundles

$$\begin{array}{c} \mathbb{T}_1 \\ \big\downarrow U \\ \mathbb{T}_0 \end{array}$$

a context map (morphism in Con)
- transforms models N of T_1
- to models NU of T_0

$$\mathcal{A} \xleftarrow{N} AU\langle \mathbb{T}_1 \rangle \xleftarrow{AU\langle U \rangle} AU\langle \mathbb{T}_0 \rangle$$

For each model M of T_0:
- think of its fibre as
"the space of models N of T_1 such that NU = M"

If U is of a particular kind (extension map)
and if M is a model in an elementary topos (with nno) S,
then this fibre exists as a generalized space in Grothendieck's sense
- get geometric theory T_1/M
- it has classifying topos

$$\begin{array}{c} \mathcal{S}[\mathbb{T}_1/M] \\ \big\downarrow \\ \mathcal{S} \end{array}$$

"Arithmetic universes and classifying toposes" (arXiv:1701.04611)

Conclusions

Con is proposed as a category of Grothendieck's generalized spaces
- but in a base-independent way
- consists of what can be done in a minimal foundational setting
- of AUs
- constructive, predicative
- includes real line; also theory of regular measures.