

# Coherence for Geometricity

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# Point-free topology

- works well in toposes

Frame = complete lattice  $A$ ,  $\wedge$  distributes over  $\vee$

Homomorphism - preserves  $\wedge, \vee$

In a topos:  $A$  a lattice,  $\vee: \mathcal{P}A \rightarrow A$

Geometric morphism  $f: \mathcal{E} \rightarrow \mathcal{F}$

$$\begin{array}{ccc} Fr_{\mathcal{E}} & \xrightarrow{f^*} & Fr_{\mathcal{F}} \\ & \xleftarrow{f^\#} & \end{array}$$

$$f^\# \neq f^*$$

$Fr$  strictly coindexed  
non-strictly indexed

over  $Top$  { elementary toposes  
+ no geometric morphisms

# Localic bundle theorem

Frames in  $\mathcal{F}$   $\simeq$  localic geometric morphisms to  $\mathcal{F}$

$f^\#$  acts as  
pseudo pullback  
on bundles

$$\begin{array}{ccc} \text{Sh}_{\mathcal{E}}(f^\# A) & \longrightarrow & \text{Sh}(A) \\ \downarrow & & \downarrow \\ \mathcal{E} & \xrightarrow{f} & \mathcal{F} \end{array}$$

# Indexed endofunctors $F$ on $\bar{F}r$

$F$  commutes with  $f^\#$  up to coherent iso

Acting on bundles - "act fibrewise"

$$\text{Sh}_{\mathcal{C}}(F f^\# A) \cong \text{Sh}_{\mathcal{C}}(f^\# FA) \longrightarrow \text{Sh}_{\mathcal{F}}(FA)$$

dependent  
type theory

$\downarrow$   
 $\mathcal{C}$

$\downarrow$   
 $\mathcal{F}$

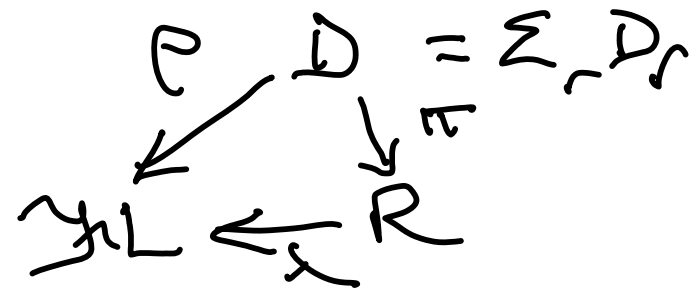
$F$  often defined using presentations

- so only up to iso

- how to get coherence?

$F_r < L$

$$| \Lambda \lambda(r) \cong \bigvee_{d \in D_r} \Lambda p(d) \quad (r \in R) |$$





# DLS Morphisms

$$(L, R, \triangleright) \rightarrow (L', R', \triangleright')$$

$$\vartheta_L : L \rightarrow L'$$

$$\vartheta_R : R \rightarrow R'$$

$$\vartheta_{\triangleright} : \triangleright \rightarrow \triangleright'$$

• preserve all structure

(e.g.  $\vartheta_L$  a DL-homomorphism)

+  $\vartheta_{\triangleright}$  fibrewise surjective

$$\pi'(d') = \vartheta_R(r) \rightarrow \exists d. (\pi(d) = r \wedge \vartheta_{\triangleright}(d) = d')$$

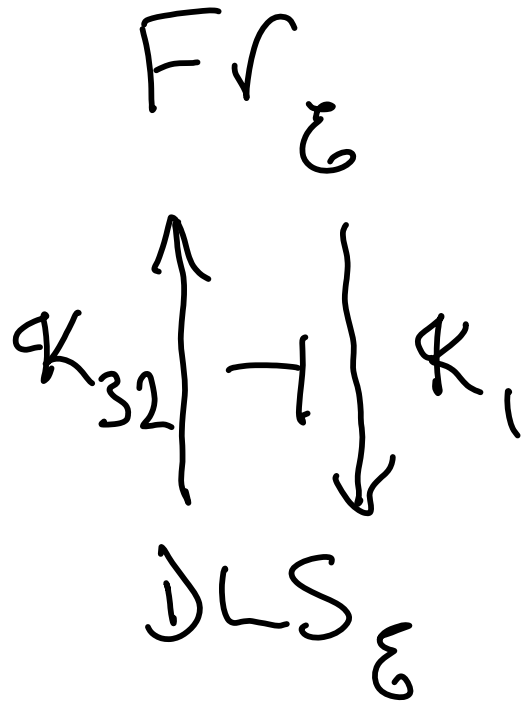
Category  $\text{DLS}_{\mathcal{E}}$  for any topos  $\mathcal{E}$ .

Another geometric theory:  
2 DL-sites  
+ morphism

# Adjunction

$$\mathcal{K}_{32}(L, R, \eta)$$

$$= \text{Fv} \langle L(\text{qua } \eta L) \mid \lambda(r) \leq \uparrow \text{rid} = r \text{ } \rho(d) \text{ } (r \in R) \rangle$$



$$\mathcal{K}_1(A) =$$

canonical presentation

$$L = A$$

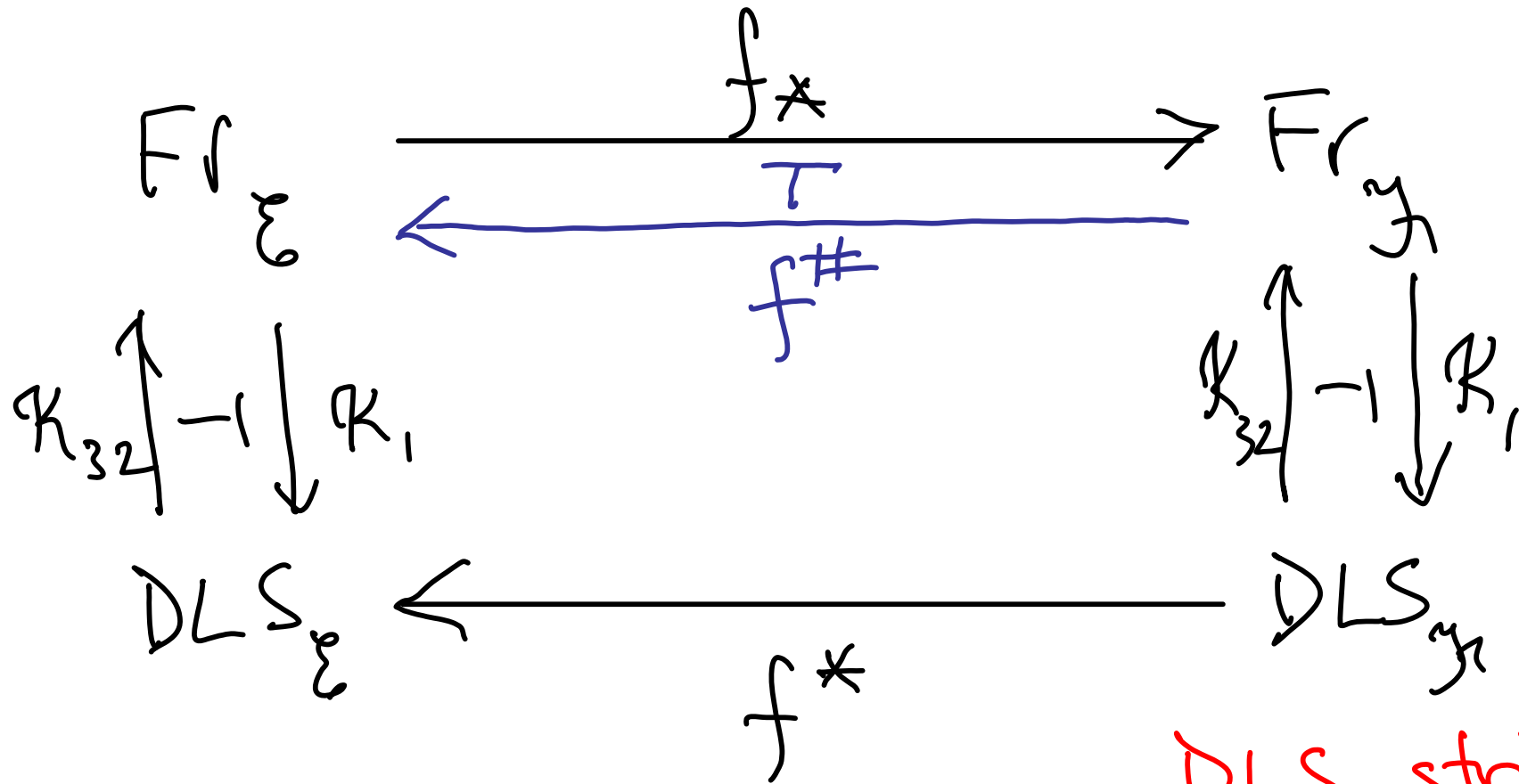
$$R = \{ (a, S) \mid a \in A, S \subseteq A \text{ directed} \}$$

$$a \leq \uparrow S \rangle$$

Unit  $\varepsilon$  an iso ( $\mathcal{K}_1(A)$  presents  $A$ )

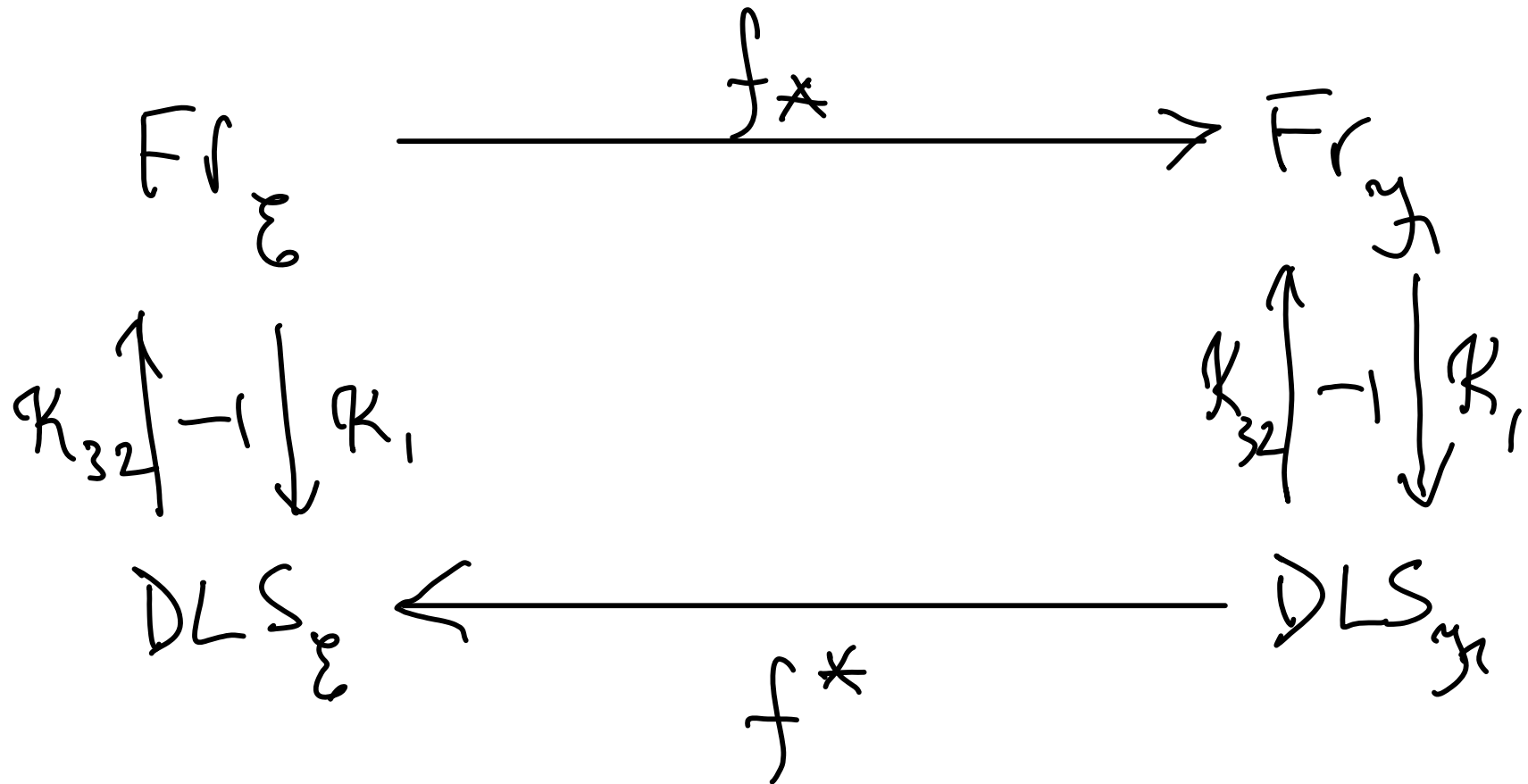
Unit  $\eta$  not an iso - but  $\mathcal{K}_{32}$  is





DLS strictly indexed using  $f^*$

①  $K_{32} f^* K_1 \rightarrow f^*$   
 $\therefore$  take  $f^\# = K_{32} f^* K_1$



②

$f^*$  has Kleisli lifting

$\therefore f^*$  preserves Kleisli isos  
(e.g.  $\eta$ )

# Indexed endofunctors: sufficient conditions

Define  $F: \mathcal{DLS} \rightarrow \mathcal{DLS}$  on generic DL-site

On  $\mathcal{F}$ , use  $\mathcal{K}_2 F' \mathcal{K}_1$

using geometric constructions  
preserved by  $f^*$ s

Then  $Ff^* \cong f^*F$

More carefully: use algebra of arithmetic universes

- finite limits, finite colimits, list objects

Then  $Ff^* \cong f^*F$  by canonical iso

Also require -

$F$  has Kleisli lifting, like  $f^*$ .

Then

$$\mathbb{K}_{32} F \mathbb{K}_1, \mathbb{K}_{32} f^* \mathbb{K}_1$$
$$\parallel \quad (\mathbb{K}_{32} F \eta)^{-1} f^* \mathbb{K}_1$$

$$\mathbb{K}_{32} F f^* \mathbb{K}_1$$
$$\parallel$$

$$\mathbb{K}_{32} f^* F \mathbb{K}_1$$
$$\parallel$$

$$\mathbb{K}_{32} f^* \eta F \mathbb{K}_1$$

$$\mathbb{K}_{32} f^* \mathbb{K}_1, \mathbb{K}_{32} F \mathbb{K}_1$$

Plan:  
have enough  
control over  
isomorphisms  
to get  
coherence

# Example Double power locale

$$\mathbb{A} = \mathbb{P}_u \mathbb{P}_L \quad \text{localic hyperspaces}$$

On frames:  $A \mapsto \text{Fr} \langle A \text{ (qua dcpo)} \rangle$

On presentations:

First, from  $(L, R, D)$  get

$$\text{Fr} \langle L \text{ (qua poset)} \mid \lambda(r) \leq \bigvee_{\pi d=r} p(d) \quad (r \in R) \rangle$$

↑  
instead of  $\Delta L$

Next, complete to  $\Delta L$ -site  $(L', R', D')$

$$L' = \Delta L \langle L \text{ (qua poset)} \rangle \quad \text{etc.}$$