Coherence for Geometricity

Steve Vickers
School of Computer Science
University of Birmingham

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Point-free topology

- works well in toposes

Frame = complete lattice A, $\wedge$ distributes over $\bigvee$

Homomorphism - preserves $\wedge, \bigvee$

In a topos: A a lattice, $\bigvee : \mathcal{P}A \to A$

Geometric morphism $f : \mathcal{E} \to \mathcal{F}$

$$\begin{tikzcd} Fr_{\mathcal{E}} \arrow[r, shift left=2, "f_*"] \arrow[r, phantom, "\top"] & Fr_{\mathcal{F}} \arrow[l, shift left=2, "f^{\#}"] \end{tikzcd}$$

$f^{\#} \neq f^{*}$

Fr strictly coindexed / non-strictly indexed over Top $\left\{ \begin{array}{l} \text{elementary toposes} \\ + \text{nno} \\ \text{geometric morphisms} \end{array} \right.$

Localic bundle theorem

Frames in $\mathcal{F}$ $\simeq$ localic geometric morphisms to $\mathcal{F}$

$$\begin{tikzcd}
\mathrm{Sh}_{\mathcal{E}}(f^{\#}A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Sh}_{\mathcal{F}}(A) \arrow[d] \\
\mathcal{E} \arrow[r, "f"'] & \mathcal{F}
\end{tikzcd}$$

$f^{\#}$ acts as pseudo pullback on bundles

Indexed endofunctors F on $\mathbf{Fr}$

F commutes with $f^\#$ — up to coherent iso

Acting on bundles – "act fibrewise"

$$\begin{tikzcd}
Sh_{\mathcal{E}}(F f^\# A) \arrow[r, phantom, "\cong"] \arrow[dr] & Sh_{\mathcal{E}}(f^\# F A) \arrow[r] \arrow[d] & Sh_{\mathcal{F}}(FA) \arrow[d] \\
& \mathcal{E} \arrow[r] & \mathcal{F}
\end{tikzcd}$$

dependent type theory

F often defined using presentations

- so only up to iso
- How to get coherence?

$$\begin{tikzcd} & D = \sum_r D_r \arrow[dl, "\rho"'] \arrow[d, "\pi"] \\ \mathcal{F}L & R \arrow[l, "\lambda"] \end{tikzcd}$$

$$\mathrm{Fr}\left\langle L \;\middle|\; \bigwedge \lambda(r) \leq \bigvee_{d \in D_r} \bigwedge \rho(d) \quad (r \in R) \right\rangle$$

DL-site

geometric theory

$$\begin{tikzcd} & D = \Sigma_r D_r \arrow[dl, "\rho"'] \arrow[d, "\pi"] \\ L & R \arrow[l, "\lambda"] \end{tikzcd}$$

$$Fr\langle L \text{ (qua DL)} \mid \lambda(r) \le \bigvee_{d \in D_r} \rho(d) \quad (r \in R)\rangle$$

- L a DL wrt. $(R, =)$
- D a poset, ρ, π monotone
  each fibre of π directed
- $\wedge : R \times L \to R$, $\wedge : D \times L \to D$ meet stability
  $\pi(d \wedge x) = \pi(d) \wedge x$, $\rho(d \wedge x) = \rho(d) \wedge x$
  $\lambda(d \wedge x) = \lambda(d) \wedge x$
- Similar "join stability"

NB Frame presented dcpo coverage theorem
$\cong dcpo\langle L \text{ (qua poset)} \mid \text{same relations}\rangle$

DLS

Morphisms

$(L, R, D) \longrightarrow (L', R', D')$

$\theta_L : L \to L'$

$\theta_R : R \to R'$

$\theta_D : D \to D'$

Another geometric theory: 2 DL-sites + morphism

- preserve all structure (e.g. θ_L a DL-homomorphism)

+ θ_D fibrewise surjective

$\pi'(d') = \theta_R(r) \longrightarrow \exists d.\,(\pi(d) = r \wedge \theta_D(d) = d')$

Category $DLS_{\mathcal{E}}$ for any topos $\mathcal{E}$.

Adjunction

$$\mathcal{K}_{32}(L,R,D) = Fr\langle L \text{ (qua } DL) \mid \lambda(r) \le \bigvee^{\uparrow}_{\rho(d)=r} \rho(d) \ (r \in R)\rangle$$

$$\begin{tikzcd} Fr_{\varepsilon} \arrow[d, bend left, "\mathcal{K}_1"] \arrow[d, phantom, "\dashv"] \\ DLS_{\varepsilon} \arrow[u, bend left, "\mathcal{K}_{32}"] \end{tikzcd}$$

$\mathcal{K}_1(A) =$ canonical presentation

$L = A$

$R = \{(a,S) \mid a \in A, S \subseteq A \text{ directed}, a \le \bigvee^{\uparrow} S\}$

Counit ε an iso ($\mathcal{K}_1(A)$ presents A)

Unit η not an iso – but $\mathcal{K}_{32}\eta$ is

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathrm{Fr}_{\mathcal{E}} \arrow[r, shift left=1.5, "f_*"] \arrow[d, shift left=1.5, "K_1"] \arrow[d, phantom, "\dashv"] & \mathrm{Fr}_{\mathcal{F}} \arrow[l, shift left=1.5, "f^\#"] \arrow[l, phantom, "\top"] \arrow[d, shift left=1.5, "K_1"] \arrow[d, phantom, "\dashv"] \\
\mathrm{DLS}_{\mathcal{E}} \arrow[u, shift left=1.5, "K_{32}"] & \mathrm{DLS}_{\mathcal{F}} \arrow[l, "f^*"] \arrow[u, shift left=1.5, "K_{32}"]
\end{tikzcd}
$$

DLS strictly indexed using f^*

(1) $K_{32} f^* K_1 \dashv f_*$

$\therefore$ take $f^\# = K_{32} f^* K_1$

$$\begin{tikzcd}
\mathrm{Fr}_{\mathcal{E}} \arrow[rr, "f_*"] \arrow[d, shift left, "K_1"] & & \mathrm{Fr}_{\mathcal{F}} \arrow[d, shift left, "K_1"] \\
\mathrm{DLS}_{\mathcal{E}} \arrow[u, shift left, "K_{32}"] \arrow[u, phantom, "\dashv"] & & \mathrm{DLS}_{\mathcal{F}} \arrow[ll, "f^*"] \arrow[u, shift left, "K_{32}"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

(2) f^* has Kleisli lifting

$\therefore$ f^* preserves Kleisli isos (e.g. η)

Indexed endofunctors: sufficient conditions

On Fr, use $\mathcal{K}_{32} F' \mathcal{K}_1$

Define $F : DLS \longrightarrow DLS$
on generic DL-site
using geometric constructions
preserved by f^*s

Then $Ff^* \cong f^* F$

More carefully: use algebra of arithmetic universes
- finite limits, finite colimits, list objects

Then $Ff^* \cong f^* F$ by canonical iso

Also require –

F has Kleisli lifting, like f^*.

Then

$$
\begin{aligned}
K_{32} F K_1 K_{32} f^* K_1 & \\
\cong \quad & (K_{32} F \eta)^{-1} f^* K_1 \\
K_{32} F f^* K_1 & \\
\cong \quad & \\
K_{32} f^* F K_1 & \\
\cong \quad & K_{32} f^* \eta F K_1 \\
K_{32} f^* K_1 K_{32} F K_1 &
\end{aligned}
$$

Plan: have enough control over isomorphisms to get coherence

Example Double powerlocale

$\mathbb{P} = P_U P_L$ localic hyperspaces

On frames: $A \mapsto \mathrm{Fr}\langle A \text{ (qua dcpo)}\rangle$

On presentations:

First, from (L, R, D) get

$$\mathrm{Fr}\langle L \text{ (qua poset)} \mid \lambda(r) \leq \bigvee\nolimits^{\uparrow}_{\pi d = r} \rho(d) \quad (r \in R)\rangle$$

instead of DL

Next, complete to DL-site (L', R', D')

$L' = DL\langle L \text{ (qua poset)}\rangle$ etc.