Dependent type theory of point-free topological spaces

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Some background in [Vic22].

Aim: dependent type theory in which –

- type = space
- space = point-free topological space
- ... even in generalized sense (topos)
- dependent type = bundle

Will be an unusual type theory

Arrow types cannot be part of the logic (because category of spaces not cartesian closed).
2-cells important, and can belong to analogues of identity types; but not invertible in general, and no path transport in general.
$\therefore$ discuss informally – no ready-made model available.

Paradigm: sets

Syntax

Terms belong to types
Terms can depend on other terms
Types can also depend on terms

Semantics

Elements belong to sets
Dependency is a function
Dependency is a bundle

What is a bundle?

1. Family of sets $Y(x)$ indexed by elements $x \in X$
2. Function $Y \; (= \coprod_{x \in X} Y(x)) \; \to X$

The sets $Y(x)$ are the fibres of the function, ie inverse images of points.
DTT syntax is (1): construction $Y(x)$ with parameter x.
Semantically, (2) makes general sense in categories, but (1) relies on set theory.

Categorically – use generalized elements

Element of object X *at stage* W = morphism $x\colon W \to X$.
Usual, *global,* elements are at stage 1.

Given a bundle $p\colon Y \to X$:

Fibre $Y(x)$ is pullback x^*Y:
It is not a set, but another bundle:
sets are bundles over 1.

$$\begin{array}{ccc} Y(x) = x^*Y & \longrightarrow & Y \\ {\scriptstyle x^*p}\downarrow & & \downarrow{\scriptstyle p} \\ W & \xrightarrow[x]{} & X \end{array}$$

A bundle is equivalent to specifying all its fibres
— at all the generalized elements.

But that's a bit of a cheat.
There's a *generic* element, identity $\mathrm{Id}_X\colon X \to X$.
Its fibre is Y, and is enough to determine all the others.

Bundles as dependent types

Syntactically

dependent type = assignment $x \mapsto Y(x)$,
base point $\mapsto$ fibre.

Categorical semantics agrees!

But in a trivial way: define generic fibre, then all others are pullbacks.
We'd prefer syntax of $Y(x)$ to capture construction over all generalized elements,
— without having to comprehend the entire category.

Some solutions are well known

Use construction of X as type, + its universal properties.
eg for elementary toposes cf. Kripke-Joyal semantics

Topologizing

Syntax

Terms belong to types
Terms can depend on terms
Types can depend on terms

Semantics

Points belong to spaces
Dependency is a (continuous) map
Dependency is a bundle

For the same reasons as before,

Point of space X *at stage* W = map $x\colon W \to X$.

What is a bundle?

1. Family of spaces $Y(x)$ indexed by points $x{:}X$
2. Map $Y\ (= \sum_{x \in X} Y(x))\ \to X$

Can we restore meaning to (1) –

... without resorting to categorical trivialities?

Example: tangent bundle of sphere S^2

Embed in $\mathbb{R}^3$.

Define tangent spaces

- Suppose $x{:}X = S^2$. $x = (x_0, x_1, x_2)$ with $x.x = 1$.
- Tangent space $Y(x)$ is space of $y{:}\mathbb{R}^3$ such that

$$(y - x).x = 0$$

How to make tangent bundle?

Solution in point-set topology – non-trivial!

- Form disjoint union of sets $|Y| = \coprod_{x \in |X|} |Y(x)|$, where $|X|$ is the set of global points of X.
- Define an appropriate topology on $|Y|$.
- Prove that projection $|Y| \to |X|$ is continuous.

In essence, proving that $x \mapsto Y(x)$ is "continuous" enough.

Topologized DTT: Desiderata

1. All term dependencies must be continuous.
2. *So too must type dependencies.*

What can (2) mean?

Point-free spaces

Point = *model of a geometric theory* $\mathbb{T}$

Think of $\mathbb{T}$ as the type, terms denote models.

Categorical semantics

Work in (2-)category of Grothendieck toposes.[1]

Semantics: Type $\mathbb{T}$ denotes classifying topos $\mathcal{S}[\mathbb{T}]$

— rather than some collection of models.
Points at stage $\mathbb{W}$
= geometric morphisms $\mathcal{S}[\mathbb{W}] \to \mathcal{S}[\mathbb{T}]$
= models of $\mathbb{T}$ in $\mathcal{S}[\mathbb{W}]$ (universal characterization of classifying toposes)
Points of $\mathcal{S}[\mathbb{T}]$ = models of $\mathbb{T}$ (at every stage).

[1] = bounded $\mathcal{S}$-toposes for some given base $\mathcal{S}$.

Dependency $x \mapsto t(x)$ or $x \mapsto Y(x)$

Say X is theory $\mathbb{T}$

- x *denotes* generic model in $\mathcal{S}[\mathbb{T}]$
- $t(x)$, $Y(x)$ then describe constructions in $\mathcal{S}[\mathbb{T}]$
- Model a of $\mathbb{T}$ (in $\mathcal{S}[\mathbb{W}]$) $=$ geometric morphism $a \colon \mathcal{S}[\mathbb{W}] \to \mathcal{S}[\mathbb{T}]$
- *Substitution* a for x in $t(x)$ is $a^* t(x)$. Similarly for Y.
- Construction must be geometric in order to be preserved by every a^*.
- That includes colimits, finite limits, free algebras; excludes exponentials, powerobjects.

Defining terms

Declare: Let x be a model of $\mathbb{T}$
... working geometrically ...
Construct all ingredients of $t(x)$, model of some theory $\mathbb{T}'$.

Outside the scope of the declaration, –

Have constructed a map (geometric morphism) $\mathcal{S}[\mathbb{T}] \to \mathcal{S}[\mathbb{T}']$.

Need syntax for geometric constructions

Will return to this later.

How to define *types*? What is an internal *space*?

Space = geometric theory

- Can always manipulate into the form of a site $(\mathcal{C}, T)$. Models of the theory = flat, continuous functors on the site.
- It has a classifying topos $\mathbf{Sh}_{\mathcal{S}}(\mathcal{C}, T) \to \mathcal{S}$ of sheaves.
- Thus we get a bundle, as desired.
- As a geometric morphism it is bounded. Every bounded geometric morphisms can be obtained this way. We take "bundle" to mean bounded.

Internal space = internal site = bundle

Apply the above principle to $\mathcal{S}[\mathbb{T}]$.

Localic case

These correspond to "ungeneralized" point-free spaces, with various representations available.

- *Frames:* [JT84] shows the equivalence between internal frames and localic bundles. Unfortunately, frame structure is not geometric, so frames are not useful for us.
- *Frame presentations:* These are geometric, so we can use them to construct spaces geometrically. See [Vic04].
- *Propositional geometric theories:* are equivalent to frame presentations.
- *Formal topologies*

Geometric theories à la Elephant [Joh02, B4.2.7]

Geometric theory built up from trivial theory $\mathbb{1}$ in finite number of primitive extension steps:

Extending theory $\mathbb{T}_0$ to $\mathbb{T}_1$

The following primitive steps are available.

1. Adjoin a sort.
2. *Simple functional extension:* Adjoin a function between two geometric constructs (of “sets”, ie objects of toposes, ie discrete spaces) on ingredients of $\mathbb{T}_0$.
3. *Simple geometric quotient:* Adjoin an inverse to an existing function between two geometric constructs.

Important advantage!

- Elephant style provides a flexible means to build up towers of theories, with forgetful maps between them, without having to force them into the first-order format of geometric theories at each stage.
- Forgetful map $\mathcal{S}[\mathbb{T}_1] \to \mathcal{S}[\mathbb{T}_0]$ defines an internal space in $\mathcal{S}[\mathbb{T}_0]$. It is $x \mapsto Y(x)$, where x is a model of $\mathbb{T}_0$, and $Y(x)$ is the theory of the extra stuff needed to make a model of $\mathbb{T}_1$.
- Extension steps are how you build dependent types.
- The extended theories are $\sum$-types. eg $\mathbb{T}_1$ is $\sum_{x:\mathbb{T}_0} Y(x)$.

What is a geometric construct?

Note these are geometric constructions of “sets”, ie discrete spaces, ie objects of toposes, and their functions.

Depends on $\mathcal{S}$!

$\mathcal{S}$ describes the infinities that can be used in “arbitrary” colimits and infinite disjunction.
Provided $\mathcal{S}$ has nno, that's enough to construct free algebras.
A useful approximation is provided by the coherent fragment (finite colimits, finite limits) + parametrized list objects.
This is enough to construct free algebras, and does not depend on choice of $\mathcal{S}$.
See [Vic19, Vic17] using arithmetic universes.

Combine this with previous slide

Then have convenient way to describe useful range of geometric theories in finitary way, and without depending on $\mathcal{S}$.

Example: space of reals $\mathbb{R}$

Theory is localic (propositional), *but* it's convenient to use the constructed sort $\mathbb{Q}$ in a first-order form.
Then can present theory of Dedekind sections directly using predicates L and R on $\mathbb{Q}$. See eg [Vic07].

Mathematical development much more natural

– than, eg, a purely logical one with propositional theories.
[NV22] shows how to construct real exponentiation and logarithms point-free in this style.

Example: tangent bundle of S^2

Need general purpose constructions of spaces

eg products, equalizers

Now we have $\mathbb{R}$:

1. Can construct $\mathbb{R}^3$.
2. Construct two maps $\mathbb{R}^3 \to \mathbb{R}$, $x \mapsto x.x$ and $x \mapsto 1$.
3. Define S^2 as equalizer.

Internally in $\mathcal{S}S^2$

Let x be a point of S^2.

1. Construct two maps $\mathbb{R}^3 \to \mathbb{R}$, $y \mapsto (y - x).x$ and $y \mapsto 0$.
2. Define tangent space $T_x(S^2)$ as their equalizer.

Externally, get tangent bundle $T(S^2) = \sum_{x:S^2} T_x(S^2) \to S^2$.

Example: tangent bundle of S^2

- We have extended the theory for S^2 to get a theory for $T(S^2)$
- Points of $T(S^2)$ are pairs (x, y) with $x{:}S^2$ and $y{:}T_x(S^2)$.
- In terms of simple extension steps it would be quite complicated, but it is packaged up in a mathematically natural way to make use of known geometricities.
- It is the geometricity that makes it enough to define the fibres. No topologies to define, no continuity proofs.

Concluding remarks

- Basic idea works for any logic for which classifying categories exist.
- For geometric logic we have classifying toposes.
- Complicated by the infinitary connectives.
- Elephant-style geometric theories, and geometric sort constructors, work well for towers of dependent types.

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