Fuzzy sets and geometric logic

Steven Vickers
School of Computer Science,
University of Birmingham,
Birmingham,
B15 2TT,
UK
s.j.vickers@cs.bham.ac.uk

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Abstract

Höhle has identified fuzzy sets, valued in a frame (complete Heyting algebra) Ω, with certain sheaves over Ω: the subsheaves of constant sheaves More general sheaves can be got as quotients of the fuzzy sets. His principal approach to sheaves over Ω, and topos-theoretic constructions on them, is via complete Ω-valued sets.

In this paper we show how the *geometric* fragment of those constructions can be described in a natural "stalkwise" manner, provided one works also with incomplete Ω-valued sets.

Our exposition examines in detail the interactions between different technical expressions of the notion of sheaf, and highlights a conceptual view of sheaf as "continuous set-valued map".

Keywords: Sheaf, Ω-valued set, locale, frame, topos.

MSC2000: 54B40; 03E72, 03G30, 06D22

1 Introduction

In his two papers [H07a] and [H07b], Ulrich Höhle has shown how frame-valued fuzzy sets may be considered as particular kinds of sheaves over the corresponding locale. There are two particular insights that underly this treatment. The first is that, of the various equivalent different ways of expressing the notion of sheaf, the one most relevant to fuzzy sets is that of the "frame-valued set" (i.e. set with frame-valued equality). The second is that fuzzy sets are then seen as subsheaves of the "constant sheaves" that correspond to standard sets, and that the general sheaf can then be got as a quotient of a fuzzy set. From this point of view, fuzzy set theory is mathematically deficient in that it does not include quotienting, and when quotienting is added one obtains sheaf theory.

As is well known, the category of sheaves over a locale is a topos and so supports categorical operations corresponding to those of (intuitionistic) set theory. These include products (cartesian products), pullbacks (fibred products), coproducts (disjoint unions), coequalizers (quotients), exponentiation (function sets), the subobject classifier (set of logical truth values) and power objects (powersets). In [H07̈a] these are described concretely in terms of the frame-valued set structure of the sheaves.

The purpose of this paper is to publicize a certain class of operations that are particularly well behaved. These are the *geometric* operations, and are known from topos theory as those operations that are preserved by the inverse image functors of geometric morphisms between toposes. Although they omit some of the topos-valid (intuitionistic) operations, they have an inherent continuity that makes it useful to restrict oneself to the geometric operations where possible. When sheaves are viewed as local homeomorphisms, the geometric operations have the important property that they can be calculated stalkwise. At first sight this approach would seem to be useful only when the locale is spatial (so that there are enough points and hence enough stalks)[1], but in fact it can be made sense of for general locales.

The geometric constructions provide a key to treating locales as spaces of points, and locale maps as geometric transformations of points. In fact, there is a sense in which continuity is just geometricity. From this point of view a sheaf is a continuous assignment of stalks to points – something which can be intuitively felt in the definition of local homeomorphism, but which has a more profound expression in topos theory.

A technical point that arises is in regard to the *completeness* of the frame-valued sets. In general, quite different frame-valued sets can present isomorphic sheaves. However, any frame-valued set can be completed to give a canonical representative, from which the presheaf can be easily extracted. Höhle describes his constructions in terms of the complete frame-valued sets, but the geometric operations can be described in particularly simple ways as constructions on the uncompleted frame-valued sets and we shall describe examples of these.

Little of the technical content of the present paper is actually new. Our aim is to present established results in a way that brings out a particular view of sheaves as applied to fuzzy sets: that they are *continuous set-valued maps*.

The recommended introduction to sheaves from a topos point of view is [MLM92]. [Gol79] is less deep, but shows well how logic and set theory translate into category theory. The ultimate comprehensive reference is [Joh02a], [Joh02b], but is not for the beginner. [Vic07] develops in more detail the relationship between continuity and geometricity, and sheaves as continuous set-valued maps.

[1] Indeed, Höhle's account describes the local homeomorphism view but consistently restricts to the spatial case.

2 Sheaves over spaces

Sheaves are the central theme of this paper. Specifically we are interested in the point-free theory of sheaves over locales, but it will be helpful first to describe sheaves over spaces. A big conceptual difficulty is that they have a variety of different but equivalent technical expressions. The two best known definitions are as *local homeomorphisms* and as *pasting presheaves*. In addition, Höhle's connection with fuzzy sets is via a lesser known, and quite different, notion, that of *frame-valued set*; and this comes in two different flavours, complete and incomplete.

We shall review the four kinds here, partly to see how bad the problem is. The four really are different, and choosing one rather than another can make a big difference to ease of calculation – but different calculations can require different choice of sheaf style.

It is usual to take the word "sheaf" to mean explicitly the representation as pasting presheaves. We shall use it more ambiguously, for any of the four representations, modulo interconversion.

However, there is a more fundamental conceptual base: a sheaf over a space X is a *continuous set-valued map* $X \to \mathbf{Set}$. This cannot be made precise by topologizing the class of sets in the ordinary way – in fact, Grothendieck's idea of topos as "generalized topological space" is what is needed to define suitable topological structure on "the space of sets". (See Section 7 for an outline of the technical details.) Instead, starting with local homeomorphisms, we shall explain how sheaves can be seen as the technical expression of a natural concept (continuous set-valued map) that cannot be derived from the ordinary topological definition of continuity. We shall eventually connect it to geometric logic.

On a matter of notation, if X is a topological space we write ΩX for its topology, its lattice of open sets. Throughout the paper, the word *map* will always be taken to imply continuity.

2.1 Local homeomorphisms

The first definition is that a sheaf over X is a local homeomorphism with X as codomain.

Definition 1 *(See Figure 1.) A map $p : Y \to X$ between topological spaces is a* local homeomorphism *if each $y \in Y$ has an open neighbourhood V for which the image $p(V)$ is open and p homeomorphically maps V to $p(V)$. The domain Y is the* display space *or* espace étalé *for the sheaf. For each $x \in X$, the fibre $p^{-1}(x)$ is the* stalk *at x.*

If $p : Y \to X$ and $q : Z \to X$ are two local homeomorphisms over X, then a morphism between them is a map $f : Y \to Z$ such that $q \circ f = p$. This final condition is equivalent to saying that f maps stalks to stalks: for each x, f will map $p^{-1}(x)$ into $q^{-1}(x)$.

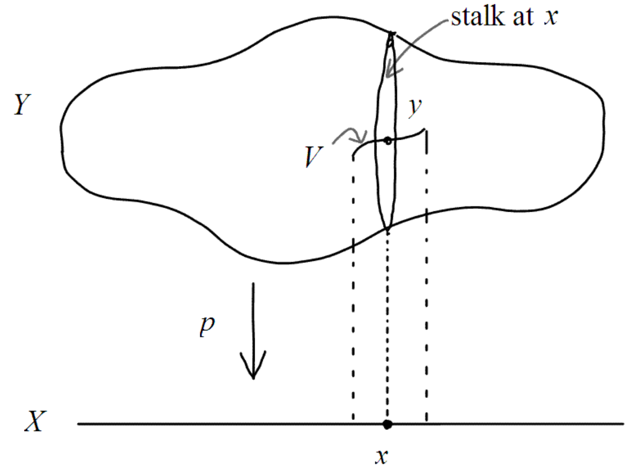

Figure 1: A local homeomorphism.

Proposition 2 *If $p : Y \to X$ is a local homeomorphism then the subspace topology on each stalk is discrete.*

Proof. Let $x \in X$ and let y be in the stalk at x, i.e. $p(y) = x$. Let V be as in the definition. It is open in Y. Since p is 1-1 on V, we get $V \cap p^{-1}(x) = \{y\}$, and so $\{y\}$ is open in the subspace topology on $p^{-1}(x)$. ■

This says that the stalk is a *set* in a strong way – we are not neglecting a topology. Our contention is that the assignment of stalks to points of X is a "continuous set-valued map" on X. Intuitively, continuity of $f : X \to Y$ says that for certain "qualities", if they hold for the result $f(x)$, then they hold for all $f(x')$ with x' in some neighbourhood of x. In the usual definition of continuity, these "qualities" are membership of open sets. Here they are existence and equality of elements of the stalks.

For existence, if $y \in p^{-1}(x)$ then the definition of local homeomorphism might be read as saying y "exists" throughout the neighbourhood $p(V)$ of x, with V giving a continuous choice of stalk elements corresponding to y as x varies.

For equality, suppose we have two such neighbourhoods V and V' of y. Then $V \cap V'$ is also a neighbourhood of y, and p restricted to either V or V' maps it to a neighbourhood of x. Hence we might say that equality of stalk elements at x extends over a neighbourhood $p(V \cap V')$ of x.

The following interesting characterization of local homeomorphism was used by [JT84] to provide a localic definition (see Section 3.2). Here $Y \times_X Y$ denotes the pullback of p against itself, i.e. the *fibred product* $\{(y_1, y_2) \in Y \times Y \mid p(y_1) = p(y_2)\}$ (with subspace topology inherited from $Y \times Y$). Recall also that a map is *open* if direct image preserves openness.

Proposition 3 *Let $p : Y \to X$ be a map of spaces. Then p is a local homeomorphism iff both p itself and the diagonal inclusion $\Delta : Y \hookrightarrow Y \times_X Y$ are open.*

Proof. First, note that $\Delta(Y)$ is open in $Y \times_X Y$ iff every $(y, y) \in \Delta(Y)$ has a basic open neighbourhood in $Y \times_X Y$ that is contained in $\Delta(Y)$, in other words we can find neighbourhoods V_1 and V_2 of y such that $(V_1 \times V_2) \cap (Y \times_X Y) \subseteq \Delta(Y)$. By restricting to $V_1 \cap V_2$ we might as well assume $V_1 = V_2 = V$ (say). The condition $(V \times V) \cap (Y \times_X Y) \subseteq \Delta(Y)$ says that if $y_1, y_2 \in V$ with $p(y_1) = p(y_2)$ then $y_1 = y_2$: in other words, p is 1-1 on V. We can deduce that Δ is open iff every $y \in Y$ has an open neighbourhood V on which p is 1-1.

$\Leftarrow$: If $y \in Y$, choose V as above. p is a continuous bijection from V onto $p(V)$, and since p is open, we deduce that this bijection is a homeomorphism.

$\Rightarrow$: Let W be open in Y. If $y \in W$, then we can find V_y playing the role of V in Definition 1, and $p(W \cap V_y)$ is open. $p(W)$ is the union of these open sets $p(W \cap V_y)$ and hence is open. Hence p is an open map. Openness of Δ follows from what we have already said. ■

Remark 4 *It is worth considering the case $X = 1$, with p the unique map $! : Y \to 1$. From the definition one can see that $!$ is a local homeomorphism iff Y is discrete. Then the proposition gives an alternative characterization of discreteness, that $!$ and the diagonal $\Delta : Y \to Y \times Y$ are both open.*[2]

2.2 Pasting presheaves

The second technical expression of sheaf is via the notion of presheaf. If p is a local homeomorphism, then for each open U in X one can consider the set $\mathrm{Sect}_p(U)$ of *local sections* of p over U, maps $\sigma : U \to Y$ such that $p \circ \sigma$ is the identity on U, so for x in U, σ continuously selects $\sigma(x) \in p^{-1}(x)$. (See Figure 2.)

In fact, these local sections correspond exactly to those opens V that are mentioned in the definition of local homeomorphism.

Proposition 5 *Let $p : Y \to X$ be a local homeomorphism. Then there is a bijection between local sections of p, and opens V of Y that are mapped homeomorphically by p to opens of X.*

Proof. If V is as stated, then the inverse of the homeomorphism is a local section on $p(V)$. Conversely, if $\sigma : U \to Y$ is a local section then we need to show

[2] Actually, ! is open for every space Y. But the corresponding result for locales is false constructively.

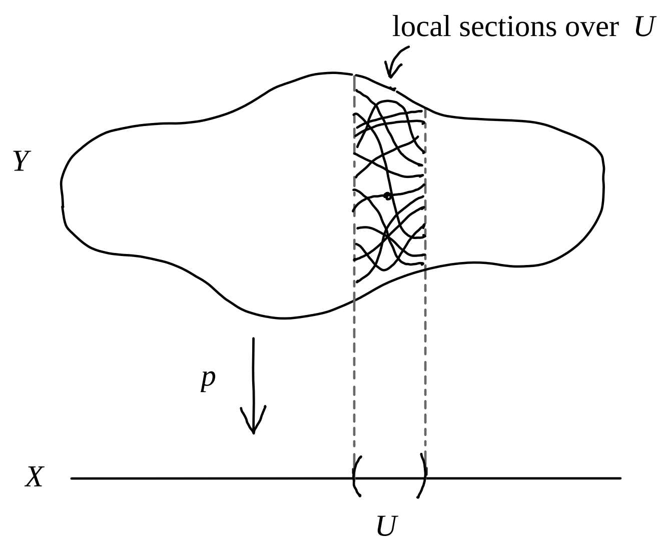

Figure 2: Local sections.

that $\sigma(U)$ is open. If $y = \sigma(x)$, then we can find V as in the definition of local homeomorphism. By intersecting with $p^{-1}(U)$, we can assume without loss of generality that $V \subseteq p^{-1}(U)$, and then it follows that $V \cap p^{-1}(\sigma^{-1}(V)) \subseteq \sigma(U)$: for if $y' \in V$ and $\sigma \circ p(y') \in V$ then $p(y') = p \circ \sigma \circ p(y')$, so $y' = \sigma \circ p(y') \in \sigma(U)$. Hence $\sigma(U)$ is open. ■

If $U \subseteq U'$ then there is a restriction map from $\mathrm{Sect}_p(U')$ to $\mathrm{Sect}_p(U)$ and this makes Sect_p a pasting presheaf according to the following definition.

Definition 6 *A* presheaf *on X is a functor $F : (\Omega X)^{op} \to \mathbf{Set}$.*[3] *If $U \subseteq U'$ and $\sigma \in F(U')$, we write $\sigma|U$ for $F(U \subseteq U')(\sigma)$, the* restriction *of σ to U.*

If F and G are presheaves, then a sheaf morphism from F to G is a natural transformation – that is to say, for each $U \in \Omega X$ a function $\theta_U : F(U) \to G(U)$ such that if $U \subseteq U'$ and $\sigma \in F(U')$ then $\theta_U(\sigma|U) = \theta_{U'}(\sigma)|U$.

F is a pasting *presheaf if it has the following "sheaf pasting" condition. Suppose U_i ($i \in I$) is a family of opens in X, and suppose we have a family of elements $\sigma_i \in F(U_i)$ such that for each pair (i, j), σ_i and σ_j have the same restriction to $U_i \cap U_j$. Then there is a unique $\sigma \in F(\bigcup_i U_i)$ that restricts to every σ_i.*

It is well known that pasting presheaves $F : (\Omega X)^{op} \to \mathbf{Set}$ are equivalent

[3]More generally, for any category $\mathcal{C}$, a contravariant functor from $\mathcal{C}$ to **Set** is called a presheaf on $\mathcal{C}$.

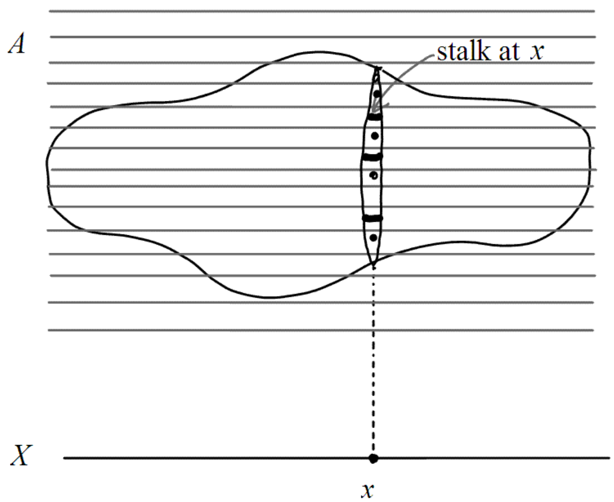

Figure 3: An ΩX set.

to local homeomorphisms over X. We have already seen how a local homeomorphism p gives rise to a pasting presheaf Sect_p. In the opposite direction, from a pasting presheaf F we get a local homeomorphism in which the stalk at x is the colimit of the $F(U)$s as U ranges over the open neighbourhoods of x.

2.3 Frame-valued sets

The definition of sheaf over X that was exploited by Höhle is that of *ΩX-valued set*. (To speak of these in generality, without specifying a particular X, we refer to "frame-valued sets". This anticipates the notion of frame as in Definition 13, of which the topology ΩX is an example.) These originated in unpublished work by Higgs, and his work was subsequently described in [Lou79] and [Sco79].

Definition 7 *Let A be a set. An ΩX-*valuation *on A is a function $E : A \times A \to \Omega X$ satisfying $E(a, b) = E(b, a)$ and $E(a, b) \wedge E(b, c) \leq E(a, c)$.*

This is the simplest of all the definitions of sheaf. Its relationship to the others is seen mostly clearly by describing the stalks. If x is a point of X, then we can define a partial equivalence relation (symmetric and transitive, but not necessarily reflexive) $\sim_x$ on A by $a \sim_x b$ if x in $E(a, b)$, and so we get a set $A/\sim_x$ of equivalence classes. If $x \in E(a, a)$, let us write $[a]_x$ for the equivalence class of a modulo $\sim_x$. Taking these sets $A/\sim_x$ as stalks, their disjoint union is the display space Y, with the evident map $p : Y \to X$, $[a]_x \mapsto x$. If we write $[a]$ for $\{[a]_x \mid x \in E(a,a)\}$, then $E(a,a)$ describes the region on which $[a]$ is defined, and $E(a,b)$ the region on which $[a]$ and $[b]$ are defined and equal. Note that $[a] \cap [b] = [a] \cap p^{-1}(E(a,b))$.

To topologize Y, we take as a subbase the sets $[a]$ and the sets $p^{-1}(U)$ ($U \in \Omega X$): then the sets $[a] \cap p^{-1}(U)$ form a base. p restricted to $[a]$ is then a homeomorphism onto $E(a,a)$, and it follows that p is a local homeomorphism. We also write $[a]$ for the corresponding local section over $E(a,a)$, $x \mapsto [a]_x$.

Although the definition of sheaf as ΩX-valued set is very simple, the corresponding definition of morphism is more complicated.

Definition 8 *Let A and B be two ΩX-valued sets. Then a* morphism *$\theta : A \to B$ is a function $\theta : A \times B \to \Omega X$ satisfying the following conditions.*

$$\begin{aligned} \theta(a,b) &\subseteq E(a,a) \cap E(b,b) \\ E(a',a) \cap \theta(a,b) \cap E(b,b') &\subseteq \theta(a',b') \\ \theta(a,b) \cap \theta(a,b') &\subseteq E(b,b') \\ E(a,a) &\subseteq \bigcup_{b \in B} \theta(a,b) \end{aligned}$$

Proposition 9 *Let A and B be two ΩX-valued sets, with corresponding local homeomorphisms $p : Y \to X$ and $q : Z \to X$. Then there is a bijection between maps $f : Y \to Z$ over X and ΩX-valued set morphisms $\theta : A \to B$.*

Proof. A function $f : Y \to Z$ over X is equivalent to a family of stalk functions

$$f_x : p^{-1}(x) \cong A/\sim_x \quad \to q^{-1}(x) \cong B/\sim_x .$$

Each stalk function is equivalent to a relation $\theta_x \subseteq A \times B$, with $a\theta_x b$ iff $a \sim_x a$ and $f_x([a]_x) = [b]_x$. An arbitrary relation θ_x corresponds to a stalk function in this way iff

$$\begin{aligned} a\theta_x b &\Longrightarrow a \sim_x a \wedge b \sim_x b \\ a' \sim_x a \wedge a\theta_x b \wedge b \sim_x b' &\Longrightarrow a'\theta_x b' \\ a\theta_x b \wedge a\theta_x b' &\Longrightarrow b \sim_x b' \\ a \sim_x a &\Longrightarrow (\exists b) a\theta_x b \end{aligned}$$

Here the first condition says that the relation θ_x relates elements only if their equivalence classes exist (strictness), the second that it relates entire equivalence classes (extensionality), and the third and fourth that it is single-valued and total.

If we define $\theta : A \times B \to \mathcal{P}X$ by $\theta(a,b) = \{x \mid a\theta_x b\}$, then the four conditions above correspond to the four in the definition of morphism. Hence it remains only to show that f is continuous iff every $\theta(a,b)$ is open. For the $\Leftarrow$ direction, note that

$$\begin{aligned} f^{-1}([b]) &= \{[a]_x \mid x \in E(a,a) \wedge f([a]_x) = [b]_x)\} \\ &= \{[a]_x \mid x \in \theta(a,b)\} \\ &= \bigcup_{a \in A} ([a] \cap p^{-1}(\theta(a,b))). \end{aligned}$$

Hence if every $\theta(a,b)$ is open, so is $f^{-1}([b])$ and it follows that f is continuous. For $\Rightarrow$, suppose $x \in \theta(a,b)$, i.e. $x \in E(a,a)$ and $[a]_x \in f^{-1}([b])$. Then we can find a basic open $[a'] \cap p^{-1}(U)$ ($a' \in A$ and $U \in \Omega X$) with

$$[a]_x \in [a'] \cap p^{-1}(U) \subseteq f^{-1}([b]).$$

The left hand part of this says that $x \in E(a,a') \cap U$. The right hand part says that $E(a',a') \cap U \subseteq \theta(a',b)$, hence

$$x \in E(a,a') \cap U \subseteq E(a,a') \cap \theta(a',b) \subseteq \theta(a,b).$$

It follows that $\theta(a,b)$ is open. ■

It can also be calculated that the identity morphism on A is $E : A \times A \to \Omega X$, and composition of $\theta : A \to B$ with $\phi : B \to C$ is defined by

$$(\phi \circ \theta)(a,c) = \bigcup_{b \in B} \theta(a,b) \cap \phi(b,c).$$

2.4 Complete frame-valued sets

In an ΩX-valued set A, each $a \in A$ gives rise to a local section $[a]$ over $E(a,a)$. These are by no means all the local sections, though they are enough to cover all the points of the display space. In some respects it is advantageous to go to *complete* ΩX-sets, in which *all* the local sections are of the form $[a]$.

To see how this works, we analyse the local sections for an arbitrary ΩX-valued set A. An open embedding $U \hookrightarrow X$ is itself a local homeomorphism, corresponding to the valuation E_U on $\{*\}$ such that $E_U(*,*) = U$. It follows from Proposition 9 that sections on U are equivalent to morphisms θ from $(\{*\}, E_U)$ to (A, E). Defining $s(a) = \theta(*, a)$, we find that sections on U are equivalent to those *singletons* (defined as follows) for which $U = \bigvee_{a \in A} s(a)$.

Definition 10 *For an ΩX-valued set A we say that a function $s : A \to \Omega X$ is a* singleton *if it satisfies*

$$\begin{aligned} s(a) &\le E(a,a) \ \ (s \textit{ is } \text{strict}) \\ s(a) \wedge E(a,b) &\le s(b) \ \ (s \textit{ is } \text{extensional}) \\ s(a) \wedge s(b) &\le E(a,b). \end{aligned}$$

(Actually, the third condition implies the first. But we separate them out in order to make explicit the properties of strictness and extensionality.)

If $a \in A$ then we can calculate the singleton $\tilde{a}$ for the section $[a]$ as follows. As a family (θ_x) of relations, we have $*\theta_x b$ iff $[b]_x = [a]_x$, i.e. iff x is in $E(a,b)$. Hence $\tilde{a}(b) = E(a,b)$.

Definition 11 *A is* complete *if for every singleton s there is a unique $a \in A$ such that $s = \tilde{a}$.*

For complete ΩX-valued sets, morphisms can be defined more simply: this is because a morphism $\theta : A \to B$ defines, for each $a \in A$, a singleton $\theta(a,-)$ and hence an element of B. In fact, the morphisms are equivalent to *functions* $\psi : A \to B$ such that $E(a,a') \leq E(\psi(a),\psi(a'))$ and $E(a,a) = E(\psi(a),\psi(a))$.

The equivalence between pasting presheaves F and complete ΩX-valued sets A is quite straightforward: A is the disjoint union of the sets $F(U)$. If $a_i \in F(U_i)$ $(i = 1,2)$, then

$$E(a_1,a_2) = \bigvee\{V \in \Omega X \mid V \subseteq U_1 \cap U_2, a_1|V = a_2|V\}.$$

Every ΩX-valued set can be completed by taking the set of all singletons, and this respects the morphisms. The valuation is then given by $E(s_1,s_2) = \bigvee_{a \in A} s_1(a) \wedge s_2(a)$.

There are some advantages in completing. The morphisms are simpler, and in addition it gives a canonical representation of the sheaf: two complete ΩX-valued sets are isomorphic as sheaves iff they are structurally isomorphic as ΩX-valued sets, whereas incomplete ΩX-valued sets can be structurally quite different but still give isomorphic sheaves. However, the completion process itself is non-trivial. In the light of Section 4, it is non-geometric. In Section 6 we shall see how for uncompleted ΩX-valued sets the geometric constructions are simple to describe.

2.5 Direct and inverse image functors

We shall write $\mathcal{S}X$ for the category of sheaves over X. This is ambiguous, since we have four different definitions of sheaf, and we get four equivalent but non-isomorphic categories. Nonetheless, we shall work with the ambiguous notation, leaving it to be interpreted according to one's current favourite definition.

If $f : X \to Y$ is a map, then we get from it *two* functors between $\mathcal{S}X$ and $\mathcal{S}Y$, forming an adjoint pair. The left adjoint $f^* : \mathcal{S}Y \to \mathcal{S}X$ is the *inverse image functor*, and the right adjoint $f_* : \mathcal{S}X \to \mathcal{S}Y$ is the *direct image functor*. An important property of f^* is that it preserves not only colimits (as does any left adjoint) but also finite limits. (Such an adjoint pair is a *geometric morphism* from $\mathcal{S}X$ to $\mathcal{S}Y$.) A key part of our discussion here will be of the "geometric" constructions, those that are preserved by every f^*.

It turns out that the ease of constructing f^* and f_* depends on which definition of sheaf one is using.

For pasting presheaves, f_* is easy. If $F : (\Omega X)^{op} \to \mathbf{Set}$ is a pasting presheaf, then $f_*(F)$ is got by composing with $f^{-1} : \Omega Y \to \Omega X$. Explicitly, $f_*(F)(V) =$

$F(f^{-1}(V))$. From the way a complete frame-valued set A comprises the local sections in a pasting presheaf, f_* can also be easily calculated for these as a pullback along f^{-1}.

$$\begin{aligned} f_*(A) &= \{(a,V) \in A \times \Omega Y \mid E(a,a) = f^{-1}(V)\} \\ E((a_1,V_1),(a_2,V_2)) &= \bigvee \{V \leq V_1 \wedge V_2 \mid f^{-1}(V) \leq E(a_1,a_2)\}. \end{aligned}$$

On the other hand, f^* is harder for these, as it involves a completion step (or "sheafification" for the presheaves).

For local homeomorphisms, f^* is easy, constructed as a pullback

$$\begin{array}{ccc} f^*(Z) & \longrightarrow & Z \\ f^*p \downarrow & & \downarrow p \\ X & \underset{f}{\longrightarrow} & Y \end{array}$$

This uses the following result.

Proposition 12 *The local homeomorphism property is preserved under pullback.*

Proof. From Proposition 3, we can deduce by purely categorical arguments that if open maps are preserved under pullback then so are local homeomorphisms. To prove the property for open maps, suppose that in the diagram above p is open. To show that f^*p is open, it suffices to consider the direct image of basic opens $(U \times W) \cap f^*(Z)$ where U and W are opens in X and Z. But

$$\begin{aligned} (f^*p)((U \times W) \cap f^*(Z)) &= \{x \in U \mid (\exists z \in W) f(x) = p(z)\} \\ &= U \cap f^{-1}(p(W)), \end{aligned}$$

which is open. ■

Note that stalks are a special case of these pullbacks. A point of Y is a map $y : 1 \to Y$, and the stalk at y is $y^*(Z)$. Now consider a point x of X. The two pullbacks fit together to make another pullback:

$$\begin{array}{ccccc} x^*(f^*(Z)) & \longrightarrow & f^*(Z) & \longrightarrow & Z \\ \downarrow & & f^*p \downarrow & & \downarrow p \\ 1 & \underset{x}{\longrightarrow} & X & \underset{f}{\longrightarrow} & Y \end{array}$$

It follows that the stalk of f^*p at x is isomorphic to the stalk of p at $f(x)$, and hence f^* matches the idea of sheaf as continuous set-valued map: from this point of view, f^* is just composition with f.

Finally, f^* is easy for frame-valued sets (not necessarily complete). If (B, E_Y) is an ΩY-valued set, then $f^*(B)$ is B again, with ΩX-valuation

$$E_X(b_1,b_2) = f^{-1}(E_Y(b_1,b_2)).$$

Note how this matches the stalks. The stalk $B/\sim_x$ of (B, E_X) over x is isomorphic to the stalk $B/\sim_{f(x)}$ of (B, E_Y) over $f(x)$, since

$$\begin{aligned} b_1 \sim_{f(x)} b_2 &\Leftrightarrow f(x) \in E_Y(b_1, b_2) \Leftrightarrow x \in f^{-1}(E_Y(b_1, b_2)) = E_X(b_1, b_2) \\ &\Leftrightarrow b_1 \sim_x b_2. \end{aligned}$$

3 Locales

3.1 Background on locales

Standard references for frames and locales are [Joh82], [Vic89] and [Pul03]. For the topos-theoretic account of constructive locales see [JT84].

Definition 13 *A* frame *is a complete lattice in which binary meet distributes over arbitrary joins. A* frame homomorphism *is a function between frames that preserves finite meets and arbitrary joins. We write* **Fr** *for the category of frames and frame homomorphisms. We generally write* $\top$ *and* $\bot$ *for the top and bottom elements of a frame.*

Frames embody the idea of "point-free topology". A frame is intended to be a "lattice of opens", except that these opens are not specified as subsets of a given set of points. Points of frames are nonetheless defined, but for some frames there are not enough of them to distinguish between all the opens – frames need not be *spatial*.

Example 14 *A typical example is the frame of regular opens of the reals* $\mathbb{R}$*. This is the image of* $\Omega\mathbb{R}$ *under a frame homomorphism that maps* $U \in \Omega\mathbb{R}$ *to the interior of its closure, which we may denote by* $\neg\neg U$*. (Note that to calculate joins in this frame we must apply* $\neg\neg$ *to the union.) A point would be a real number* x *such that for every open* U *of the reals, if* x *is in* $\neg\neg U$ *then it is already in* U*. But this is not possible – consider* $U = (x-1, x) \cup (x, x+1)$*, for which* $\neg\neg U = (x-1, x+1)$*.*

One might wonder therefore what virtue there is in the point-free approach to topology: not only does it obfuscate the topology by converting it to lattice theory, it does not even capture the established theory. However, it turns out that in constructive mathematics (for example, in the topos-valid mathematics one can obtain by replacing sets by sheaves) it gives a theory that is better behaved than point-set topology, retaining classical theorems such as Heine-Borel and Tychonoff that otherwise are lost.

We shall use the language of *locales*. For present purposes, we may think of a locale as "a frame pretending to be a topological space" and define the category **Loc** of locales to be the opposite of the category of frames. That is to say, the objects are the same, but a morphism (a *continuous map*, or just *map*) $f : X \to Y$ between locales is a frame homomorphism (the *inverse image function*) in the opposite direction. We shall write ΩX for the frame corresponding to X, and $\Omega f : \Omega Y \to \Omega X$ for the frame homomorphism. The purpose of this duplication of notation is to allow us to use a language that supports spatial intuitions in point-free topology.

The Ω may seem otiose if the locale X just "is" the frame ΩX. However, it makes it immediately apparent what kind of morphisms are being used. $f : X \to Y$ is a locale map, $f : \Omega X \to \Omega Y$ is a function between frames, and $f : X \to \Omega Y$ is a type error.

It is usual to define a *point* of a locale X to be a map $1 \to X$, where $\Omega 1$ (or just Ω, the same notation as for the subobject classifier in a topos) is the frame of truth-values. However, we shall generalize this: a *point at stage* W (sometimes emphasized as a *generalized* point) is a map $W \to X$. Then the ordinary points (at stage 1) are called *global* points.

The idea that an *arbitrary* map into X should be considered a "generalized point" of X may seem reckless at first. Of course, there are other, quite different, ways to view maps – for example, when one views a local homeomorphism $p : Y \to X$ as a set (the stalk) parameterized by a point of X. The view as generalized point matches the fact (developed in Section 4.1) that points of a locale X are models of a logical theory (in propositional geometric logic). When the models are sought in the internal logic of sheaves over W, it turns out that they are equivalent to maps $W \to X$.

Composition with a map $f : X \to Y$ transforms points of X to points of Y, just as with an ordinary continuous map. Can one still go in the opposite direction, from point transformer to locale map? If one just considers global points here, then there is a problem from non-spatiality. That is to say, the action on global points does not in general define the locale map. This is clearest in those locales that are non-trivial, but have no global points at all.

However, composition with f also transforms points at any given stage W. Now, we do have enough points. In fact, consider the *generic* point, the identity map $\mathrm{Id} : X \to X$, which is a point at stage X. Transforming this immediately gives f, as a point of Y at stage X. An additional property of this point transformation is that it respects *change of stage*. Suppose we have $\alpha : W_1 \to W_2$. Then composition with α transforms points at stage W_2 to points at stage W_1. Associativity of composition, i.e. $f \circ (x \circ \alpha) = (f \circ x) \circ \alpha$, says that the point transformer f commutes with change of stage.

In general, suppose we have a point transformer that commutes with change of stage. More precisely, –

1. For each stage W, we have a function F_W that transforms points $x : W \to X$ of X to points $F_W(x) : W \to Y$ of Y (both at stage W).

2. If $\alpha : W_1 \to W_2$, then $F_{W_2}(x) \circ \alpha = F_{W_1}(x \circ \alpha)$.

Let $f = F_X(\mathrm{Id}) : X \to Y$. Then if $x : W \to X$ we have

$$F_W(x) = F_W(\mathrm{Id} \circ x) = F_W(\mathrm{Id}) \circ x = f \circ x.$$

From this it follows that morphisms are equivalent to "generalized point transformers that commute with change of stage".

This is in fact not at all deep. It is a completely general argument that applies in any category, and derives from the notion of "generalized element" in categorical logic. For locales, however, we shall later see how the logical principle of "geometricity" can provide a guarantee that a point transformer commutes with change of stage.

To illustrate already some of the usefulness of generalized points, consider as an example *pullbacks* (fibred products).

$$\begin{array}{ccc} Y \times_X Z & \rightarrow & Z \\ \downarrow & & \downarrow g \\ Y & \underset{f}{\rightarrow} & X \end{array}$$

For spaces, the points of $Y \times_X Z$ are those pairs $(y, z) \in Y \times Z$ such that $f(y) = g(z)$, and then the topology is the coarsest such that the projections to Y and X are both continuous. For locales, the frame-theoretic description of the pullback is complicated. But it does exist, and the universal characterization of pullback then says precisely that, at any stage W, the points of $Y \times_X Z$ are those pairs (y, z) such that $f \circ y = g \circ z$ where y, z are points of Y and Z respectively. Note that there is no need to mention the topology explicitly – it is enough to describe the points. For example, the projection maps to Y and Z are *defined* as the pair of points at stage $Y \times_X Z$ that corresponds to the generic point of $Y \times_X Z$. They are automatically continuous, because all locale maps are.

3.2 Sheaves over locales

Let X be a locale. Clearly the presheaf and ΩX-valued set definitions of sheaf transfer directly from spaces to locales, since they are expressed in terms of the frame ΩX and do not mention points. The same goes for the morphisms between sheaves in those styles, and for the interconversion between the styles.

It is less clear that the local homeomorphism definition also transfers, and one might be tempted to give up on them – surely, three different definitions of sheaf are already more than adequate. But the local homeomorphisms are a rather interesting case because of our intuition that the stalk map is a "continuous set-valued map". Fortunately, there is a good localic definition based on Proposition 3. This relies on having a localic definition of open map. A good development of these ideas can be found in [JT84]; we have sketched proofs in order to provide some acquaintance with the techniques underlying these key ideas of locale theory.

Definition 15 *A locale map* $p : Y \to X$ *is* open *if* Ωp *has a left adjoint* $\exists_p : \Omega Y \to \Omega X$ *satisfying the* Frobenius condition

$$\exists_p(V \wedge \Omega p(U)) = \exists_p(V) \wedge U.$$

p *is a* local homeomorphism *if both* p *and the diagonal* $\Delta : Y \hookrightarrow Y \times_X Y$ *are open.*

If $p : Y \to X$ and $q : Z \to X$ are two local homeomorphisms over X, then a morphism between them is a map $f : Y \to Z$ such that $q \circ f = p$.

Though the definition of open map looks abstract, it turns out that it is equivalent to the property that direct images of open sublocales are open. $\exists_p$ describes the direct images of opens. The adjunction $\exists_p \dashv \Omega p$ is equivalent to $\Omega p(\exists_p(V)) \geq V$ and $\exists_p(\Omega p(U)) \leq U$ for all $V \in \Omega Y$, $U \in \Omega X$.

Remark 16 *Open* inclusions *into X are equivalent to opens of X. (A map p is an inclusion if Ωp is surjective. These are also known as* sublocales *of X.) If $i : U \hookrightarrow X$ is an open inclusion, then the corresponding open is $\exists_i \top \in \Omega X$. Given $U \in \Omega X$, it can be made into a locale by defining ΩU to be the downset $\downarrow U$ of U in ΩX, and defining $i : U \hookrightarrow X$ by $\Omega i(U') = U \wedge U'$, $\exists_i U'' = U''$. As we shall see (Proposition 21), open inclusions are also local homeomorphisms and (Corollary 22) correspond to subsheaves of* 1.

The following two propositions are of central importance, as can be seen from [JT84]. They are the localic analogues of Remark 4 and Proposition 12.

Proposition 17 *A locale X is discrete (i.e. its frame is isomorphic to a powerset) iff the unique map $! : X \to 1$ is a local homeomorphism.*

Proof. We sketch a proof. A more detailed proof is given for Theorem 29, which deals with local homeomorphisms with arbitrary codomain.

$\Rightarrow$: Suppose $\Omega X = \mathcal{P}A$. (We might as well assume equality here.) The functions $\exists_! : \mathcal{P}A \to \Omega$ and $\exists_\Delta : \mathcal{P}A \to \mathcal{P}A \otimes \mathcal{P}A \cong \mathcal{P}(A \times A)$ are expected to give direct images, so they are defined by

$$\exists_!(S) = \textbf{true} \text{ if } S \text{ is inhabited}$$
$$\exists_\Delta(S) = \Delta(S) = \{(a, a) \mid a \in S\}.$$

$\Leftarrow$: Since the topology is supposed to be discrete, one must identify those opens that are singletons. Thinking spatially, we seek opens $a \in \Omega X$ such that if $x, y \in a$ then $x = y$: in other words, $a \times a \leq \exists_\Delta \top$. These are to be the opens with at *most* one element. On the other hand, the opens with at *least* one element, in other words the "positive" opens, are those opens a satisfying $\exists_! a$. (The fact that $\exists_! : \Omega X \to \Omega$ serves as a *positivity predicate* on ΩX is brought out well in [Joh84] and is also a standard feature in predicative formal topology.) We define A to be the set of opens a satisfying both those conditions. It remains to show that every open U is uniquely expressible as a join of elements of A. ■

Proposition 18 *The open property of locale maps is preserved under pullback. In other words, if p in the following pullback diagram is open, then so is f^*p.*

$$\begin{array}{ccc} f^*(Z) = X \times_Y Z & \stackrel{p^*f}{\longrightarrow} & Z \\ f^*p \downarrow & & \downarrow p \\ X & \underset{f}{\longrightarrow} & Y \end{array}$$

Moreover, the diagram satisfies the following Beck-Chevalley *condition, that for every* $W \in \Omega Z$

$$\exists_{f^*p}(\Omega p^* f(W)) = \Omega f(\exists_p(W)).$$

Proof. [4]By analogy with the proof of Proposition 12, we should like to define

$$\exists_{f^*p}(U \otimes W) = U \wedge \Omega f(\exists_p(W)), \qquad (1)$$

where $U \otimes W$ denotes the image of $U \times W$ in $\Omega(X \times_Y Z)$. The function $\exists_{f^*p}$, as a left adjoint, is expected to preserve all joins, but not necessarily finite meets. There is therefore a non-trivial technical question about how to define this function from the frame pushout $\Omega(X \times_Y Z)$. Equation 1 suffices to determine $\exists_{f^*p}$ on the whole of $\Omega(X \times_Y Z)$, since an arbitrary element is a joins of ones of the form $U \otimes W$; but there is still the question of whether it is well-defined. It turns out that $\Omega(X \times_Y Z)$, a *pushout* with respect to frame homomorphisms, is a *tensor product* with respect to join-preserving functions. To check the well-definedness of Equation 1 it suffices to check (i) it preserves joins in U, (ii) it preserves joins in W, and (iii) it gives the same answer for $(U \wedge \Omega f(V)) \otimes W$ as for $U \otimes (\Omega p(V) \wedge W)$. (These should be familiar from the analogous conditions for the tensor products of vector spaces.) For a justification see [JT84]. It can also be derived as a consequence of Johnstone's description [Joh82] of the frame of C-ideals, given a site.[5]

Once we know that $\exists_{f^*p}$ is well defined, it is readily checked that it has the properties needed to give openness of f^*p. The Beck-Chevalley condition follows from

$$\exists_{f^*p}(\Omega p^* f(W)) = \exists_{f^*p}(\top \otimes W) = \top \wedge \Omega f(\exists_p(W)).$$

■

Proposition 19 *The local homeomorphism property of locale maps is preserved under pullback.*

Proof. Just as in the spatial case (Proposition 3), we can deduce by purely categorical arguments that if open maps are preserved under pullback then so are local homeomorphisms. Let us be explicit about these categorical arguments this time. Suppose we have a pullback diagram as in the statement of Proposition 18 with p a local homeomorphism. By Proposition 18, f^*p is open. We still want $\Delta : f^*(Z) \to f^*(Z) \times_X f^*(Z)$ to be open; but it is obtained from a pullback

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{p^* f} & Z \\ \Delta \downarrow & & \downarrow \Delta \\ (X \times_Y Z) \times_X (X \times_Y Z) & \underset{p^* f \times p^* f}{\longrightarrow} & Z \times_Y Z \end{array}$$

[4]There is a more geometric proof using *powerlocales*, i.e. localic hyperspaces. It is based on a result in [Vic95], that the unique map $! : X \to 1$ is open iff the lower powerlocale $P_L X$ has a top point in a suitably strong sense. Using the hyperspace nature of $P_L X$, this top point corresponds to X as sublocale of itself. The idea is explained further in other papers such as [Vic04] and [Vic09].

[5]The technology is developed extensively for other kinds of functions beyond those that preserve joins; see e.g. [Vic04].

so that openness of the left-hand Δ follows from that of the right-hand Δ.

Proving that this is a pullback can be done by categorical diagram chasing, but it is illuminating to do it by considering generalized points as mentioned in Section 3.1. A point of $X \times_Y Z$ is a pair (x, z) of points of X and Z, such that $f(x) = p(z)$; then a point of $(X \times_Y Z) \times_X (X \times_Y Z)$ is a triple (x, z_1, z_2) such that $f(x) = p(z_1) = p(z_2)$. (Actually, we get two X-components, but they are required to be equal.) Now a point of the pullback we are calculating is a quadruple (x, z_1, z_2, z) such that (x, z_1, z_2) is in $(X \times_Y Z) \times_X (X \times_Y Z)$, z is in Z, and $(z_1, z_2) = \Delta(z) = (z, z)$. Hence $z_1 = z = z_2$ and the points are equivalent to pairs (x, z) in $X \times_Y Z$. ■

With the localic definition, local homeomorphisms are again equivalent to pasting presheaves. In one direction, from local homeomorphism p to pasting presheaf Sect_p, the construction follows the spatial idea. If $U \in \Omega X$, then by Remark 16 the open U corresponds to an open inclusion $U \hookrightarrow X$, and we define the local sections over U to be the locale maps $\sigma : U \to Y$ such that $p \circ \sigma$ is the inclusion. $\mathrm{Sect}_p(U)$ is the set of local sections over U.

The elegant argument that completes the proof that local homeomorphisms are equivalent to pasting presheaves can be found in [JT84]. First, it relies on the standard topos-theoretic result that $\mathcal{S}X$, understood specifically as the category of pasting presheaves over X, is a topos. That gives access to the ability to internalize mathematics in $\mathcal{S}X$, provided the mathematics is conducted intuitionistically. In particular that gives an internal treatment of frames and locales, and it turns out that an internal locale in $\mathcal{S}X$ is equivalent to an external locale map with codomain X (a "locale over X"). They give an intuitionistic proof that internally discrete locales are equivalent to external local homeomorphisms. But "discrete" means the frame is isomorphic to a powerset (this is where the importance of Proposition 17 comes in), and it follows that internally discrete locales are equivalent to objects of $\mathcal{S}X$, i.e. pasting presheaves.

In Section 5 we shall give a more direct proof of the equivalence between local homeomorphisms over X and ΩX-valued sets.

Change of base functors can be calculated using formulae analogous to those in Section 2.5. Suppose $f : X \to Y$ is a locale map. For pasting presheaves, f_* is got using composition with Ωf, explicitly $f_*(F)(V) = F(\Omega f(V))$. For complete frame-valued sets A, f_* can be calculated as a pullback along Ωf:

$$f_*(A) = \{(a, V) \in A \times \Omega Y \mid E(a, a) = \Omega f(V)\}$$
$$E((a_1, V_1), (a_2, V_2)) = \bigvee \{V \leq V_1 \wedge V_2 \mid \Omega f(V) \leq E(a_1, a_2)\}.$$

For local homeomorphisms (see Section 3.2), and making use of Proposition 19, f^* is a pullback

$$\begin{array}{ccc} f^*(Z) & \longrightarrow & Z \\ f^*p \downarrow & & \downarrow p \\ X & \underset{f}{\longrightarrow} & Y \end{array}$$

Finally, for frame-valued sets (not necessarily complete), if (B, E_X) is an ΩY-valued set, then $f^*(B)$ is B again, with ΩX-valuation $E_X(b_1, b_2) = \Omega f(E_Y(b_1, b_2))$.

3.3 Key results on local homeomorphisms

We summarize here some miscellaneous and well known facts about sheaves over locales, especially when considered as local homeomorphisms.

Proposition 20 *1. Any homeomorphism is a local homeomorphism.*

2. *The identity map* $\mathrm{Id}_X : X \to X$ *is a terminal object in the category of local homeomorphisms over* X*.*

3. *The composite of two local homeomorphisms is again a local homeomorphism.*

4. *The locale pullback of two local homeomorphisms over* X *is again a local homeomorphism over* X*.*

5. *Let* $q : Z \to X$ *and* $p : Y \to X$ *be local homeomorphisms, and* $f : Z \to Y$ *a map over* X*. Then* f *is a local homeomorphism.*

Proof. (1), (2) are easy.

(3): Let $q : Z \to Y$ and $p : Y \to X$ be maps. It is not hard to verify that if they are both open, then so is $p \circ q$. Now suppose they are both local homeomorphisms. The diagonal $\Delta : Z \to Z \times_X Z$ is a composite of $\Delta : Z \hookrightarrow Z \times_Y Z$, which is open by hypothesis, and the map $(q \times_X q)^*\Delta$, which is open as a pullback of an open.

$$\begin{array}{ccc} Z \times_Y Z & \longrightarrow & Y \\ (q \times_X q)^*\Delta \downarrow & & \downarrow \Delta \\ Z \times_X Z & \underset{q \times_X q}{\longrightarrow} & Y \times_X Y \end{array}$$

(To see that the diagram is a pullback, consider generalized points. A point of the pullback is a triple (z_1, z_2, y) such that $p(q(z_1)) = p(q(z_2))$ and $(q(z_1), q(z_2)) = (y, y)$, and this is equivalent to a pair (z_1, z_2) such that $q(z_1) = q(z_2)$.)

(4): Consider the pullback

$$\begin{array}{ccc} Y \times_X Z & \overset{q^*p}{\longrightarrow} & Z \\ p^*q \downarrow & & \downarrow q \\ Y & \underset{p}{\longrightarrow} & X \end{array}$$

The map from $Y \times_X Z$ to X is the composite $p \circ (p^*q)$. Using Proposition 19, both of these are local homeomorphisms; hence, by part (3), so is the composite.

(5): We can decompose f as a composite of local homeomorphisms,

$$Z \cong Z \times_Y Y \overset{Z \times_Y \Delta}{\longrightarrow} Z \times_Y Y \times_X Y \cong Z \times_X Y \overset{p^*q}{\longrightarrow} Y.$$

We can see these compose to f by considering (generalized) points,

$$z \longmapsto (z, f(z)) \longmapsto (z, f(z), f(z)) \longmapsto (z, f(z)) \longmapsto f(z).$$

Note that $\Delta : Y \to Y \times_X Y$ is a local homeomorphism. It is open from the fact that p is a local homeomorphism, and we have $Y \times_{(Y \times_X Y)} Y \cong Y \times_X Y$. ■

The following proposition is vital for analysing subsheaves.

Proposition 21 *Let $p : Y \to X$ be a local homeomorphism (considered as sheaf over X). Then subsheaves of p are equivalent to opens of Y.*

Proof. From Remark 16 we know that opens of Y are equivalent to open inclusions $i : V \hookrightarrow Y$. Inclusions are monic, so by considering the pullback square that defines $V \times_Y V$ we can deduce that the two projections $p_i : V \times_Y V \to V$ are equal to each other and inverse to $\Delta : V \to V \times_Y V$. Hence Δ, being a homeomorphism, is a local homeomorphism. Hence i is a local homeomorphism. The locale map i thus gives a morphism of local homeomorphisms over X, from $p \circ i$ to p. Since i is an inclusion as locale map, it must also be monic in the category of local homeomorphisms over X. Hence we have a subsheaf of p.

Now let $q : Z \to X$ be a local homeomorphism with a monic sheaf morphism $i : Z \to Y$ over X. By Proposition 20 (5), i is a local homeomorphism (and hence open). The pullback diagram

$$\begin{array}{ccc} Z \times_Y Z & \xrightarrow{p_2} & Z \\ p_1 \downarrow & & \downarrow i \\ Z & \xrightarrow[i]{} & Y \end{array}$$

is also a pullback diagram of sheaves over X, and so by sheaf monicity of i we deduce that $p_1 = p_2$. Now from Proposition 18 (the Beck-Chevalley condition and Equation 1) we see

$$\Omega i(\exists_i(W)) = \exists_{p_1}(\Omega p_1(W)) = \exists_{p_1}(W \otimes \top) = W \wedge \Omega i(\exists_i(\top)) = W$$

and it follows that Ωi is surjective, so i is an inclusion. ■

Corollary 22 *Let X be a locale. Then subsheaves of the terminal sheaf over X are equivalent to opens of X.*

3.4 Sheaves as set-valued maps

Now let us return to the idea that a sheaf over X (in the form of local homeomorphism) is a set-valued map on X. If $p : Y \to X$ is a local homeomorphism (between locales) and $x : 1 \to X$ is a global point of X, then we can construct the stalk $p^{-1}(x)$ using a pullback

$$\begin{array}{ccc} p^{-1}(x) & \longrightarrow & Y \\ x^*p \downarrow & & \downarrow p \\ 1 & \xrightarrow[x]{} & X \end{array}$$

By Proposition 19 x^*p is a local homeomorphism, and then by Proposition 17 $p^{-1}(y)$ is a discrete locale. Thus again the stalks provide an assignment of sets to points of X.

Since X might not have enough global points, it by now seems less plausible that a sheaf can be sensibly viewed as a continuous set-valued map. The above argument for a global point x also works for generalized points, but only provided that we think of "sets at stage W" as the sheaves over W. Each point $x : W \to X$ gives us by pullback a "generalized stalk" $x^*p : W \times_X Y \to W$. Although in this generalized sense the stalk is not a set in the standard sense, as a local homeomorphism it plays the role of "set" in the internal mathematics of sheaves over W.

However, so far the argument is completely circular: we try to present the intuition that a sheaf over X is a map from points of X to sets; yet when it comes to the generalized points $x : W \to X$ (which we need because the possible non-spatiality means there might not be enough global points) we find that "set" has to be interpreted as sheaf over the stage W. Hence in order to motivate sheaves as set-valued maps, we apparently have to presuppose a technical definition of sheaf in order to explain the generalized notion of set. Since an important generalized point is the generic point $\mathrm{Id}_X : X \to X$ at stage X, we end up using sheaves over X in order to explain sheaves over X.

The central subtlety lies in how we should think of a "set-valued map". It is wrong to think of it *extensionally* as somehow listing, for each point (whether global or generalized), what the corresponding set is. With that approach, if we just use global points then we may have too few, while if we use generalized points then we beg the question of what generalized sets are. Instead we should think of a set-valued map *intensionally*, as providing a uniform description of how the set is constructed from the point.

Every point of a locale X can be described as a certain structure of sets. For example, a point of the locale $\mathbb{R}$ of reals (as described, for example, in [Joh82]) can be described as a pair of subsets of the rationals $\mathbb{Q}$, satisfying properties to constrain them to form a Dedekind cut. (See, e.g., [Vic07] for a detailed discussion of this.) From these two sets one can construct other sets in a uniform way. If this is done formally, one ends up with the category of "sets that can be constructed out of a generic real number".[6] This, then, is the idea of set-valued map: a sheaf over X is, modulo equivalence, a formal way of constructing sets out of points of X.

To make this precise, the central question is what set constructions are allowed. They are the "geometric" constructions examined in the following section (and specifically Section 4.2).

[6]Note that the formality referred to here is not that of set theory, with a logical account of $\in$ as a binary relation on the universe of sets. Rather, our manipulation of sets uses mathematical constructions such as product, disjoint union, etc. that are characterized up to isomorphism by universal properties. It is formalized using category theory rather than set theory.

4 Geometric logic

The geometric constructions mentioned at the end of the previous section are best approached though a particular "geometric" logic, to which we now give an introduction. For fuller details, see [MR77], [Joh02b] or (for a conceptual overview) [Vic07].

Geometric logic is a positive logic matched to topological structure. For example, the logical connectives in its propositional fragment are finite conjunction and arbitrary disjunction, matching the finite intersections and arbitrary unions with which one can combine open sets. A far-reaching consequence of having arbitrary disjunctions is that the logic goes along with a certain notion of "geometric type construction" (Section 4.2), and our claim here is that this is the essence of continuity. For locales at least, a continuous map $X \to Y$ is a geometric construction of points of Y out of points of X. This is surprising for two reasons. First, no continuity proof is needed. The geometricity constraints mean that geometric constructions are intrinsically continuous. In a sense, obeying those constraints means foregoing the ability to construct discontinuous functions. Second, it applies even though there might not be enough points. We then generalize this to the situation where we might be constructing not points of a locale but more elaborate set-based structures, and this will include the notion of sheaf as continuous set-valued map.

Let us outline the main argument of this section.

In Section 4.1 we describe propositional geometric theories. These are equivalent to presentations of frames by generators and relations (see, e.g., [Vic89]), and each open (in the frame) may be seen as a way to transform the ingredients of a point geometrically – using joins and finite meets – into a truth value (or, equivalently, a subsingleton set).

In Section 4.2 we generalize to predicate geometric theories and describe how out of the logic we also obtain a notion of geometric *type constructors.*

In Section 4.3 we outline the notion of *classifying topos.* This is central to our argument, because it generalizes the idea of presenting a frame. The frame comprises the formal ways to construct a subsingleton set geometrically out of a point; in the classifying topos this is generalized to constructing arbitrary sets geometrically, using the geometric type constructors. It is fundamental to topos theory that for any propositional geometric theory, the classifying topos is equivalent to the topos of sheaves over the frame presented. Thus sheaves become ways to construct sets out of points.

In Section 4.4 we bring this to bear on constant sheaves, and Höhle's identification of their subsheaves with fuzzy sets.

4.1 Propositional geometric logic

Let Σ be a propositional signature, i.e. a set of propositional symbols. A geometric *formula* is built out of them using finitary conjunction ($\wedge$) and arbitrary disjunction ($\bigvee$).[7] We shall not go into the logical rules here, but they include distributivity of $\wedge$ over $\bigvee$ and are enough to ensure that each formula is equivalent to one expressed as a disjunction of finite conjunctions of propositional symbols.

Compared with ordinary classical logic, geometric logic lacks implication ($\rightarrow$) as a connective in formulae and so cannot be presented in the Hilbert style. A geometric theory is a set not of sentences (formulae with no free variables), as in classical logic, but of geometric *sequents* of the form $\phi \rightarrow \psi$, where ϕ and ψ are geometric formulae. A geometric *theory* is a pair (Σ, T) where Σ is a signature and T a set of sequents – or we say that T is a theory *over* Σ.

The notion of *model* will be important here. A model first requires each propositional symbol in Σ to be interpreted as a truth value. That interpretation can then be extended to arbitrary formulae ϕ in an obvious way, and then the interpretation is a *model* of (Σ, T) if, for every sequent $\phi \rightarrow \psi$ in T, if ϕ is interpreted as **true** then so is ψ.

In fact a geometric theory (Σ, T) is structurally the same as a presentation $\mathrm{Fr}\langle \Sigma \mid T \rangle$ of a frame algebraically using generators (the propositional symbols in the signature Σ) and relations (the sequents in the theory T). A model of the theory (Σ, T) is exactly what is required to define a frame homomorphism from $\mathrm{Fr}\langle \Sigma \mid T \rangle$ to Ω (by a function $\Sigma \rightarrow \Omega$ that respects the relations), and so is the same as a global point of the locale $[\Sigma, T]$ defined by $\Omega[\Sigma, T] = \mathrm{Fr}\langle \Sigma \mid T \rangle$. Hence one can think of a locale as "the space of models for a propositional theory".

The notion of model can be generalized. A standard model interprets propositional symbols as truth values, in Ω. But this still makes sense if the symbols are interpreted in any other frame ΩW, and for each sequent $\phi \rightarrow \psi$ in T the interpretation of ϕ (as element of ΩW) is below that of ψ. Then the models of (Σ, T) in ΩW are the same as frame homomorphisms $\Omega[\Sigma, T] \rightarrow \Omega W$, i.e. locale maps $W \rightarrow [\Sigma, T]$, i.e. generalized points of $[\Sigma, T]$ at stage W.

Hence we can say in generality that *the points of the locale* $[\Sigma, T]$ *are the models of* (Σ, T).

An important generalized model of (Σ, T) is the *generic* point, at stage $[\Sigma, T]$ (see Section 3.1). As locale map, this is the identity map on $[\Sigma, T]$, and it corresponds to the injection of generators $\Sigma \rightarrow \Omega[\Sigma, T]$.

Let us now look more carefully at the opens. The following definition will be important.

Definition 23 *The* Sierpiński *locale* $\mathbb{S}$ *is presented by the geometric theory with one generator* P *and no relations.*

It has (in classical mathematics) two global points, got by interpreting P as either **true** or **false**, and three opens, $\bot$ (bottom), P and $\top$ (top). A point of $\mathbb{S}$ at stage W (a map $W \rightarrow \mathbb{S}$), in other words a model of the theory in ΩW, is just an open of W. That is equivalent to a subsheaf of 1 over W, or, in the internal mathematics of sheaves over W, a subsingleton set. But truth values ϕ are also

[7] [MLM92] do not allow for infinitary disjunctions. What they define as geometric logic is not the full generality, but *coherent* logic.

equivalent to subsingleton sets $\{* \mid \phi\} \subseteq \{*\}$, and this equivalence is still valid in the intuitionistic mathematics of sheaves. (Truth values are elements of the subobject classifier, and they are equivalent to subobjects of 1.) Thus quite generally we may think of the points of $\mathbb{S}$ as being truth values, or subsets of 1.

Now the opens of $\Omega[\Sigma, T]$ are built up from the generators using joins and finite meets: they are the geometric formulae modulo provable equivalence. (Logically, the frame is the *Lindenbaum algebra* for the theory.) The generators are the ingredients that define a point (as model of (Σ, T)), so we may think of the opens as formal geometric constructions of truth values – i.e. subsingleton sets – out of points, thus partially fulfilling the ambitions set out in Section 3.4. This seems to match well an intuition that opens, as maps $W \to \mathbb{S}$, map points of W to points of $\mathbb{S}$, i.e. subsingleton sets. In fact, this is an instance of a very general principle.

Consider maps $[\Sigma_1, T_1] \to [\Sigma_2, T_2]$. These are models of (Σ_2, T_2) in $\Omega[\Sigma_1, T_1]$. But all the ingredients of such a model are made geometrically (i.e. using $\wedge$ and $\bigvee$) from the symbols of Σ_1 and one deduces that maps are equivalent to geometrically defined transformations of models of (Σ_1, T_1) into models of (Σ_2, T_2). To define a map of locales $f : X \to Y$ we declare "let x be a point of X" (technically this is then going to be the generic point of X) and then, geometrically, define a point $f(x)$ of Y. Maps can be defined pointwise, even if X does not have enough global points, and no continuity proof is needed!

Comparing this with what was said in Section 3.1, we see that the geometricity provides a guarantee that the point transformation commutes with change of stage. This is because the stage changes α correspond to applying $\Omega\alpha$, which preserves the geometric connectives.

Conceptually, therefore, –

- a locale is the "space of models" of a propositional geometric theory, and
- a map is a geometric transformation of models.

To summarize it as a slogan, *continuity is geometricity.*

There is a subtle issue concerning the role of geometricity. A frame is a complete Heyting algebra, and has intuitionistic but non-geometric structure such as the Heyting arrow and negation. Suppose we say "let x be a point of X", and then, *non*-geometrically, define a point $f(x)$ of Y. We can apply this to the generic point, the identity map on X, and thus get a point of Y at stage X, in other words a map $X \to Y$. However, in terms of the discussion in Section 3, the non-geometricity means this does not commute with change of stage. This is because change of stage for α is achieved by applying the frame homomorphism $\Omega\alpha$, and non-geometric operations are not preserved by frame homomorphisms.

Example 24 *Consider the point transformer F that transforms points of $\mathbb{S}$ to points of $\mathbb{S}$ by applying Heyting negation $\neg$ in frames. At stage W, the points of $\mathbb{S}$ are the elements of ΩW, and we define $F_W(U) = \neg U$. The generic point* $\mathrm{Id} : \mathbb{S} \to \mathbb{S}$*, as element of $\Omega\mathbb{S}$, is the generator P, and in $\Omega\mathbb{S}$ we have $\neg P = \bot$.*

The corresponding frame homomorphism takes $\top$ to $\top$, and P and $\bot$ both to $\bot$. When we use composition with this to give a point transformer, we find it always takes any open U of W to $\bot$, and not to $\neg U$ as intended. It just happens that for the generic point P, we have that $\neg P$ and $\bot$ are equal.

4.2 Predicate geometric logic and geometric type constructions

There is also *predicate* geometric logic. For this, we allow the signature Σ to include sorts, and function symbols and predicates together with their arities (including the sorts of the arguments and results). Then terms can be built from sorted variables and the function symbols in the usual way, and geometric formulae are built from terms and predicate symbols using not only $\wedge$ and $\bigvee$, but also equality $=$ and existential quantification $\exists$. Then a geometric sequent is of the form $(\forall xyz\cdots)(\phi \to \psi)$, where "$xyz\cdots$" is a finite list of sorted variables, and ϕ and ψ are geometric formulae in which every free variable is in the list $xyz\cdots$. A geometric theory is again a set of sequents.

The infinitary disjunctions make this an unusual logic, with a natural type theory (type constructions) associated with it. They give us the power within geometric theories to characterize certain sorts up to isomorphism. Suppose, for example, we want to characterize a sort N as the natural numbers. We can do this with a constant 0, a successor map s, and sequents

$$(\forall n)(s(n) = 0 \to \mathbf{false})$$
$$(\forall mn)(s(m) = s(n) \to m = n)$$
$$(\forall n)(\mathbf{true} \to \bigvee_{i \in \mathbb{N}} n = s^i(0))$$

(Here, the exponent i in $s^i(0)$ is not part of the logical syntax, but is meant to suggest an inductive definition of formulae ϕ_i in which ϕ_0 is the formula $n = 0$, ϕ_1 is $n = s(0)$, ϕ_2 is $n = s(s(0))$ and so on.)

Because of this ability, which is impossible in finitary logic, geometric logic embodies a "geometric type theory". The geometric type constructions include finite limits (products, pullbacks, equalizers, ...), arbitrary colimits (coproducts, quotients, ...) and also all free algebra constructions.

Because these type constructions can already be characterized within the logic, it makes no difference to the expressive power of the logic if we treat them as being freely available. As we shall see in Section 4.3, this has a radical effect on how we deal with geometric theories and their models in the predicate case. It takes in the geometric construction of new types and moves from a pure logic to a "geometric mathematics". Some explicit applications of this to predicate geometric logic and toposes can be found in [Vic99] and [Vic07].

4.3 Geometric logic of sheaves

The geometric type constructions can be built up using finite limits and arbitrary colimits, at least if these are assumed to be "geometrically well behaved" in a way that matches the rules of geometric logic.[8] This therefore suggests the following process, of using a geometric theory (Σ, T) (still propositional here, but influenced by the consideration of predicate theories) to generate a "category of formal geometric sets" in an analogous way to how we generate the frame $\Omega[\Sigma, T]$ of "formal geometric truth values". This category is the *classifying topos*, but it turns out to be equivalent to the category $\mathcal{S}[\Sigma, T]$ of sheaves.

Consider the nature of a point x of $[\Sigma, T]$. It comprises a set of subsingleton sets, one for each element of Σ, or, equivalently, a subset of Σ (containing those elements $P \in \Sigma$ for which the subsingleton set is inhabited). Either way, the axioms of T must be respected.

Remark 25 *How T needs to be structured, and what "respecting the axioms" then means, is examined in detail in the* (G,R,D)-systems *of [Vic04]. Technically, a (G, R, D)-system comprises three sets G, R and D, equipped with functions*

$$\lambda : R \to \mathcal{F}G$$
$$\rho : D \to \mathcal{F}G$$
$$\pi : D \to R$$

where $\mathcal{F}$ denotes the finite powerset. The system represents a theory (G, R) in which each $r \in R$ corresponds to a geometric sequent

$$\bigwedge \lambda(r) \to \bigvee_{\pi(d)=r} \bigwedge \rho(d).$$

Once one has the point x of $[\Sigma, T]$, as subset of Σ, one can start constructing finite limits and arbitrary colimits of its constituents. Suppose this is done in a formal way, making no assumptions about x other than that it is a point, and ensuring that everything is geometrically well behaved. This is analogous to generating the frame $\Omega[\Sigma, T]$, but more general. The frame restricts itself to constructing subsingleton sets (truth values) out of x. The process can be made precise in categorical logic, and generates what is called the *classifying topos* of the theory (Σ, T). One may write it $\mathcal{S}[\Sigma, T]$.

From this point of view, the objects of the classifying topos can be quite naturally understood as maps from points to sets: from a point, and the sets that it comprises, each object provides a construction that yields another set.

On the other hand, two very fundamental results of topos theory relate classifying toposes to sheaves.

1. For any propositional geometric theory (Σ, T), the classifying topos $\mathcal{S}[\Sigma, T]$ is equivalent to the topos of sheaves over the locale $[\Sigma, T]$. (Note that our

[8] Categorically, it means that we are working in a Grothendieck topos.

notation here for classifying toposes, modulo equivalence, thus matches our earlier notation of $\mathcal{S}X$ for the category of sheaves over X.)

2. Given two theories (Σ_1, T_1) and (Σ_2, T_2), maps $f : [\Sigma_1, T_1] \to [\Sigma_2, T_2]$ are equivalent to geometric morphisms (f^*, f_*) from $\mathcal{S}[\Sigma_1, T_1]$ to $\mathcal{S}[\Sigma_2, T_2]$. (See Section 2.5.) Under the equivalence with categories of sheaves they agree with the geometric morphisms (f^*, f_*) defined before.

The proofs are present in standard texts of topos-theory, such as [MLM92]. In [Vic07] there is an exposition more carefully adapted to the point of view described here. In one direction, the equivalence works by generalizing the notion of model from sets to sheaves, in the same way as for propositional theories it was generalized from truth values to opens. Sorts are interpreted as sheaves, function symbols as sheaf morphisms from a sheaf product to another sheaf, and predicates as subsheaves of sheaf products. Once that is done, terms can be interpreted as sheaf morphisms and formulae as subsheaves. Specific categorical structure in the category of sheaves is needed for this; for instance, equalizers are needed in order to interpret $=$. In fact, all can be done using finite limits and arbitrary colimits. Moreover, particular properties of the way these limits and colimits interact with each other ensure that the rules of geometric logic are valid in the sheaf interpretations. This gives a functor from the classifying topos to the topos of sheaves, but it takes somewhat more work to show that the functor is an equivalence.

Once we know that sheaves are equivalent to objects of the classifying topos, this now provides the deepest sense in which a sheaf is a set-valued map: a sheaf is equivalent to a geometric recipe for constructing sets out of points.

Note how this works for local homeomorphisms. The inverse image functors f^*, which act by pullback, preserve finite limits and all colimits, and hence all geometric constructions. (For other geometric constructions, such as free algebra constructions, preservation by the f^*s can also be proved directly.) But for a global point x, we know that $x^*(S)$ constructs the stalk of S at x, as well as preserving the point-to-set recipe by which S was specified; hence the sets that the recipe constructs are the stalks. *A sheaf can be presented as a uniform, geometric recipe for constructing the stalks.* If we are willing to extend the notion in Section 4.1 that "continuity is geometricity", then a sheaf becomes a "continuous set-valued map".[9] This is given some technical substance in Section 7.

The same argument also applies to sheaf morphisms: to describe one, it suffices to give a uniform geometric description of its action on stalks.

[9]The technique in Section 4.1 for defining a locale map $f : X \to Y$ as a geometric transformation of points can often be fruitfully broadened from opens to sheaves, using the fact that every open is a subsheaf of 1. The geometric set constructions can also be legitimately used in transforming points of X to points of Y. [Vic07] illustrates this with examples from the real numbers.

4.4 Constant sheaves as geometric constructions

Let us examine in more detail the *constant* sheaves, since they are in some ways the starting point. The geometric constructions include arbitrary colimits, "arbitrary" here meaning that the colimit diagram is small – its nodes and arrows are indexed by elements of sets. Hence for any set A we can form a coproduct $\coprod_{a \in A} 1$ of an A-indexed family of copies of the terminal object 1. In ordinary sets this characterizes A (as a disjoint union of its singletons), and by this means A can be represented in any context in which we have the geometric constructions. In sheaves over X, the construction is got by taking the coproduct construction of A in ordinary sets, and translating it to a sheaf construction by the inverse image $!_X^*$ where $!_X : X \to 1$ is the unique map. Hence we shall write it as $!_X^* A$

In the context of sheaves over X, we should think of this construction "stalk-wise" as a construction of set from point. But the recipe $\coprod_{a \in A} 1$ for performing the construction makes no reference to the point x and hence is independent of it. For this reason, $!_X^* A$ is a *constant* sheaf. Consequently, calculating the stalk for any global point x always gives A, regardless of x. For a generalized point $x : W \to X$, it gives the $!_W^* A$ appropriate to W, regardless of the map x.

In terms of local homeomorphisms, the general topos results tell us that inverse image functors are applied by taking pullback. Hence $!_X^* A$ is the projection $p : X \times A \to X$, got by pulling back $A \to 1$ along $!_X : X \to 1$. (See Propositions 17 and 19.)

In a constant sheaf it is the *stalks* that are constant. By contrast the pasting presheaf is almost never a constant functor; recall that for any pasting presheaf F we have $F(\emptyset)$ a singleton. In the case of a constant sheaf $p : X \times A \to X$, the pasting presheaf Sect_p can be calculated as follows. If U is an open in X then a section over U is a map $U \to X \times A$ such that the first component is just the inclusion, and these are equivalent to maps $U \to A$.

We now prove a result that establishes the importance of ΩX-valued fuzzy sets (functions from a set A to ΩX) in two different ways. They provide a useful concrete representation of opens of $X \times A$, but also capture exactly the subsheaves of constant sheaves. This result will be used pervasively in Section 6.[10] (After this proposition, it is a simple exercise to verify directly that $X \times A$ is the locale coproduct of A copies of X.)

Theorem 26 *Let X be a locale and A a set. Then the following are in order-preserving bijection.*

1. *Opens of $X \times A$. (Here, A is understood as the corresponding discrete locale, with ΩA the powerset $\mathcal{P}A$.)*
2. *Functions $A \to \Omega X$.*
3. *Subsheaves of $!_X^* A$.*

[10]One of the referees has remarked that in the fuzzy community $(\Omega X)^A$ is interpreted as the X-fuzzy powerset of A, and the inverse image function Ωp of the projection map $p : X \times A \to X$ plays an important role in fuzzy topology, as in the work of R. Lowen.

If $p : X \times A \to X$ is the projection map then the inverse image function Ωp takes $U \in \Omega X$ to the constant function $a \mapsto U$ – because $U \times A = \bigvee_{a \in A} U \times \{a\}$.

Proof. Proposition 21 gives us the equivalence between (1) and (3).

Suppose $a \in A$. By pulling back $a : 1 \to A$ we get $!_X^* a : 1 \to !_X^* A \cong \coprod_{a' \in A} 1$, and it is the corresponding coproduct injection. For any subsheaf $Y \hookrightarrow !_X^* A$ we get pullbacks

$$\begin{array}{ccc} [a \in Y] & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 1 & \underset{!_X^* a}{\longrightarrow} & !_X^* A \end{array}$$

and in fact Y can be expressed as a coproduct $\coprod_{a \in A} [a \in Y]$. (This property of the interaction between coproducts and pullbacks, which is true in **Sets**, is also true in any Grothendieck topos.) It follows that the subsheaves Y are equivalent to the assignments $a \mapsto [a \in Y]$, a subsheaf of 1. By Corollary 22 subsheaves of 1 are equivalent to opens of X, so such assignments are equivalent to functions from A to ΩX. ■

5 The local homeomorphism of an ΩX-valued set

Let (A, E) be an ΩX-valued set. Höhle defines the frame $P(A, E)$ to have as its elements the strict, extensional functions $A \to \Omega X$ (see Definition 10). We shall use the notation $P(A, E)$ for the locale rather than the frame. It is the display locale for the sheaf. Höhle shows that if X is spatial then so is $P(A, E)$, and the display map $\mathrm{pt}(P(A, E)) \to \mathrm{pt}\, X$ is a local homeomorphism corresponding to the sheaf for (A, E). We shall now show how to define the local homeomorphism even in the non-spatial case.

Following Höhle's notation we write $\mathbb{E}$ for the top strict extensional function, $\mathbb{E}a = E(a, a)$, and $(U \wedge \mathbb{E})$ for the one defined by $(U \wedge \mathbb{E})(a) = U \wedge E(a, a)$. Recall also (Section 2.4) the singleton $\tilde{a}$, defined by $\tilde{a}(b) = E(a, b)$.

For any strict extensional s we have

$$s = \bigvee_{a \in A} (s(a) \wedge \mathbb{E}) \wedge \tilde{a}, \qquad (2)$$

in other words $s(b) = \bigvee_{a \in A} s(a) \wedge E(b, b) \wedge E(a, b) = \bigvee_{a \in A} s(a) \wedge E(a, b)$ for all $b \in A$. The $\geq$ direction follows from extensionality, while the $\leq$ direction follows from strictness, taking $a = b$.

Note also that

$$s \wedge \tilde{a} = (s(a) \wedge \mathbb{E}) \wedge a, \qquad (3)$$

for on applying both sides to b the equation becomes $s(b) \wedge E(a, b) = s(a) \wedge E(a, b)$.

In the following lemma, note how the generators correspond to the subbasic opens used to topologize the display space Y in Section 2.3. The lemma is also a generalization of the first part of Theorem 26, which covers the case of the crisp valuation on A. This is because the frame $\mathcal{P}A$ can be presented by formal generators $\{a\}$, subject to relations $\top \leq \bigvee_{a\in A}\{a\}$ and $\{a\}\wedge\{b\} \leq \bigvee\{\top \mid a = b\}$.

Lemma 27 *$\Omega P(A, E)$ can be presented by generators and relations as*

$$\begin{aligned}\mathrm{Fr}\langle \Omega X \ \textit{(qua frame)},\ [a]\ (a \in A) \mid \top \leq \bigvee_{a\in A}[a]\\ [a] \wedge [b] \leq E(a, b)\\ E(a, b) \wedge [b] \leq [a]\rangle.\end{aligned}$$

Proof. Let us write F for the frame presented as stated. To define a homomorphism $\alpha : F \to \Omega P(A, E)$ we must describe its action on the generators and show that it respects the relations (including that it preserves the frame structure on ΩX). It will map $U \in \Omega X$ to $(U \wedge \mathbb{E})$, and the formal generator $[a]$ to $\tilde{a}$. This respects the relations. Next we define $\beta : \Omega P(A, E) \to F$ by $\beta(s) = \bigvee_{a\in A} s(a) \wedge [a]$. This is easily seen to be a homomorphism, using the fact that the joins and binary meets on the strict, extensional maps are defined argumentwise. Finally, it remains to show that α and β are mutually inverse. The composite $\alpha \circ \beta$ is the identity on $\Omega P(A, E)$ because of Equation 2. For $\beta\circ\alpha$, it suffices to check its action on the generators. If $U \in \Omega X$ then $\beta\circ\alpha(U) = \bigvee_{a\in A} U \wedge E(a, a) \wedge [a]$. From the second relation (and putting $b = a$) we see $[a] \leq E(a, a)$, so it suffices to show $U = \bigvee_{a\in A} U \wedge [a]$, which follows from the first relation. For $[a]$ we have $\beta \circ \alpha([a]) = \bigvee_{b\in A} E(a, b) \wedge [b]$ and this equals $[a]$ by the third relation. ■

The advantage of this is that the generators and relations give us a direct description of the points at all stages W: a point is a function from the generators to ΩW that respects the relations. On the generators U this gives us a point x of X. On the generators $[a]$ we find a subset $S \subseteq A$ (we are working in the internal mathematics of sheaves over W, so S is actually a subsheaf of $!_W^* A$) such that (i) S is inhabited, (ii) if $a, b \in S$ then x satisfies $E(a, b)$, and (iii) if x satisfies $E(a, b)$ and $b \in S$ then $a \in S$: in other words, S is an equivalence class for $\sim_x$. Hence a point of $P(A, E)$ can be described geometrically as a pair (x, u) where x is a point of X and u is an element of its stalk $A/\sim_x$.

(Actually, the use of the frame ΩX is non-geometric. However, this can be circumvented by using a presentation of it by generators and relations.)

The projection map $p : P(A, E) \to X$ is defined on points (x, u) by forgetting u. However, its inverse image function Ωp is also clear enough; it is the inclusion of generators U, or the function $U \longmapsto (U \wedge \mathbb{E})$.

The presentation of $P(A, E)$ by generators and relations also allows us to calculate the local sections of p very easily. Suppose U is an open of X. A section of p over U is a map $\sigma : U \to P(A, E)$ such that $p \circ \sigma$ is the inclusion $U \hookrightarrow X$. On generators of $\Omega P(A, E)$, $\Omega\sigma$ must take $U' \in \Omega X$ to $U \wedge U'$ – this is because $p \circ \sigma$ is the inclusion. On the generators $[a]$, $\Omega\sigma$ determines a function $s : A \to \Omega X$, and the relations tell us that s must be a singleton for which $\bigvee_{a\in A} s(a) = U$. Hence the local sections of p are the singletons of (A, E).

Theorem 28 *$p : P(A, E) \to X$ is a local homeomorphism.*

Proof. First, we show that p is open. Define $\exists_p : \Omega P(A, E) \to \Omega X$ by $\exists_p(s) = \bigvee_{a \in A} s(a)$. Then

$$\exists_p(s) \leq U \Leftrightarrow (\forall a) s(a) \leq U \Leftrightarrow s \leq (U \wedge \mathbb{E})$$

and so $\exists_p(s)$ is left adjoint to Ωp. For the Frobenius condition,

$$\exists_p(s \wedge (U \wedge \mathbb{E})) = \bigvee_a (s(a) \wedge U \wedge E(a, a)) = \bigvee_a s(a) \wedge U = \exists_p(s) \wedge U.$$

Now we show that $\Delta : P(A, E) \to P(A, E) \times_X P(A, E)$ is open. We define $\exists_\Delta(s) = \bigvee_{a \in A}((s(a) \wedge \mathbb{E}) \wedge \tilde{a}) \otimes \tilde{a}$. Then

$$\Omega\Delta \circ \exists_\Delta(s) = \bigvee_{a \in A} s(a) \wedge \tilde{a} \wedge \tilde{a} = s$$

by Equation 2. Also,

$$\begin{aligned}
\exists_\Delta \circ \Omega\Delta(s \otimes t) &= \bigvee_{a \in A} (s(a) \wedge t(a) \wedge \mathbb{E}) \wedge \tilde{a} \otimes \tilde{a} \\
&= \bigvee_{a \in A} (s(a) \wedge \mathbb{E}) \wedge \tilde{a} \otimes (t(a) \wedge \mathbb{E}) \wedge \tilde{a} \\
&\leq \bigvee_{a, b \in A} (s(a) \wedge \mathbb{E}) \wedge \tilde{a} \otimes (t(b) \wedge \mathbb{E}) \wedge \tilde{b} = s \otimes t
\end{aligned}$$

and it follows that $\exists_\Delta$ is left adjoint to $\Omega\Delta$. For Frobenius,

$$\begin{aligned}
\exists_\Delta(s \wedge \Omega\Delta(t \otimes u)) &= \bigvee_{a \in A} (s(a) \wedge t(a) \wedge u(a) \wedge \mathbb{E}) \wedge \tilde{a} \otimes \tilde{a} \\
&= \bigvee_{a \in A} (s(a) \wedge t(a) \wedge \mathbb{E}) \wedge \tilde{a} \otimes (u(a) \wedge \mathbb{E}) \wedge \tilde{a} \\
&= \bigvee_{a \in A} (s(a) \wedge \mathbb{E}) \wedge t \wedge \tilde{a} \otimes u \wedge \tilde{a} = \exists_\Delta(s) \wedge (t \otimes u).
\end{aligned}$$

■

To complete the standard proof [JT84] of the equivalence between local homeomorphisms and other forms of sheaf, one uses Proposition 17 interpreted in the internal mathematics of the topos of sheaves. However, it is perhaps illuminating to see directly how to get from a local homeomorphism to a (complete) frame-valued set. Suppose $p : Y \to X$ is a local homeomorphism, with functions $\exists_p : \Omega Y \to \Omega X$ and $\exists_\Delta : \Omega Y \to \Omega(Y \times_X Y)$ the left adjoints needed for openness of p and Δ. We shall construct from this the complete ΩX-valued set of local sections. The construction and proof are analogous to Proposition 17, but with normal truth values ("whether" something is true) replaced by elements of ΩX ("where" something is true). The function $\exists_p$ still plays the role of positivity predicate, but now says where a local section is defined.

Theorem 29 *Let $p : Y \to X$ be a local homeomorphism between locales. We say an element $a \in \Omega Y$ is* subatomic *if $a \otimes a \leq \exists_\Delta \top$ in $\Omega(Y \times_X Y)$, and define A to be the set of subatoms. We also define $E : A \times A \to \Omega X$ by $E(a, a') = \exists_p(a \wedge a')$.*

1. *(A, E) is an ΩX-valued set.*
2. *Y is homeomorphic to $P(A, E)$ over X.*
3. *(A, E) is complete.*

Proof. Let us first prove some lemmas about the situation; these develop the ideas that were already present in Proposition 17. Note that by the adjunction we have $\Omega\Delta \circ \exists_\Delta \circ \Omega\Delta = \Omega\Delta$; and since $\Omega\Delta$ is onto (because Δ is an inclusion), we deduce that $\Omega\Delta \circ \exists_\Delta = \mathrm{Id}_{\Omega Y}$.

The subatoms cover Y. For we can express $\exists_\Delta \top$ as a join $\bigvee_i V_i \otimes V_i'$ of basics, and then $\top = \Omega\Delta(\exists_\Delta \top) = \bigvee_i V_i \wedge V_i'$ is a join of subatoms.

Using Proposition 18, Equation 1, we see that if $p_1 : Y \times_X Y \to Y$ is the first projection then $\exists_{p_1}(V_1 \otimes V_2) = V_1 \wedge \Omega p(\exists_p(V_2))$. If $V \in \Omega Y$, then in $\Omega(Y \times_X Y)$ we have

$$\begin{aligned}\exists_\Delta \top \wedge (V \otimes \top) &= \exists_\Delta(\top \wedge \Omega\Delta(V \otimes \top)) = \exists_\Delta(\top \wedge V \wedge \top) = \exists_\Delta V \\ &= \cdots = \exists_\Delta \top \wedge (\top \otimes V).\end{aligned}$$

Hence, if a is any subatom, then $(a \wedge V) \otimes a = a \otimes (a \wedge V)$. Applying $\exists_{p_1}$, we obtain

$$a \wedge V = a \wedge \Omega p(\exists_p(a \wedge V)). \tag{4}$$

If we also have $V' \in \Omega Y$, then

$$a \wedge V \wedge \Omega p(\exists_p(a \wedge V')) = a \wedge V \wedge V'.$$

Applying $\exists_p$, and using Frobenius, we obtain

$$\exists_p(a \wedge V) \wedge \exists_p(a \wedge V') = \exists_p(a \wedge V \wedge V'). \tag{5}$$

Part 1: Clearly E is symmetric; transitivity follows from Equation 5.

Part 2: We first define $\alpha : \Omega P(A, E) \to \Omega Y$ using the presentation in Lemma 27. For the generators from ΩX we define α to agree with Ωp, which is given to be a frame homomorphism (i.e. it preserves the frame structure of ΩX, which is what "qua frame" requires in the presentation). This will suffice to show that the locale map from α is over X. For the generators $\tilde{a}$, corresponding to subatoms a, we define $\alpha(\tilde{a}) = a$. The three relations come down to $\top \leq \bigvee_{a \in A} a$ (we have already noted that the subatoms cover Y), $a \wedge b \leq \Omega p(\exists_p(a \wedge b))$ (which follows from the adjunction $\exists_p \dashv \Omega p$) and $\Omega p(\exists_p(a \wedge c) \wedge c \leq a$ for any subatomics a, c (for which Equation 4 tells us that the left-hand side is $a \wedge c$).

For $\beta : \Omega Y \to \Omega P(A, E)$, we define $\beta(V)$ to be the strict, extensional function $s_V : A \to \Omega X$ with $s_V(a) = \exists_p(a \wedge V)$. Strictness is evident, since

$\exists_p(a \wedge V) \leq \exists_p a = E(a,a)$. Extensionality follows from Equation 5, taking $V' = b$. We must show that $V \mapsto s_V$ is a frame homomorphism. Preservation of joins is immediate from the fact that $\exists_p$ and $a \wedge -$ both preserve joins; and $s_\top$ is the top element $\mathbb{E}$. For binary meets, $s_{V \wedge V'} = s_V \wedge s_{V'}$ is just Equation 5. To show that the corresponding locale map is over X, we need $s_{\Omega p(U)} = (U \wedge \mathbb{E})$, i.e. $\exists_p(a \wedge \Omega p(U)) = U \wedge E(a,a)$, which works out as the Frobenius condition.

Now we must show that α and β are mutually inverse. First,

$$\begin{aligned}
\alpha \circ \beta(V) = \alpha(s_V) = \alpha(\bigvee_{a \in A} (s_V(a) \wedge \mathbb{E}) \wedge \tilde{a}) &= \bigvee_{a \in A} \Omega p(\exists_p(a \wedge V)) \wedge a \\
&= \bigvee_{a \in A} (a \wedge V) \qquad \text{(using equation 4)} \\
&= V \qquad \text{(because the subatoms cover } Y\text{)}
\end{aligned}$$

For $\beta \circ \alpha$, we check the actions on the generators. For those derived from elements of ΩX, this has already been done when we verified that the maps were over X. For $\tilde{a}$, we have $\beta \circ \alpha(\tilde{a}) = s_a = \tilde{a}$ by definition.

Part 3: We now know that the strict extensionals on A are in bijection with the elements of ΩY. We show further that amongst those, the singletons correspond to the subatomic elements of ΩY. Let $V \in \Omega Y$. Its strict extensional s_V is a singleton iff for all subatoms a and b we have

$$\exists_p(a \wedge V) \wedge \exists_p(b \wedge V) \leq \exists_p(a \wedge b).$$

If V is subatomic, we know this formula holds – it is transitivity of E. We prove the converse. The subatoms cover Y, and so $V = \bigvee_{a \in A} a \wedge V$, and $V \otimes V = \bigvee_{a,b \in A} (a \wedge V) \otimes (b \wedge V)$. Now for each a, b we have

$$\begin{aligned}
(a \wedge V) \otimes (b \wedge V) &= (a \wedge \Omega p(\exists_p(a \wedge V)) \otimes (b \wedge \Omega p(\exists_p(b \wedge V))) \\
&\qquad \text{(by equation 4)} \\
&= (a \wedge \Omega p(\exists_p(a \wedge V)) \wedge \Omega p(\exists_p(b \wedge V))) \otimes b \\
&\qquad \text{(elements } \Omega p(U) \text{ can be moved across the } \otimes \text{)} \\
&= (a \wedge \Omega p(\exists_p(a \wedge V) \wedge \exists_p(b \wedge V))) \otimes b \\
&\leq (a \wedge \Omega p(\exists_p(a \wedge b))) \otimes b \\
&= (a \wedge b) \otimes b \leq b \otimes b \leq \exists_\Delta \top.
\end{aligned}$$

■

6 Geometric constructions on ΩX-valued sets

Throughout this section X will be a locale.

[H07a] describes a full range of topos-theoretic constructions on complete ΩX-valued sets. For some of them, completeness is a very convenient part of the construction. However, our contention here is that for the geometric constructions (which are performed stalkwise on the local homeomorphisms) there are simpler and more natural constructions that work directly on the uncompleted ΩX-valued sets and avoid the need to complete.

At the same time, we shall also illustrate the benefits of the geometric reasoning. The aim is to give localic validity to pointwise reasoning such as in Proposition 9, which showed the bijection between sheaf morphisms $A \to B$ and functions $\theta : A \times B \to \Omega X$ satisfying certain conditions. The argument comprised three steps.

1. The stalkwise setting was examined, with a family of stalk functions $f_x : A/\sim_x \to B/\sim_x$ corresponding to a family of relations $\theta_x \subseteq A \times B$ satisfying certain conditions.

2. The family (θ_x) of relations was translated into an external function $\theta : A \times B \to \mathcal{P}X$, defined by $\theta(a,b) = \{x \mid a\theta_x b\}$, and the conditions on θ_x translate into certain conditions on θ.

3. It was shown that continuity of f (between the display spaces) corresponded to θ taking its values in ΩX.

We shall examine this example in more detail later, but we can already use it to illustrate the general pattern. Step (1), the stalkwise reasoning, will generally be the same as for the spatial analogue. However, care must be taken to do it geometrically so that it applies to generalized points x. Steps (2) and (3) are covered by Theorem 26, which implies that subsheaves of constant sheaves are equivalent to fuzzy sets. The theorem embodies two principles. The first, corresponding to Step (2), is that a subsheaf $Y \hookrightarrow !_X^* A$ can be specified by the truth values $[a \in Y]$ for the *external* elements $a \in A$. This is essentially because $Y = \coprod_{a \in A} [a \in Y]$. The second, deriving ultimately from Corollary 22, and corresponding to Step (3), is that each $[a \in Y]$, a sub*sheaf* of 1, corresponds to an open of X (saying in effect where the stalks are inhabited).

6.1 Constant sheaves and their subsheaves

The constant sheaf $!_X^* A$ can be presented as the ΩX-valued set A with *crisp* equality, $E(a,b) = \bigvee \{\top \mid a = b\}$. The set of disjuncts here, $\{\top \mid a = b\}$, is a subset of $\{\top\}$, so classically there are two possibilities: $E(a,b) = \top$ if $a = b$, and $\bot$ $(= \bigvee \emptyset)$ if $a \neq b$. If a point x is in $E(a,b)$ then it must be in one of the disjuncts: so $a = b$ and x is in $\top$ (which it is anyway). In other words, $a \sim_x b$ iff $a = b$, so the stalk $A/\sim_x$ is A, as expected. This argument is geometric, so we do not have to worry about whether there are enough points. We know that this ΩX-valued set presents the right sheaf.

Note that the equivalence "$a \sim_x b$ iff $a = b$" is interpreted internally. As an extreme example, at the inconsistent stage $W = \emptyset$ (the empty locale, with no points and one open) the equation $a = b$ takes the value $\top$ for all elements $a, b \in A$, and $A/\sim_x \cong 1$. But that does not matter since all sheaves over $\emptyset$ are isomorphic to 1 and that includes $!^* A$. As local homeomorphism, the projection $\emptyset \times A \to \emptyset$ is a homeomorphism.

Let us check how the argument appears for the generic point, in sheaves over X. We have two constructions of a subsheaf of $!_X^* A \times !_X^* A \cong !_X^*(A \times A)$, namely "$x \mapsto \sim_x$" and $!_X^*(=_A)$. Our argument above purports to show that they are isomorphic as subsheaves, so that the corresponding subquotients agree. From Theorem 26 we see that describing a subsheaf of $!_X^*(A \times A)$ is equivalent to describing, for each pair $(a, b) \in A \times A$, a subsheaf of 1, i.e. an open of ΩX. Since this is equivalent to a map $X \to \mathbb{S}$ (see Definition 23), it can be understood as a geometric transformation from points x to truth values, or subsets of 1 (points of $\mathbb{S}$). Hence describing a subsheaf of $!_X^*(A \times A)$ resolves into geometrically transforming triples (x, a, b) into truth values. The argument was phrased as though it said, for each x, that the corresponding sets of pairs (a, b) were equal. That is the "stalkwise" view of it, and it does tell us the actual stalks for global points x. However, it can be turned round and viewed as a proof that for each pair (a, b) the open $E(a, b)$ is the constant open corresponding to the equation $a = b$.

Theorem 26 shows further that fuzzy sets are subsheaves of constant sheaves. In order to exploit it, we shall need to examine how internal geometric constructions on the subsheaves correspond to external constructions on the fuzzy sets. If $\phi : A \to \Omega X$ is a fuzzy set, then we shall write the corresponding subsheaf of $!_X^* A$ stalkwise as $\{a \in A \mid x \vDash \phi(a)\}$.[11] In the particular case when x is a global point, this is a literal subset of A and "$x \vDash \phi(a)$" asserts that $\Omega x(\phi(a)) = \mathbf{true}$. More generally, if $x : W \to X$ is any point of X, then "$x \vDash \phi(a)$" denotes (in sheaves over W) the sheaf $x^*(\phi(a))$ where $\phi(a)$ is considered as a subsheaf of 1 over X, and the stalk $\{a \in A \mid x \vDash \phi(a)\}$ denotes $\coprod_{a \in A} x^*(\phi(a))$ as a sheaf over W.

The "stalkwise reasoning" says that if we manipulate expressions such as $\{a \in A \mid x \vDash \phi(a)\}$ in a conventional way but restricting to geometric manipulations, then the manipulation can be interpreted as a construction in sheaves over X and it is preserved by inverse image functors ("taking generalized stalks").

Lemma 30 *Let $f : B \to A$ be a function between sets.*

1. *Unions and finite intersections of subsheaves of $!_X^* A$ correspond to argumentwise joins and finite meets of the corresponding fuzzy sets.*

2. *Let Y be a subsheaf of $!_X^* A$, corresponding to fuzzy set $\phi : A \to \Omega X$. Then the inverse image $f^{-1}Y$ of Y along $!_X^* f : !_X^* A \to !_X^* B$ corresponds to the composite $\phi \circ f$. Logically, if Y is thought of as an internal predicate $Y(a)$ on $!_X^* A$, then the inverse image corresponds to substituting the term $f(b)$ for a.*

[11] The symbol "$\vDash$" to denote when a point "is in" an open is adopted from [Vic89]. It circumvents the notational problem in locales that a point, considered as a completely prime filter of opens, has opens as its elements rather than the other way round.

3. *Let Z be a subsheaf of $!_X^* B$, corresponding to fuzzy set $\psi : B \to \Omega X$. Then the direct image of Z under $!_X^* f$ corresponds to the fuzzy set $a \mapsto \bigvee_{f(b)=a} \psi(b)$. Logically, if Z is thought of as an internal predicate $Z(b)$ on $!_X^* B$, then the image corresponds to $(\exists b)(f(b) = a \wedge Z(b))$.*

Proof. 1. Suppose $\phi_i : A \to \Omega X$ $(i = 1, 2)$ are two fuzzy sets, and let $\phi(a) = \phi_1(a) \wedge \phi_2(a)$ be their argumentwise meet. Then

$$\{a \in A \mid x \vDash \phi(a)\} = \{a \in A \mid x \vDash \phi_1(a) \wedge x \vDash \phi_2(a)\}$$
$$= \{a \in A \mid x \vDash \phi_1(a)\} \cap \{a \in A \mid x \vDash \phi_2(a)\}.$$

The argument is similar for unions.

2. A stalk of the inverse image is

$$\{b \in B \mid f(b) \in \{a \in A \mid x \vDash \phi(a)\}\} = \{b \in B \mid x \vDash \phi(f(b))\}.$$

3. A stalk of the direct image is

$$\{a \in A \mid (\exists b \in B)(f(b) = a \wedge x \vDash \psi(b))\}$$
$$= \{a \in A \mid x \vDash \bigvee_{f(b)=a} \psi(b)\}.$$

■

6.2 Subquotients of constant sheaves

By *subquotient*, we mean quotient of a subobject. In sets, a subquotient of A is given by a partial equivalence relation $\sim$ on A. The subquotient is then $A/\sim$, i.e. the set of equivalence classes. This construction is geometric. The subobject is got by image factorization (which is geometric) of $\sim \hookrightarrow A \times A \xrightarrow{p_1} A$, and then the quotient is got by a coequalizer.

Proposition 31 *Let A be a set. Then there is a bijection between subquotients of $!_X^* A$ and ΩX-valuations on A.*

Proof. Let Y be the subquotient, corresponding to a partial equivalence relation $\sim$ on $!_X^* A$ and hence, by Theorem 26, to a fuzzy set $E : A \times A \to \Omega X$. Working stalkwise, we see that the stalk $x^* Y$ is given by a partial equivalence relation $\sim_x = \{(a, b) \in A \times A \mid x \vDash E(a, b)\}$ on A. We use Lemma 30 to show that $\sim_x$ is a partial equivalence relation for every point x iff E is an ΩX-valuation.

We start with transitivity. Taking the inverse image of $\sim_x$ along the function $A \times A \times A \to A \times A$ given by the first two projections, we get $\{(a, b, c) \mid a \sim_x b\}$ corresponding to the fuzzy set $E_1 : A \times A \times A \to \Omega X$, given by $E_1(a, b, c) = E(a, b)$. Similarly,

$\{(a, b, c) \mid b \sim_x c\}$	corresponds to	$E_2(a, b, c) = E(b, c)$
$\{(a, b, c) \mid a \sim_x b \sim_x c\}$	"	$E_3(a, b, c) = E(a, b) \wedge E(b, c)$
$\{(a, c) \mid (\exists b) a \sim_x b \sim_x c\}$	"	$E_4(a, c) = \bigvee_b E(a, b) \wedge E(b, c)$

The last of these is the relational composition $\sim_x \circ \sim_x$, and so transitivity, $\sim_x \circ \sim_x \subseteq \sim_x$, is equivalent to $E(a,b) \wedge E(b,c) \leq E(a,c)$ for all a, b, c. The rule for symmetry is similar, but simpler. ■

This kind of translation between internal properties (e.g. stalkwise transitivity) and external (e.g. $E(a,b) \wedge E(b,c) \leq E(a,c)$) is quite general, and we shall use it in future without comment.

6.3 Subsheaves of ΩX-valued sets

If A is a set and E an ΩX-valuation on it, we shall write $P(A,E)$ for the corresponding subsheaf of $!_X^* A$. This is a mild abuse of the notation of Section 5, which identified a locale $P(A,E)$ as the display locale for the corresponding local homeomorphism. The following result shows directly what we could already deduce from Lemma 27 (and the remarks following it about the points) and Proposition 21: that the opens of the display locale are the strict extensional functions.

Proposition 32 *Let A be a set and E an ΩX-valuation on it. Then there is a bijection between subsheaves of $P(A,E)$ and strict, extensional functions $s : A \to \Omega X$.*

The subsheaf is described by an ΩX-valuation E' on A, $E'(a,b) = E(a,b) \wedge s(a)$; note that $E'(a,a) = s(a)$.

Proof. Let Y be the subsheaf. Stalkwise, each x^*Y is a subset of $A/\sim_x$ and hence can be represented as a subset Z_x of A that is a union of equivalence classes. This corresponds to two conditions

$$a \in Z_x \Longrightarrow a \sim_x a$$
$$a \sim_x b \in Z_x \Longrightarrow a \in Z_x$$

In Theorem 26, let $s : A \to \Omega X$ be the fuzzy set corresponding to Z. Then the two conditions above are equivalent to s being strict and extensional. If $\sim'_x$ is the partial equivalence relation corresponding to E' then $a \sim'_x b$ iff $a \sim_x b$ and $a \in Z_x$, so $A/\sim'_x \cong x^*Y$ as required. ■

6.4 Morphisms between ΩX-valued sets

We now turn to the localic version of Proposition 9.

Proposition 33 *Let A and B be two ΩX-valued sets, with corresponding local homeomorphisms $p : Y \to X$ and $q : Z \to X$. Then there is a bijection between maps $f : Y \to Z$ over X and morphisms $\theta : A \to B$.*

Proof. Reasoning stalkwise just as in the spatial case, we get a stalk function $f_x : A/\sim_x \to B/\sim_x$ and that is equivalent to a relation $\theta_x \subseteq A \times B$, with $a\theta_x b$

iff $a \sim_x a$ and $f_x([a]_x) = [b]_x$. An arbitrary relation θ_x corresponds to a stalk function in this way iff

$$\begin{aligned} a\theta_x b &\Longrightarrow a \sim_x a \wedge b \sim_x b \\ a' \sim_x a \wedge a\theta_x b \wedge b \sim_x b' &\Longrightarrow a'\theta_x b' \\ a\theta_x b \wedge a\theta_x b' &\Longrightarrow b \sim_x b' \\ a \sim_x a &\Longrightarrow (\exists b) a\theta_x b \end{aligned}$$

The θ_xs define a subobject of the constant sheaf $!_X^*(A \times B)$ and hence correspond to a function $\theta : A \times B \to \Omega X$, and then the four conditions above correspond to the four in the definition of ΩX-valued set morphism. ■

Proposition 34 *1. Let $\theta : A \to B$ and $\phi : B \to C$ be two morphisms of ΩX-valued sets. Then their composite is given by*

$$(\phi \circ \theta)(a, c) = \bigvee_{b \in B} \theta(a, b) \wedge \phi(b, c).$$

2. The identity morphism on an ΩX-valued set A is given by $\mathrm{Id}(a, a') = E(a, a')$.

Proof. (1) Let f_x and g_x be the corresponding stalk functions at point x. Their functional composite is given by the relational composite of θ_x and ϕ_x. By a calculation similar to that used in Proposition 31 to calculate $\sim_x \circ \sim_x$, it is given by the stated formula for $\phi \circ \theta$.

(2) Follows because $[a]_x = [a']_x$ iff $E(a, a')$. ■

Proposition 35 *Let $\theta : A \to B$ be a morphism of ΩX-valued sets.*

1. *θ is monic (all stalk functions f_x are 1-1) iff $\theta(a, b) \wedge \theta(a', b) \le E(a, a')$ for all a, a', b.*
2. *θ is epi (all stalk functions f_x are onto) iff $E(b, b) \le \bigvee_a \theta(a, b)$.*
3. *θ is an isomorphism iff both the above conditions hold.*

Proof. (1) Monicity is characterized geometrically, so θ is monic if we can prove geometrically that every stalk function is monic (1-1). Let x be a point of X. f_x is monic iff $f_x([a]_x) = f_x([a']_x) \Rightarrow a \sim_x a'$, i.e. if $x \vDash \theta(a, b) \wedge \theta(a', b)$ for some b then $x \vDash E(a, a')$. The result follows.

(2) is a similar argument, and (3) combines the first two. ■

Proposition 36 *Let $\theta : A \to B$ be a morphism of ΩX-valued sets. Its image is given by (according to Proposition 32) a strict, extensional function s on B,*

$$s(b) = \bigvee_{a \in A} \theta(a, b).$$

The epi-mono factorization of θ is then $\theta_m \circ \theta_e$ where

$$\theta_e(a,b) = \theta(a,b),$$
$$\theta_m(b,b') = s(b) \wedge E_B(b,b').$$

Proof. From the previous results it is now straightforward to check that s is strict and extensional, that $\theta_e : (A, E_A) \to (B, E')$ and $\theta_m : (B, E') \to (B, E_B)$ are morphisms, with θ_e epi and θ_m mono, and that $\theta = \theta_m \circ \theta_e$. But it is also easy to see that they give the image stalkwise. ■

6.5 Other geometric constructions

We give some more examples to illustrate the techniques. Note that in each case the construction is close to what is familiar from set theory. This depends on the fact that we are happy to work with incomplete ΩX-valued sets. If we had to complete, the constructions would be made much more complicated.

Products: Let A and B be ΩX-valued sets. How can we define the product as ΩX-set? It would seem natural use the set product $A \times B$, with equality defined componentwise – $(a_1, b_1) \sim_x (a_2, b_2) = a_1 \sim_x a_2$ and $b_1 \sim_x b_2$. This gives us the definition of E on $A \times B$, namely

$$E((a_1,b_1),(a_2,b_2)) = E(a_1,a_2) \wedge E(b_1,b_2).$$

To check that that does indeed define the sheaf product, it suffices to check the stalks. This is because binary product is a geometric construction, so sheaf product exists and is calculated stalkwise. In other words, we must check $(A \times B)/\sim_x \cong A/\sim_x \times B/\sim_x$, which is clear. We also need the product projections $p : A \times B \to A$ and $q : A \times B \to B$. Clearly we want $p_x([a,b]) = [a']$ iff $a \sim_x a'$ (and $b \sim_x b$), which translates into

$$p((a,b),a') = E(a,a') \wedge E(b,b).$$

One can then check that this is a morphism and gives the correct stalk functions.

Equalizers: Let A and B be ΩX-valued sets and let $\theta, \phi : A \to B$ be morphisms. The equalizer is a subsheaf of A, given by the same set A and a strict extensional map s. The elements of the corresponding substalk are those $[a]$ such that $\theta_x([a]) = \phi_x([a])$, so we get

$$s(a) = \bigvee_b \theta(a,b) \wedge \phi(a,b).$$

Again, one must check that this gives the right stalks.

Coproducts: Let (A_i, E_i) $(i \in I)$ be ΩX-valued sets. (Note that we can deal with arbitrary set-indexed coproducts. This is by contrast with the finitary products above, which would not generalize to the infinite case; geometric constructions include arbitrary colimits but only finite limits.) Let A be the coproduct (disjoint union) $\coprod_i A_i$, and define

$$E(\langle i,a\rangle, \langle j,a'\rangle) = \bigvee \{E_k(a,a') \mid i = k = j\}.$$

The disjunction on the right is set up carefully to be geometric. Classically, one might have been expecting to split it into cases as $E_i(a, a')$ (if $i = j$), or $\perp$ (if $i \neq j$). Stalkwise, $A/\sim_x \cong \coprod_i (A_i/\sim_x)$.

Coequalizers: This is a good example of a case where the geometric type construction is harder to characterize as logic. Let A and B be ΩX-valued sets, and let $\theta, \phi : (A, E_A) \to (B, E_B)$ be morphisms. Stalkwise, we need to calculate the coequalizer as a quotient for a partial equivalence relation $\approx_x$ on B that includes $\sim_x$ (though without making any more elements self-related). $\approx_x$ is generated by the relation that relates $\theta_x([a]_x)$ to $\phi_x([a]_x)$ for each a. (We have abused notation slightly by writing θ_x, ϕ_x as functions as well as relations.) In order to capture the transitive closure part of this, we need to consider chains of related links. We find $b \approx_x b'$ iff for some $n \geq 0$ there are sequences $b_0, \dots, b_n \in B$ and $a_1, \dots, a_n$ such that $b_0 \sim_x b$, $b_n \sim_x b'$, and for each i $(1 \leq i \leq n)$ we have either $a_i \theta_x b_{i-1}$ and $a_i \phi_x b_i$, or $a_i \phi_x b_{i-1}$ and $a_i \theta_x b_i$. The appropriate ΩX-valuation on B to give this quotient is

$$
\begin{aligned}
E'_B(b, b') = \bigvee_{n \geq 0} \bigvee_{b_0, \dots, b_n \in B} \bigvee_{a_1, \dots, a_n \in A} & (E_B(b, b_0) \wedge E_B(b_n, b') \\
& \wedge \bigwedge_{i=1}^{n} ((\theta(a_i, b_{i-1}) \wedge \phi(a_i, b_i)) \vee (\phi(a_i, b_{i-1}) \wedge \theta(a_i, b_i)))).
\end{aligned}
$$

The proof is essentially standard once one accepts that the geometric construction described above does indeed calculate the symmetric, transitive closure of $\sim_x \cup \{(b, b') \mid (\exists a)(a \theta_x b \wedge a \phi_x b')\}$. However, it is an interesting exercise to prove the coequalizer property directly. It comes down to showing that if $\psi : (B, E_B) \to (C, E_C)$ has $\psi \circ \theta = \psi \circ \phi$, then the same ψ is also a morphism $(B, E'_B) \to (C, E_C)$. The hardest part is verifying the condition that $E'_B(b, b') \wedge \psi(b', c) \leq \psi(b, c)$. Given $n, b_0, \dots, b_n, a_1, \dots, a_n$ as in the definition of $E'_B(b, b')$, one must show that $D \wedge \psi(b', c) \leq \psi(b, c)$ where D is the corresponding disjunct of $E'_B(b, b')$. By induction on n one shows that $D \wedge \psi(b', c) \leq \psi(b_i, c)$ $(0 \leq i \leq n)$. This follows from a calculation that

$$
\begin{aligned}
\theta(a_i, b_{i-1}) \wedge \phi(a_i, b_i) \wedge \psi(b_i, c) &\leq \theta(a_i, b_{i-1}) \wedge (\psi \circ \phi)(a_i, c_i) \\
&= \theta(a_i, b_{i-1}) \wedge (\psi \circ \theta)(a_i, c_i) \\
&= \bigvee_{b''} \theta(a_i, b_{i-1}) \wedge \theta(a_i, b'') \wedge \psi(b'', c) \\
&\leq \bigvee_{b''} E_B(b_{i-1}, b'') \wedge \psi(b'', c) \leq \psi(b_{i-1}, c).
\end{aligned}
$$

List sets: If A is a set, we write A^* for the set of finite lists of elements of A. How can we make the analogous construction for ΩX-valued sets A? It would seem natural to use the set A^*, with $(a_i)_{i=0}^{m-1} \sim_x (b_i)_{i=0}^{n-1}$ if $m = n$ and for each index i we have $a_i \sim_x b_i$. This translates into

$$
E((a_i)_{i=0}^{m-1}, (b_i)_{i=0}^{n-1}) = \bigvee \{ \bigwedge_{i=0}^{m-1} E(a_i, b_i) \mid m = n \}.
$$

(Note that the expression on the right evaluates to $\bot$ if $m \neq n$, since the set of disjuncts is then empty.) This gives the correct stalks. Associated structure, such as the concatenation operation that, with the empty list as unit, makes the list set into a monoid, can also be checked stalkwise.

Finite powersets: (Note that the full powerset is not geometric.) The finite powerset $\mathcal{F}A$ can be constructed as a quotient of the list set A^*, using $(a_i)_{i=0}^{m-1} \sim_x (b_i)_{i=0}^{n-1}$ if for each index i there is some index j with $a_i \sim_x b_j$, and vice versa. This translates into

$$E((a_i)_{i=0}^{m-1}, (b_i)_{i=0}^{n-1}) = \bigwedge_{i=0}^{m-1} \bigvee_{j=0}^{n-1} E(a_i, b_j) \wedge \bigwedge_{j=0}^{n-1} \bigvee_{i=0}^{m-1} E(a_i, b_j).$$

7 The object classifier

We have stressed the idea that a sheaf over a space (or locale) X is a continuous set-valued map on X, but also that there is no conventional topology on the class of sets that realizes this idea. Here we briefly explain the topos-theoretic idea that the *object classifier* $\mathcal{S}[U]$ expresses the "space of sets".[12] This somewhat technical section is less relevant to the understanding of fuzzy sets as sheaves.

In the category **Loc** of locales we have the idea that morphisms $W \to X$ are "points of X at stage W", with points at stage 1 (the terminal locale) distinguished as *global* points. In fact this abstract idea makes sense in any category, and a fruitful one to generalize to is the category **Top** of Grothendieck toposes and geometric morphisms. The reason that this is a generalization is that **Loc** embeds fully and faithfully in **Top**, taking each locale X to the topos $\mathcal{S}X$ of sheaves over X. Locale maps $X \to Y$ are equivalent to geometric morphisms $\mathcal{S}X \to \mathcal{S}Y$, so a geometric morphism should be thought of as a "continuous map between toposes" and the global points of a topos $\mathcal{E}$ are the geometric morphisms from $\mathcal{S}1 \simeq$ **Set** to $\mathcal{E}$.

Since **Loc** embeds in **Top**, it would be possible to conduct locale theory entirely in terms of the categories of sheaves, but of course one does not normally do this – it is much simpler to work in terms of the lattices of opens. What makes locales localic (as opposed to the more general toposes) is that there are enough opens. For a general topos there are not enough opens and so sheaves have to be used instead.

The distinction can also be understood in terms of geometric logic. A topos in general is "the space of models" for a predicate geometric theory, while for a locale the theory is propositional – no sorts (so no terms[13]).

We argued that sheaves over X were to be thought of as continuous maps from X to "the space of sets", but with nothing in the usual world of topology that can serve as this space of sets. Now in the generalized world of toposes,

[12] The object classifier is a particular topos, and is not to be confused with the *subobject classifier* Ω, a particular object in each topos.

[13] Actually, it does not matter if the geometric type constructions are used to make "constant types" such as $\mathbb{N}$ and $\mathbb{Q}$ (for the natural numbers and the rationals), and their terms are used.

it seems we must seek a topos $\mathcal{E}$ such that sheaves over X are equivalent to geometric morphisms from $\mathcal{S}X$ to $\mathcal{E}$. Our $\mathcal{E}$ is the object classifier $\mathcal{S}[U]$. It is a topos generalization of the Sierpiński locale $\mathbb{S}$ (Definition 23), which classifies subsingleton sets.

$\mathcal{S}[U]$ is defined to be the functor category $[\mathbf{Set}_f, \mathbf{Set}]$, where $\mathbf{Set}_f$ is the category of finite sets. In other words, the objects of $\mathcal{S}[U]$ are the functors from $\mathbf{Set}_f$ to $\mathbf{Set}$, and the morphisms are the natural transformations. But any set is a filtered colimit of finite sets, and it can be proved that any functor from $\mathbf{Set}_f$ to $\mathbf{Set}$ can be extended, uniquely up to isomorphism, to a functor from $\mathbf{Set}$ to $\mathbf{Set}$ that preserves filtered colimits. "Preservation of filtered colimits" is a categorical generalization of preservation of directed joins, i.e. Scott continuity.

The key property of $\mathcal{S}[U]$ is that it *classifies* the geometric theory with one sort and no functions, predicates or axioms. A model of this theory is simply a set (the carrier of the single sort), or, in a general topos $\mathcal{F}$, an object of $\mathcal{F}$. The "classifier" property is as follows. First note there is a special object of $\mathcal{S}[U]$, the *generic object* $U : \mathbf{Set}_f \to \mathbf{Set}$ defined by $U(S) = S$. (Of course, this is just a fancy name for the inclusion functor.) Now for any Grothendieck topos $\mathcal{F}$, and for any object X of $\mathcal{F}$, there is a unique (up to isomorphism) geometric morphism $f : \mathcal{F} \to \mathcal{S}[U]$ such that $f^*(U) = X$. We sketch a proof of this; more details can be found in standard texts such as [MLM92]. First note that the Yoneda embedding $\mathcal{Y} : (\mathbf{Set}_f)^{op} \to [\mathbf{Set}_f, \mathbf{Set}]$ is a *free cocompletion*. Any functor $F : (\mathbf{Set}_f)^{op} \to \mathcal{C}$, where $\mathcal{C}$ is cocomplete (i.e. has all small colimits), factors uniquely (up to isomorphism) as $\overline{F} \circ \mathcal{Y}$ where $\overline{F} : [\mathbf{Set}_f, \mathbf{Set}] \to \mathcal{C}$ is cocontinuous (preserves all colimits). Moreover, if F preserves finite limits then so does $\overline{F}$, and in that case, $\overline{F}$ is the inverse image part of a geometric morphism. Now given X in $\mathcal{F}$, we can define $F : (\mathbf{Set}_f)^{op} \to \mathcal{F}$ by $F(S) = X^{|S|}$. This preserves finite limits, and so gives us our geometric morphism. It is easily calculated that $U = \mathcal{Y}(1)$, and so $\overline{F}(U) \cong F(1) = X^1 = X$ as required.

From this we see that the points of $\mathcal{S}[U]$ at stage $\mathcal{F}$, in other words the geometric morphisms from $\mathcal{F}$ to $\mathcal{S}[U]$, are just the objects of $\mathcal{F}$, and in particular the global points are just sets (since $\mathbf{Sets} \simeq \mathcal{S}1$). Thus in the world of Grothendieck toposes we can understand $\mathcal{S}[U]$ as the "space of sets".

It is interesting to calculate the opens – in other words, the subobjects of 1 – for $\mathcal{S}[U]$. We shall do the calculation classically. The terminal object 1 is the functor $\mathbf{Set}_f \to \mathbf{Set}$ that takes every finite set S to a singleton 1. A subobject of this is a functor F that takes every S to a subset of 1, and (classically, at least) there are only two possible such subsets, namely 1 itself and the empty set $\emptyset$. Clearly there are two constant functors corresponding to 1 and $\emptyset$. Also, if $F(1) = \emptyset$ then for every S the unique function $! : S \to 1$ gives $F(!) : F(S) \to F(1) = \emptyset$ and it follows that F is constant $\emptyset$. Now suppose $F(1) = 1$. For every *inhabited* S there is a function $1 \to S$, and hence a function $1 = F(1) \to F(S)$, so $F(S) = 1$. It follows that the only possible non-constant functor F takes $\emptyset$ to $\emptyset$, and every inhabited S to 1.

Thus, classically, $\mathcal{S}[U]$ has three opens. As subclasses of the class of sets, they correspond to the empty class, the class of all sets, and the class of inhabited sets. Topologically, this cannot be distinguished from the Sierpiński locale $\mathbb{S}$.

In fact, $\mathbb{S}$ is the *localic reflection* of $\mathcal{S}[U]$, what you get when you try to deal only with its opens and without using sheaves.

8 Conclusions

Their range of different technical expressions can make sheaves daunting to the newcomer. However, there is a simple unifying intuition: a sheaf over a locale X is a continuous set-valued function, the value at a point x being the stalk. We have described why continuity may be thought of as geometricity of the construction. Geometric constructions on sheaves, such as finite limits, arbitrary colimits and free algebra constructions, can be performed stalkwise. When carried out on ΩX-valued sets, they can often be formulated simply if one does not require the ΩX-valued sets to be complete, and can be verified by checking the actions on stalks.

It practice it is not always obvious to a beginner just what constitutes a geometric type construction. Nonetheless, I hope to have demonstrated three points.

1. The stalkwise reasoning is intuitively valuable where valid.
2. There is a definite criterion (geometricity) for its validity.
3. Despite appearances, it can provide rigorous arguments even for non-spatial locales.

We have seen how geometric logic enables us to use spatial language for *frame*-valued sets. This works very well, but it should be pointed out that it is not at all clear how to extend this to sets valued in other kinds of structures. The techniques deserve further investigation in the setting of quantal sets, particularly where the quantale is the real line (with + as the quantale multiplication). This relates to generalized metric spaces in the sense of [Law73].

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