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FORMAL TOPOLOGY and GEOMETRIC LOGIC

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I Space = geometric theory

II Map = geometric transformation of points to points

• III Bundle = geometric transformation of points to spaces

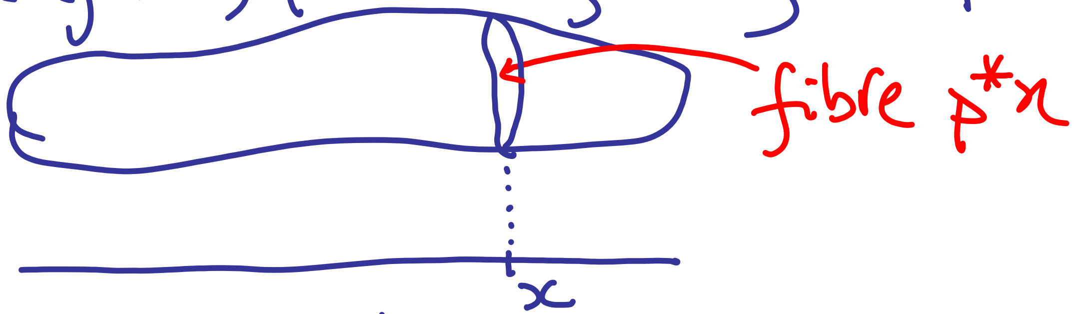
Definition:

A bundle is a map over X with codomain X

For simplicity: work with locales (all theories propositional)

thought of as space (fibre) parametrized by base point

$$\begin{array}{c} Y \\ \downarrow p \\ X \end{array}$$



- get fibrewise topology of bundles

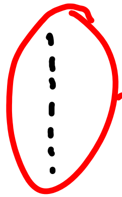
fibre p^*x = pullback of bundle along point x

Bundle = space-valued map

$p: Y \to X$

X is given

$\boxed{\text{Let } x \text{ be a point of } X. \text{ Then } p^* x = \text{[drawing] is a space}}$



geometric construction

Y = space of pairs (x, y), y a point of $p^* x$

Goal . Justify this

. What is a geometric construction of a space?

Discrete case: set-valued map

Say $X = [\mathbb{T}_1]$

Let x be a point of X

Then $p^* x =$ $\vdots$

is a set

geometric construction

- sheaf over X
- map $S : X \longrightarrow [set]$
- sort characterized geometrically over $\mathbb{T}_1$

$[\mathbb{T}_1] \simeq [\mathbb{T}_1, S]$

$Y = [\mathbb{T}_2] = [\mathbb{T}_1, S, x \in S]$

$$\begin{array}{c} Y \\ \downarrow \\ [\mathbb{T}_1, S] \simeq X \end{array}$$

Y as pullback

discrete space for $S(x)$

$$
\begin{tikzcd}
p^*x \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, "p"] & {[\mathrm{set}, \mathrm{elt}]} \arrow[d] \\
1 \arrow[r, "x"'] & X \arrow[r, "S"'] & {[\mathrm{set}]}
\end{tikzcd}
$$

Geometric constructions on sets

Characterizable geometrically

Constructible using known geometric primitives

- finite limits, finite colimits
- list types

  } AUs

- set-indexed colimits

Preserved by inverse image functors

As constructions on bundles (local homeomorphisms)

- preserved by pullback
- hence work fibrewise

Space-valued maps

Constructing space directly

$$\begin{tikzcd} & D \arrow[dl, "\rho"'] \arrow[dr, "\pi"] & \\ \mathcal{F}G & & R \arrow[ll, "\lambda"] \end{tikzcd} \qquad \Gamma$$

e.g. space = GRD-system

G, R, D – set-valued maps

$\mathcal{F}$ = finite powerset – geometric

λ, ρ, π – set-valued functions

$$X = [\mathbb{T},] \simeq [\mathbb{T},, \Gamma]$$

$$Y = [\mathbb{T},, \Gamma, \text{pt of } \Gamma]$$

$$p \downarrow$$

$$[\mathbb{T},, \Gamma] \simeq X$$

$$F \subseteq G$$

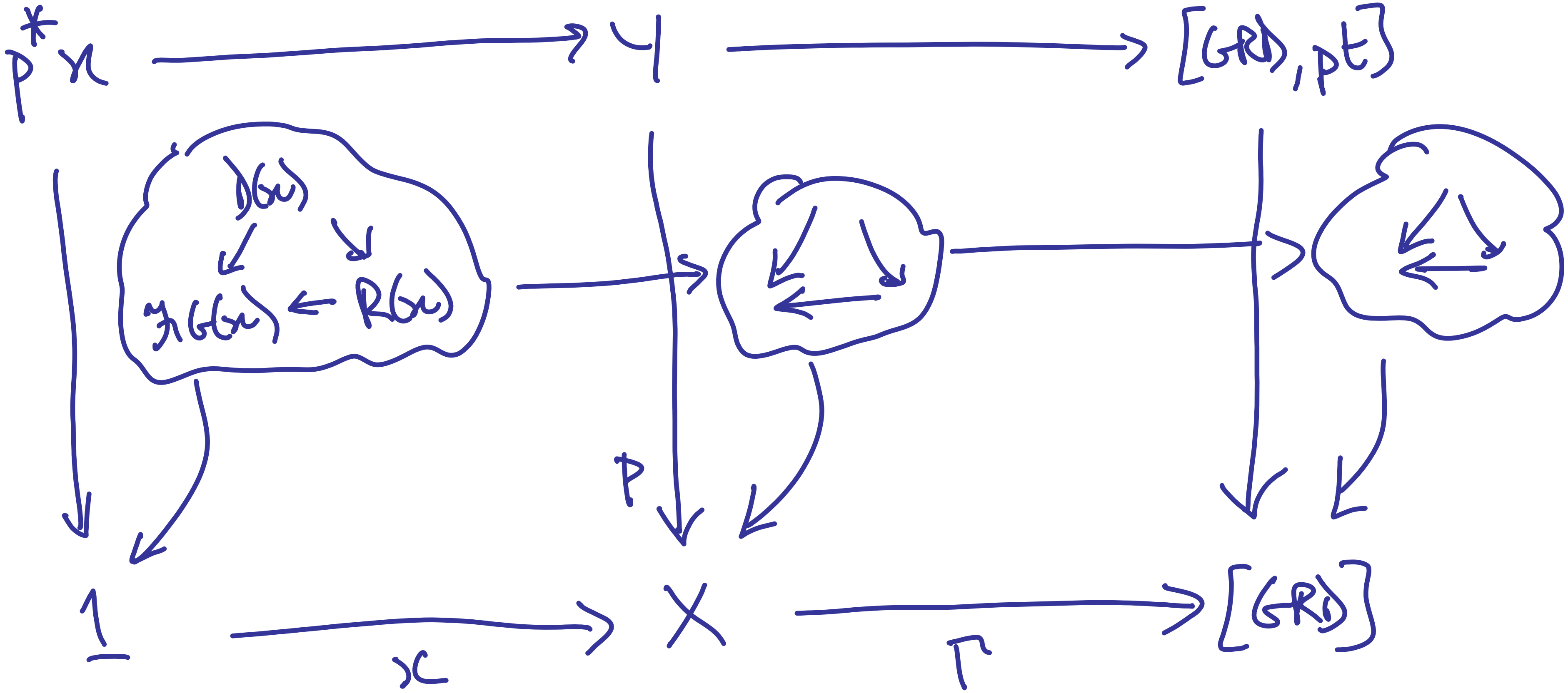
$$\left\{ \forall g \in \lambda(r).\, F(g) \vdash_{r:R} \exists d:D.\, \big(\pi(d) = r \wedge \forall g \in \rho(d).\, F(g)\big) \right.$$

Construction $\Gamma \mapsto$ bundle preserved by pullback

- because <u>defined</u> by pulling back generic construction
- works fibrewise

Fibre p^*x constructed as described from

$$
\begin{tikzcd}
 & D(x) \arrow[dl, "p(x)"'] \arrow[dr, "\pi(x)"] & \\
\mathrm{Fr}\,G(x) & & R(x) \arrow[ll, "\lambda(x)"]
\end{tikzcd}
$$



$p^*\mathcal{X}$
Y
$[G(R), pt]$
$\mathcal{D}(x)$
$\mathcal{J}G(x)$
$R(x)$
p
1
X
$[G(R)]$
x
Γ

Think: everything is a bundle

Geometricity = preservation under pullback

$\Rightarrow$ done fibrewise

e.g. - for sets fibrewise discrete, local homeomorphisms

- finite limits, colimits etc.

e.g. $\Gamma \longmapsto$ bundle

e.g. Anything else?

e.g. powerlocale (lower - P_L)

Impredicatively:

$\Omega P_L X$ = frame freely generated by ΩX qua suplattice

Map $W \to P_L X$ = function $\Omega W \leftarrow \Omega X$ preserving all joins

Point of $P_L X$ = overt, weakly closed sublocale of X

$P_L X$: localic hyperspace

Impredicatively

Localic bundle theorem:

Joyal & Tierney
Fourman & Scott

Internal frames in $\mathcal{S}X$ $\simeq$ localic bundle over X

$Y \mapsto P_L Y$ topos-valid

$\therefore$ translates to construction on bundles

It's geometric!

$$\begin{tikzcd}
Y \arrow[dr, "p"'] & & P_L^X Y \arrow[dl] \\
& X &
\end{tikzcd}$$

Predicative results

overt – positivity predicate on opens

- Point of $P_L Y$ = overt, weakly closed subspace of Y

- Y overt $\iff$
$$\begin{tikzcd} 1 \arrow[r, dashed, shift right, "T"'] & P_L Y \arrow[l, shift right, "!"'] \end{tikzcd}$$
with $! \dashv T$

  As subspace: T must be Y

- Bundle p fibrewise overt $\exists$ ...

$$\begin{tikzcd} Y \arrow[dr, "p"'] & & P_L^X Y \arrow[dl, shift right, "q"'] \\ & X \arrow[ur, shift right, "T"'] \arrow[ur, phantom, "\dashv"] & \end{tikzcd}$$

with $q \circ T = \mathrm{Id}_X$

Fibrewise overt $\Longleftrightarrow$ open

Impredicatively: Joyal & Tierney

$$\begin{tikzcd} V \arrow[r, hook, "\text{open}"] & Y \arrow[d, "p"] \\ & X \end{tikzcd}$$

Idea x in $p(V)$

$\Longleftrightarrow p^* x \wedge V$ positive

Exercise

X discrete $\Leftrightarrow$ $1 \xleftarrow{!} X \xrightarrow{\Delta} X \times X$ both open

Hence fibrewise discreteness for $\begin{array}{c} Y \\ p\downarrow \\ X \end{array}$:

p and $\Delta : Y \longrightarrow Y \times_X Y$ both open

Other geometric constructions on spaces

Upper powerlocale P_u : point = compact fitted subspace

- used to characterize compactness

Double powerlocale $\mathbb{P} = P_u P_L \cong P_L P_u$

$\mathbb{P}X \cong \$^{\$^X}$

Johnstone, Vickers, Townsend

Valuation locale $\mathfrak{V}$: point = regular measure

- localic theory of integration
- central role in topos approach to quantum foundations

Heckmann Vickers Coquand Spitters

Quantum foundations

Isham Butterfield Döring
Heunen Landsman
Spitters

Spectral bundle

$$\begin{tikzcd} \Sigma \arrow[d, "p"'] \\ B \end{tikzcd}$$

— spectra

— contexts

$$\begin{tikzcd} \Sigma \arrow[dr, "p"'] & & \bigsqcup_B \Sigma \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

no cross-sections (Kochen Specker) — p

q — cross-sections got from quantum states

qbit ...

Fibrewise topology of bundles

Topology "parametrized by point of X"

in box

i.e. in $\mathcal{S}X$

$\boxed{\begin{array}{l}\text{Let } x \text{ be a point of } X \\ \quad\vdots\end{array}}$

- Must be topos-valid
- For good topology must be point-free
- To work fibrewise want geometricity

  space = geometric theory

  map = geometric transformation of models

  bundle = space-valued map

Selected bibliography

Geom. logic, cat. semantics

Johnstone: Sketches of an elephant vol 2
Stone Spaces

frames

Joyal & Tierney: An extension of the Galois theory of Grothendieck

Bundles

Coquand & Spitters: Integrals and valuations

Vickers: Locales & toposes as spaces (chapter in Handbook of Spatial Logics)

geom. types

Topical categories of domains

geometricity = continuity, bundles

The double powerlocale and exponentiation

etc. etc.

– with Maietti: An induction principle for consequence in arithmetic universes