**Talk given 27 Nov 2011 PSSL 91 Amsterdam**

**Starting point:** Arithmetic universes **"**AUs**, Joyal**

**Preprint on web:** "An induction principle for consequence in arithmetic universes".

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**Problem:** AUs not cartesian closed

**Propositional Predicate Example:** $\mathbb{R}$

**Finite list of sorted variables**

**Finite order, many sorted, positive, infinitary**

**Signature $\Sigma$:** Sorts, functions, predicates

**Formulae $\phi$:** use $T, \wedge, \top, V, =, \exists$

**Dilemmas can be infinite**

**Formalisation in context $(x, \phi)$**

**Sequents** $\phi \vdash \psi, (x, \phi), (x, \psi)$ both formalised in context

**Formalisation of axioms**

**Theory $\Sigma$ over $\Sigma$:** set of sequents

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**Weak locatedness $\Rightarrow$ strong**

**Given $\varepsilon > 0$**

**By induction:**

$\forall n \in \mathbb{N} \quad (\exists q < 2^n \exists (L(q), R(q)) \land q - q_0 < 2^n \varepsilon)

\to \exists q \in \mathbb{R} (L(q), R(q)) \land q - q_0 < 3 \varepsilon$)

**Base case $n = 0$: immediate**

**Suppose $L(q) \land R(q) \land q - q_0 < 2^n \varepsilon$**

**Cases**

- Define $q_1$ by $q_1 = \frac{q + q_0}{2} < 2^{n+1} \varepsilon$
- $L(s_2)$
- $L(s_3)$
- $L(s_5)$
- $L(s_6)$

**Use induction**
For every $A : \text{list}(A)$, there is a map $\varnothing \to \text{list}(A)$ such that $A \times \text{list}(A) \to \text{list}(A)$ and $\varnothing \times \text{list}(A) \to \text{list}(A)$.

1. $\varnothing \to \text{list}(A)$
2. $A \times \text{list}(A) \to \text{list}(A)$
3. $\varnothing \times \text{list}(A) \to \text{list}(A)$

\[ A \times \text{list}(A) \cong \text{list}(A) \]

Let $\varnothing = \text{cons}(\varnothing, \varnothing, \ldots, \varnothing)$.

\[ \text{cons}(a, [a_1, \ldots, a_n]) = [a, a_1, \ldots, a_n] \]

Theory of AU is cartesian and can present with generators and relations.

For example, AU freely generated by a Dedekind section similar to $\text{Sh}(R)$.

Arithmetic space $X = \text{AU} \times X$ has AU functor in reverse.

**Strictness**

- AU has a canonical structure.
- Strict AU functors preserve it on the nose.

- AU functor preserves up to iso.

Universal algebra uses strict AU functor.

- Homomorphisms for cartesian theory of AU.

We need to use non-strict AU.

- Characterized by $A [u : u]$, where $u$ is a function.

**Locatedness**

- "AU freely generated by Dedekind section".

- Which locatedness axiom?

Equivalence proof relies on cartesian closedness to interpret $\Rightarrow$ as formula connective.

BUT AU are not cartesian closed in general.

Are axioms equivalent in AU?
**Induction in \(\mathbb{N}\)**

\[ N = \text{List}(\mathbb{N}) \]

1. \( \phi \rightarrow N \quad \phi(0), \quad \phi(n) \rightarrow \phi(n+1) \Rightarrow \tau \rightarrow \phi(n) \)
   - \( \phi \) a subset of \( N \) closed under \( 0, \text{succ} \)

\( \vdash: \text{whole of } N \)

2. \( \psi: \phi \rightarrow N \quad \phi(0) \rightarrow \psi(0) \quad ? \quad \phi(n) \rightarrow \psi(n) \)
   - **induction step?**
   - **for that fixed \( n \)**

**Induction hypothesis:**

- Fix \( n \) (generically), assume \( \phi(n) \rightarrow \psi(n) \)
- Working in \( \forall \mathcal{A}[n: \mathcal{A}] [\phi(n) \rightarrow \psi(n)] = \mathcal{A}' \) (say)

**Induction step:** In \( \mathcal{A}' \) have \( \phi(n+1) \rightarrow \psi(n+1) \)
- Can we deduce \( \phi(n) \rightarrow \psi(n) \) in \( \mathcal{A} \)? **Yes!**

**Proof outline**

- **Structure theorems ** \( \mathcal{A}[n: u] \approx \mathcal{A}/u \)
  - \( \phi \rightarrow 1 \approx \mathcal{A}[\phi] \approx \text{category of sheaves} \)
  - **subspace** open \( \mathcal{A}[\tau \rightarrow \psi] \), closed \( \mathcal{A}[\phi \rightarrow 1] \)
  - generate lattice \( = \mathcal{B} \langle \text{Sub}_{\mathcal{A}}(1) \rangle \)
  - classical logic of subspaces conservative over coherent logic of subobject
  - use Boolean manipulation of induction step to find properties in \( \mathcal{A} \)
  - new induction lemma to deduce conclusion from those properties

**Structure theorems**

\( \mathcal{A}[\phi \rightarrow 1] \) is a Boolean algebra

- \( B_\phi = \mathcal{B} \langle \text{Sub}_{\mathcal{A}}(1) \rangle \)

- Finitary sheaf: only finitary pasting

- Closed subspace is Stone over superspace

- \( x: \phi \rightarrow \text{Clop} \quad x \rightarrow (x \rightarrow \phi) \)

- \( x \rightarrow (x \rightarrow \phi) \)

- Presheaf \( F(d) \rightarrow F(0) \), iso if \( \phi \)

- Sheaf \( \approx F(1) \)

- \( \approx \text{object } U \text{ of } \mathcal{A} \text{ s.t. } U \rightarrow 1 \text{ iso if } \phi \)

- For any \( U \): coequalize \( U \rightarrow u \rightarrow u + \phi \rightarrow V(u) \)

- \( \approx \text{Va is monad, } U \text{ iso } \)

- \( \approx \text{finitely pasting} \)
Subspaces

A \[ m \cdot 1 \] m monic in A

Preorder: A \[ m \cdot 1 \] \leq A \[ m \cdot 1 \]

If m_2 invertible in A \[ m \cdot 1 \]

Semidirect :

A \[ m \cdot 1 \] \cdot A \[ m \cdot 1 \] = A \[ (m + m_2) \cdot 1 \] \leq A \[ m \cdot 1 \] \cdot A \[ m \cdot 1 \]

If \( \phi, \psi \mapsto 1 \):

- open \( \psi = A \[ \phi \mapsto 1 \] \cdot A \[ \psi \mapsto 1 \] \)
- closed \( \neg \phi = A \[ \psi \mapsto 1 \] \cdot A \[ \phi \mapsto 1 \] \)
- crescent \( \neg \phi \mapsto \phi \)
- co-crescent \( \phi \mapsto \neg \phi \)

Booleans algebra of subspaces generated by

opens & closeds

If \( a = \bigwedge (\neg \phi, \psi) \in \mathcal{BA}(\text{Sub}_A(1)) \) write

\[ 0(a) = \bigwedge \mathcal{A}[\phi_i \mapsto 1] \] - meet of co-crescents

- preserves joins
- is order embedding (conservativity)
- closed subspace \( \neg \phi \) is Boolean complement of \( \phi \)
- \( A[\phi \mapsto 1] = \neg \phi \mapsto \psi \)

Induction hypothesis

\( A\[n \cdot 1 \] \neq \psi(n) \)

- as subspace of \( A[n \cdot 1 \] \)
- \( \neg \phi(n) \mapsto \psi(n) \)

Induction step \( \neg \phi(n) \mapsto \psi(n) \leq \neg \phi(n) \mapsto \psi(n + 1) \)

\( \neg \phi(n) \leq \neg \phi(n) \mapsto \psi(n + 1) \)

\( \phi(n + 1) \leq \phi(n) \mapsto \psi(n + 1) \)

\( \phi(n + 1) \mapsto \psi(n + 1) \)

\( \neg (n + 1) \mapsto \psi(n + 1) \)

in A[n \cdot 1 \]

Induction lemma

If \( \phi(n + 1) \mapsto \psi(n) \)

\( \phi(n + 1) \mapsto \psi(n + 1) \) then \( \phi(n) \mapsto n \cdot 1 \)

For \( k \cdot 1 \) can define \( A(k) = \{ j \in \mathbb{N} | j \in k, \phi(j), \ldots, \phi(k) \} \)

\( f_k : A(k) \rightarrow \psi(k) \) by recursion on \( j + k \)

Cases for \( f_k(j) \): \( j = k \leq 0 \)

If \( f_k(j) \rightarrow \phi(k) \)

Given \( \phi, f_k(n) \rightarrow \psi(n) \)

If \( j < k \):

\( f_{k \cdot 1}(j) = \psi(k - 1) \)

\( \phi(k) \mapsto \psi(k - 1) \mapsto \psi(k) \) (done)
Conclusions

- Can prove implications by induction even though not cartesian closed
- More general induction principles too
- Some results analogous to those for lattice of sublocales
- Some structure theorems for some classifying AUs
- More plausibility to general idea: use AUs to provide finitary fragment of geometric logic, strictly stronger than coherent